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Decision Procedure for a Class of $(L_{\omega_1\omega})_t$-Types of T_3 Spaces

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The $(L_{\omega\omega})_t$-types of T_3 spaces are introduced in [1]. An effective procedure is then obtained to decide whether a type is satisfiable in some T_3 space. The expressibility of $(L_{\omega_1\omega})_t$ for T_3 spaces is studied in [2]. For this purpose a class of $(L_{\omega_1\omega})_t$-types is introduced and in this way we obtain a characterization of the $(L_{\omega_1\omega})_t$-equivalence for a wide class of T_3 spaces. In the present paper, we prove that there is a decision procedure for this class of types.

1 Preliminaries Suppose that A is a T_3 space and A^* is a subset of A. The *n-move game* $G_n(A^*, A)$ between two players, I and II, is defined as follows. In his i-th move ($i = 1 \ldots n$) player I chooses an arbitrary finite sequence $a_1, \ldots, a_r$ of points in A and then in his i-th move player II chooses a sequence of r neighborhoods U_1 of $a_1, \ldots, U_r$ of a_r in A. Let $U_1', \ldots, U_m'$ be all the neighborhoods chosen by II during the game. Player I wins if $A^* \subset U_1' \cup \ldots \cup U_m'$; otherwise, player II wins. Then, A^* is *accessible* (in the space A) if for some $n \in \omega$ player I has a winning strategy in the game $G_n(A^*, A)$. With this notion we can study the behavior of convergence. If $a \in A$ we say that A^* converges to a, $A^* \to a$, if a is an accumulation point of A^*. If $A^* \to a$ the following two types of convergence are considered:

(i) $A^* \xrightarrow{0} a$, if for every neighborhood U of a we have that $A^* \cap U$ is not accessible.
(ii) $A^* \xrightarrow{1} a$, if there is a neighborhood U of a with $A^* \cap U$ accessible.

The set S_n of n-types is then defined by induction on n:

$$S_0 = \{*\},\ S_{n+1} = P\left(\bigcup_{\lambda=0,1} \{(\alpha, \lambda) : \alpha \in S_n\}\right),$$

where $P(X)$ denotes the power set of X.

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The n-type of $a \in A$ is defined inductively by:

$$s_0(a, A) = *, \qquad s_{n+1}(a, A) = \bigcup_{\lambda=0,1} \{(\alpha, \lambda): \alpha \in S_n \text{ and } A_\alpha \xrightarrow{\lambda} a\},$$

where $A_\alpha = \{a \in A: s_n(a, A) = \alpha\}$.

For $m, n \in \omega$ with $m \leq n$ and $\alpha \in S_n$ the m-type $(\alpha)_m$ is defined in a way such that if A is a T_3 space and $a \in A$, $s_m(a, A) = (s_n(a, A))_m$ (cf. [2]). So, for $m \leq n$, the n-type of a determines the m-type of a.

By means of these types we obtain a characterization of the $(L_{\omega_1\omega})_t$-equivalence for the class of T_3 spaces of a-finite type. A space A is of *a-finite type* if for some $n_0 \in \omega$: $s_{n_0}(a, A) = s_{n_0}(a', A)$ implies $s_n(a, A) = s_n(a', A)$ for all $n > n_0$, $a, a' \in A$. Two T_3 spaces A and B are *a-type equivalent* if for every n-type α we have:

(i) A and B have the same number of points of n-type α
(ii) A_α is accessible iff B_α is accessible.

Then, if A and B are T_3 spaces of a-finite type we have: A and B are $(L_{\omega_1\omega})_t$-equivalent iff A and B are a-type equivalent (cf. [2]).

An n-type α is *satisfiable* in A if there is an $a \in A$ with $s_n(a, A) = \alpha$. The set of satisfiable n-types in A is denoted by $S_n(A)$. In this paper, we find an effective procedure to decide whether for a nonempty set S of n-types and f: $S \to \{0, 1\}$ there exists a T_3 space A such that $S = S_n(A)$ and, for any $\alpha \in S$, $f(\alpha) = 1$ iff A_α is accessible.

Ziegler's notion of an ω-tree employed in [1] to find a decision procedure for the $(L_{\omega\omega})_t$-types of T_3 spaces will also be useful in our case. This notion can be found in [1] and [2]. If $(T, \leq)$ is an ω-tree and $\sigma_\leq$ is the topology induced by $\leq$, we say that $(T, \sigma_\leq)$ is an *ω-topological tree.*

In the present paper, we presuppose acquaintance with [2] on the basic properties of the accessible sets. We refer to that paper for examples and basic ideas.

2 The decision procedure Suppose that A is a T_3 space and A_1^*, A_2^* are subsets of A. If $A_1^* \to a$ for every $a \in A_2^*$, we write $A_1^* \to A_2^*$. If $A_1^* \xrightarrow{\lambda} a$ for every $a \in A_2^*$, we write $A_1^* \xrightarrow{\lambda} A_2^*$ $(\lambda = 0, 1)$. The easy proof of the next lemma is left to the reader.

Lemma 1 *Suppose that A is a T_3 space, A_1^*, A_2^*, A^* are subsets of A and $a \in A$. We have:*
(a) *If $A_1^* \xrightarrow{0} A_2^*$ and $A_2^* \to a$, $A_1^* \xrightarrow{0} a$.*
(b) *If $A_1^* \to A_2^*$ and $A_2^* \xrightarrow{0} a$, $A_1^* \xrightarrow{0} a$.*
(c) *If $A^* \to A^*$, $A^* \xrightarrow{0} A^*$.*

We say that $(P, \prec, \rho, \mu)$ is an *accessibility relation* (in the sequel we shall say a-relation) if P is a nonempty finite set, $\prec$ is a transitive binary relation on P, ρ: $\{(p, q): p, q \in P \text{ with } p \prec q\} \to \{0, 1\}$ and μ: $P \to \{0, 1\}$ are functions such that the following three conditions hold (we write $p \leqq q$ if $p \prec q$ or $p = q$):

(i) For every $p \in P$:
$p \prec p$ implies $\rho(p, p) = 0$.
(ii) For every $p, q, p', q' \in P$ with $p \leqq p' \prec q' \leqq q$:
$\rho(p, q) = 1$ implies $\rho(p', q') = 1$.
(iii) For every $p, q \in P$ with $p \prec q$:
$\mu(q) = 1$ implies $\mu(p) = 1$ and $\rho(p, q) = 1$.

We say that $q \in P$ is minimal if there is no $p \in P$ with $p \prec q$.

If $(P, \prec, \rho, \mu)$ is an a-relation, it is very easy to check:

(i) For all $p, q, r \in P$ with $p \leqq r \leqq q$ and $p \neq q$:
$\rho(p, q) = 1$ implies $r \nprec r$.
(ii) For all $p, q \in P$ with $p \leqq q$:
$\mu(q) = 1$ implies p $\nprec p$.

Now suppose that A is an ω-topological tree. Note that the following hold:

(i) Any infinite path of A is a nonaccessible set.
(ii) For $a \in A$ and $n \in \omega$, the set of all the points of the paths of origin a and length $\leq n$ is accessible.

If $a \in A$ the set of immediate successors of a is denoted by $N(a)$.

Lemma 2 *Suppose that $(P, \prec, \rho, \mu)$ is an a-relation. Then, there is an ω-topological tree A and a partition $(A_p)_{p \in P}$ of A such that for every $p, q \in P$ the following hold*:

(a) $A_q \to a$ *for some* $a \in A_p$ *implies* $A_q \to A_p$
(b) $A_q \to A_p$ *iff* $p \prec q$
(c) $A_q \to A_p$ *implies* $A_q \xrightarrow{\rho(p,q)} A_p$
(d) A_p *accessible iff* $\mu(p) = 1$.

Proof: We are going to construct pairwise disjoint sets A_p^n for $p \in P$ and $n \in \omega$ by induction on n.

If $\mu(p) = 1$ and p is minimal, A_p^0 is a nonempty finite set. If $\mu(p) = 1$ and p is not minimal, $A_p^0 = \varnothing$. If $\mu(p) = 0$, A_p^0 is a denumerable infinite set.

Suppose that A_p^n is defined for all $p \in P$. Assume that $p, q \in P$, $p \prec q$, and $a \in A_p^n$. If $\rho(p, q) = 1$, we consider a denumerable infinite set $A_{q,a}^n$. We suppose that $A_{q,a}^n \subset N(a)$. If $\rho(p, q) = 0$ we consider a denumerable infinite set $A_{q,a}^{n,k}$ for each $k \in \omega$; then, the following are assumed:

(i) $A_{q,a}^{n,0} \subset N(a)$
(ii) For every $b' \in A_{q,a}^{n,k+1}$ there is a $b \in A_{q,a}^{n,k}$ such that $b' \in N(b)$
(iii) For every $b \in A_{q,a}^{n,k}$ there is only a $b' \in A_{q,a}^{n,k+1}$ with $b' \in N(b)$.

We put

$$A_{q,a}^n = \dot{\bigcup_{k \in \omega}} A_{q,a}^{n,k}.$$

For each $q \in P$ we set

$$A_q^{n+1} = \dot{\bigcup} \; \{A_{q,a}^n \colon a \in A_p^n \text{ and } p \prec q \text{ for some } p\}.$$

Suppose that $a \in A_p^n$. If $n \geq 1$ and there are b and k such that $a \in A_{p,b}^{n-1,k}$, we consider the immediate successor a' of a in $A_{p,b}^{n-1,k+1}$ and set

$$N(a) = \{a'\} \cup \bigcup_{\substack{p \lessdot q \\ \rho(p,q)=1}} A_{q,a}^n \cup \bigcup_{\substack{p \lessdot q \\ \rho(p,q)=0}} A_{q,a}^{n,0}.$$

Otherwise,

$$N(a) = \bigcup_{\substack{p \lessdot q \\ \rho(p,q)=1}} A_{q,a}^n \cup \bigcup_{\substack{p \lessdot q \\ \rho(p,q)=0}} A_{q,a}^{n,0}.$$

Now we put $A_p = \bigcup_{n \in \omega} A_p^n$ and $A = \bigcup_{p \in P} A_p$.

If $(P, <, \rho, \mu)$ is an a-relation and $p \in P$ we define *the n-type of p in P*, $s_n(p, P)$, by induction on n as follows:

$s_0(p, P) = {}^*$,
$s_{n+1}(p, P) = \{(\beta, \lambda_\beta)$: (a) the set J of all $q \in P$ with $p < q$ and $s_n(q, P) = \beta$ is nonempty, and (b) $\lambda_\beta = 0$ if there is a $q \in J$ with $\rho(p, q) = 0$, $\lambda_\beta = 1$ otherwise$\}$.

Proceeding by induction on n it is easy to prove the following lemma.

Lemma 3 *Let $(P, <, \rho, \mu)$ be an a-relation. Suppose A is an ω-topological tree with a partition $(A_p)_{p \in P}$ satisfying* (a)–(d) *of Lemma* 2. *Then, for every $p \in P$, $a \in A_p$ and $n \in \omega$ we have that $s_n(a, A) = s_n(p, P)$.*

Theorem *Suppose that S is a nonempty set of n-types and $f: S \to \{0, 1\}$. The following two conditions are equivalent*:
(a) *There is a T_3 space A with $S = S_n(A)$ and such that, for every $\gamma \in S$, $f(\gamma) = 1$ iff A_γ is accessible.*
(b) *There is an a-relation $(S', \lessdot, \rho, \mu)$ such that*:
 (i) *$S' \subset S_{n+1}$ and $S = \{(\alpha)_n: \alpha \in S'\}$*
 (ii) *$\alpha = s_{n+1}(\alpha, S')$ for all $\alpha \in S'$*
 (iii) *For each $\gamma \in S$: $f(\gamma) = 1$ iff $\mu(\alpha) = 1$ for every $\alpha \in S'$ with $(\alpha)_n = \gamma$.*

Proof: By using Lemma 3, it is easy to prove that (b) implies (a).

Conversely, let A be a T_3 space with $S = S_n(A)$ and such that, for every $\gamma \in S$, $f(\gamma) = 1$ iff A_γ is accessible. Put

$$S' = S_{n+1}(A).$$

We define the binary relation $\vdash$ on S' by

$$\alpha \vdash \beta \text{ iff } A_\beta \to a \text{ for some } a \in A_\alpha.$$

Let $(\alpha_1, \ldots, \alpha_k)$ be a finite sequence of $n + 1$-types of S' with $k \geq 2$. We say that $(\alpha_1, \ldots, \alpha_k)$ is a *chain* if $\alpha_i \vdash \alpha_{i+1}$ for $1 \leq i \leq k - 1$.

If $\alpha \vdash \beta$, we define $\rho'(\alpha, \beta)$ by

$$\rho'(\alpha, \beta) = \begin{cases} 0, & \text{if } A_\beta \xrightarrow{0} a \text{ for some } a \in A_\alpha. \\ 1, & \text{otherwise.} \end{cases}$$

If $w = (\alpha_1, \ldots, \alpha_k)$ is a chain, we define $\rho'(w)$ by

$$\rho'(w) = \begin{cases} 0, \text{ if there is an } i \text{ with } 1 \leq i \leq k-1 \text{ and } \rho'(\alpha_i, \alpha_{i+1}) = 0. \\ 1, \text{ otherwise.} \end{cases}$$

Now we introduce the transitive binary relation $\prec$ on S' as follows:

$\alpha \prec \beta$ iff one of the following two conditions holds:

(i) There is a chain w of the form $(\alpha, \ldots, \beta)$ with $\rho'(w) = 0$.

(ii) There is a chain of the form $(\alpha, \ldots, \beta)$ and there is no chain of the form $(\beta, \ldots, \beta)$.

If $\alpha \prec \beta$ we define $\rho(\alpha, \beta)$ by

$$\rho(\alpha, \beta) = \begin{cases} 0, \text{ if there is a chain } w \text{ of the form } (\alpha, \ldots, \beta) \\ \quad \text{with } \rho'(w) = 0. \\ 1, \text{ otherwise.} \end{cases}$$

Suppose that $\alpha \in S'$ and $\gamma \in S$. We need the following four statements:

(1) If $\alpha \prec \beta$, then $A_{(\beta)_n} \to A_\alpha$.

(2) If $A_\gamma \to A_\alpha$, then there is a $\beta \in S'$ with $(\beta)_n = \gamma$ and $\alpha \prec \beta$.

(3) If $A_\gamma \xrightarrow{0} A_\alpha$, then there is a $\beta \in S'$ with $(\beta)_n = \gamma$, $\alpha \prec \beta$ and $\rho(\alpha, \beta) = 0$.

(4) If $A_\gamma \xrightarrow{1} A_\alpha$, then for any $\beta \in S'$ with $(\beta)_n = \gamma$ and $\alpha \prec \beta$ we have that $\rho(\alpha, \beta) = 1$.

Clearly, if $\beta, \beta' \in S'$ and $A_{(\beta)_n} \to a$ for some $a \in A_{\beta'}$ then $A_{(\beta)_n} \to A_{\beta'}$. So, we obtain (1).

To verify (3), note that if $A_\gamma \xrightarrow{0} A_\alpha$ then for every $a \in A_\alpha$ there is a $\beta \in S'$ with $(\beta)_n = \gamma$ and $A_\beta \xrightarrow{0} a$.

By Lemma 1 (a) and (b) we see that if $\alpha \prec \beta$ and $\rho(\alpha, \beta) = 0$ then $A_{(\beta)_n} \xrightarrow{0} A_\alpha$. Therefore, (4) holds.

To prove (2), we may assume that $A_\gamma \xrightarrow{1} A_\alpha$ (otherwise, it would be enough to apply (3)). Consider

$$C = \{\beta \in S' \colon (\beta)_n = \gamma \text{ and there is a chain of the form } (\alpha, \ldots, \beta)\}.$$

It is easy to see that $C \neq \varnothing$. Now we put

$$D = \{\beta \in S' \colon (\beta)_n = \gamma \text{ and } \alpha \prec \beta\}.$$

Suppose that $D = \varnothing$. Then we would have that for every $\beta \in C$ there is a chain of the form $(\beta, \ldots, \beta)$. Thus, if $\beta \in C$,

$$A_\gamma \to A_\beta.$$

Since $A_\gamma \xrightarrow{1} A_\alpha$ and there is a chain of the form $(\alpha, \ldots, \beta)$,

$$A_\gamma \xrightarrow{1} A_\beta.$$

Therefore,

$$\bigcup_{\substack{\beta \vdash \beta' \\ (\beta')_n = \gamma}} A_{\beta'} \xrightarrow{1} A_\beta.$$

Consequently,

$$\bigcup_{\beta \in C} A_\beta \xrightarrow{1} \bigcup_{\beta \in C} A_\beta,$$

which contradicts Lemma 1 (c).

We define $\mu: S' \to \{0, 1\}$ as follows:

$$\mu(\alpha) = \begin{cases} 1, \text{ if } A_\alpha \text{ is accessible and for any } \alpha' \lessdot \alpha \ A_{\alpha'} \text{ is accessible and} \\ \quad \rho(\alpha', \alpha) = 1. \\ 0, \text{ otherwise.} \end{cases}$$

So, if α is minimal we have that $\mu(\alpha) = 1$ iff A_α is accessible.

Note that $(S', \lessdot, \rho, \mu)$ is an a-relation and $S = \{(\alpha)_n: \alpha \in S'\}$. By (1) . . . (4) we can prove by induction on m that if $m \leq n + 1$ and $\alpha \in S'$:

$$(\alpha)_m = s_m(\alpha, S').$$

Hence,

$$\alpha = s_{n+1}(\alpha, S') \quad \text{for every } \alpha \in S'.$$

One can check that if $\gamma \in S$:

$$A_\gamma \text{ accessible iff } \mu(\alpha) = 1 \text{ for all } \alpha \in S' \text{ with } (\alpha)_n = \gamma.$$

We immediately obtain from the theorem that there is an effective procedure to decide whether a given n-type is satisfiable in some T_3 space. This result was announced in [2]. Note that, for any n-type α, if α is satisfiable in some T_3 space then α is satisfiable in some T_3 space of a-finite type.

Corollary *There is an effective procedure to decide whether for* $S \subset S_0 \cup \ldots \cup S_n$ *with* $S \cap S_k \neq \varnothing$ $(k \leq n)$ *and* $f: S \to \{0, 1\}$ *there is a* T_3 *space* A *such that*:

$S \cap S_k = S_k(A) \quad (k \leq n)$,

$f(\gamma) = 1$ *iff* A_γ *accessible* $\quad (\gamma \in S)$.

Proof: If such a space A exists, for $k < n$ we have:

(i) $S \cap S_k = \{(\alpha)_k: \alpha \in S \cap S_n\}$

(ii) If $\gamma \in S \cap S_k$,
$f(\gamma) = 1$ iff $f(\alpha) = 1$ for every $\alpha \in S \cap S_n$ with $(\alpha)_k = \gamma$.

Remark: If A is a T_3 space, $E_n^A: S_n \to \omega \cup \{\infty\}$ is defined in [2] by $E_n^A(\alpha) =$ number of $a \in A$ with $s_n(a, A) = \alpha$. By a method similar to the one we have been using, we can find an effective procedure to decide whether for $h: S_n \to \omega \cup \{\infty\}$ and $f: \{\gamma \in S_n: h(\gamma) \neq 0\} \to \{0, 1\}$ there is a T_3 space A such that $h = E_n^A$ and, for any $\gamma \in S_n$ with $h(\gamma) \neq 0, f(\gamma) = 1$ iff A_γ is accessible. Then, in the definition of the accessibility relation, we have to include a function H: $P \to \{n: n \geq 1\} \cup \{\infty\}$ such that for every $p \in P$:

p nonminimal implies $H(p) = \infty$

p minimal implies $(H(p) = \infty$ iff $\mu(p) = 0)$.

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