Notre Dame Journal of Formal Logic
Volume 28, Number 2, April 1987

Survey of Generalizations of Urquhart Semantics

R. A. BULL

0 Introduction It has often been felt that the Intuitionist account of implication is too inclusive, the most thoroughly argued case against it being that of Anderson and Belnap. In my opinion the problem is not the irrelevance of antecedents but the relationship between them, which Intuitionism takes to be conjunction. Thus it has

$$((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$$
$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C),$$

which are equivalent[1] to $A \rightarrow (B \rightarrow (A \wedge B))$ and $A \rightarrow (B \rightarrow A)$, and to $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$, respectively. Consequences of the second formula, which I consider to be counterintuitive, include $(A \wedge (A \rightarrow B)) \rightarrow B$, $((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$, and, together with the contrapositive, $(A \rightarrow B) \rightarrow (\bar{A} \vee B)$ and $(A \rightarrow \bar{A}) \rightarrow \bar{A}$. If the relationship between antecedents is not conjunction, one can introduce some other symbol for it, say $\circ$, referred to as fusion or intensional conjunction. In view of these remarks it is appropriate to consider the logic often called $R^+ - W^2$, with axiom schemas

$\vdash A \rightarrow A$
$\vdash A \rightarrow ((A \rightarrow B) \rightarrow B)$
$\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
$\vdash (A \wedge B) \rightarrow A$ $\vdash (A \wedge B) \rightarrow B$
$\vdash ((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
$\vdash A \rightarrow (A \vee B)$ $\vdash (B \rightarrow (A \vee B)$
$\vdash ((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
$\vdash (A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
$\vdash A \rightarrow (B \rightarrow (A \circ B))$
$\vdash (A \rightarrow (B \rightarrow C)) \rightarrow ((A \circ B) \rightarrow C)$

and the rules

if $\vdash A$ and $\vdash B$ then $\vdash A \wedge B$
if $\vdash A$ and $\vdash A \rightarrow B$ then $\vdash B$.

Received August 9, 1984; revised July 26, 1985

All the papers considered in this survey omit one or other of these formulas to which I object. In fact their arguments work just as well with both omitted, so I shall trivially alter the originals and consider this weaker system throughout.

Urquhart [11] introduces a compelling model of truth in terms of 'pieces of information', which he takes to be sets of formulas. With the amendment just suggested, I shall take instead lists of formulas. Here lists are collections which may be permuted, but in which the number of repetitions of each item must remain fixed, unlike sets. Write O for the empty list, and take $a \cdot b$ to be the list obtained by combining lists a and b in the natural way. (That is, the number of occurrences of an item in $a \cdot b$ is the sum of the numbers of its occurrences in a and in b.) Then the structure $\mathfrak{F} = (W, 0, \cdot)$ satisfies the conditions

$$a \cdot 0 = a$$
$$a \cdot b = b \cdot a$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for each $a, b, c \in W$, but not $a \cdot a = a$. A valuation V is defined on any structure $\mathfrak{F}$ satisfying these conditions by taking it to be a function into $\{T, F\}$ such that

$$\begin{aligned}
V(A \to B, c) &= T \text{ iff, for each } a, \text{ if } V(A,a) = T \text{ then } V(B, a \cdot c) = T \\
V(A \wedge B, c) &= T \text{ iff } V(A, c) = T \text{ and } V(B, c) = T \\
V(A \vee B, c) &= T \text{ iff } V(A, c) = T \text{ or } V(B, c) = T \\
V(A \circ B, c) &= T \text{ iff, for some } a, b \text{ such that } a \cdot b = c,\ V(A, a) = T \\
&\quad\ \text{and } V(B, b) = T.
\end{aligned}$$

A formula A is verified by the model $(\mathfrak{F}, V)$ iff $V(A, 0) = T$, and is verified by the structure $\mathfrak{F}$ iff it is verified by all models $(\mathfrak{F}, V)$. it is easy to check that the theses of $R^+ - W$ are verified by any structure $\mathfrak{F}$ which satisfies the conditions above. ([11] also quotes a formula which is verified by any structure, although it is known to not be in R^+. Inspection will show that this verification requires the condition $a \cdot a = a$ which I have omitted.)

A beautifully simple completeness proof for the fragment in $\to$, alone, is provided by the following canonical model $(\mathbf{T}, 0, \cdot, V)$. Take $\mathbf{T}$ to consist of all finite lists of formulas, define $\cdot$ on lists as in the previous paragraph, and take

$$V(A,a) = T \text{ iff } \vdash A_1 \to (\ldots(A_m \to A)\ldots),$$
$$\text{where } A_1, \ldots, A_m \text{ are the items in } a.$$

It is easy to show that V satisfies the conditions of the previous paragraph, and clearly if $V(A, 0) = T$ then $\vdash A$ in the logic. One would like to extend this completeness proof to the whole of $R^+ - W$ by taking $\mathbf{T}$ to be the set of principal theories of $R^+ - W$, with

$$\text{V(A,a)} = \text{T iff } A \in a.$$

Here a theory a is a set of formulas such that

if $A \in a$ and $B \in a$ then $A \wedge B \in a$
if $A \in a$ and $\vdash A \to B$ then $B \in a$

(in the second condition, $\vdash A \to B$ cannot be replaced by $A \to B \in a$ in this context). Define $\cdot$ on theories by taking

$$a \cdot b = \{C\colon \text{for some } A, B, A \in a \ \& \ B \in b \ \& \vdash A \to (B \to C)\}$$

or, equivalently,

$$a \cdot b = \{C\colon \text{for some } B, B \to C \in a \ \& \ B \in b\}.$$

However it is not clear that principal theories are prime theories, satisfying

if $A \vee B \in a$ then $A \in a$ or $B \in a$,

and a completeness proof with respect to (w.r.t) this semantics has yet to be found.

In order to obtain a completeness proof for $R^+ - W$, three approaches have emerged. The first, due independently to Routley and Meyer, and to Kit Fine, is to work with prime theories instead of principal theories. Since $a \cdot b$ need not be prime for prime a and b, a more complicated notion of structure is necessary. The second, due to Abraham (only recently published after a long delay), is to retain principal theories but find a more complicated definition for $V(A \vee B, c)$. The precedent here was Beth's semantics for Intuitionist disjunction. However this method does seem to make considerable requirements on the properties of negation, holding not for $R^+ - W$ but for the logic $CR^+ - W$ which I shall describe later. It will be noticed that none of these methods appeals to proof theory, in marked contrast to Intuitionist logic. The reason for this at last became clear when Urquhart ([12] and [13]) used a sophisticated geometrical argument to show that Cut Elimination is impossible for many logics in the neighborhood of $R - W$ (or rather, strictly speaking, the deducibility problem is unsolvable). However this Curse of the Urquharts can be avoided if the usual distributivity of $\wedge$ and $\vee$ is omitted (though I am uneasy as to the philosophical justification for this, in spite of the precedent of the so-called quantum logics). The third approach, then, due to Ono and Komori, is to drop Distributivity, prove Cut Elimination, and exploit it to establish the properties of

$$V(A,a) = T \text{ iff } A_1, \ldots, A_m \vdash A,$$
$$\text{where } A_1, \ldots, A_m \text{ are the items in } a.$$

In all these approaches there is a compromise, between loss of the intuitive content of Urquhart semantics and the complexity necessary for obtaining completeness proofs.

Since it is difficult to conceive other approaches to proving completeness for logics related to $R^+ - W$, the time seems to be ripe for a survey of the area. Of course canonical models are not the only topic of interest. Ono and Komori have unified the area by producing a common generalization of the semantics used in the first and third approaches mentioned. Further, one would like to see a metatheory for these implicational logics and their semantics, such as that developed for modal logic and Kripke frames in the 1970s. However, establishing an analogue of, say, the modal metatheory of R. I. Goldblatt seems to be quite impossible. The greater generality of the semantics of Ono and Komori makes it a more promising candidate for some sort of metatheory.

So far I have concentrated on positive logics, which I shall describe in Part I below. Urquhart in [11] goes on to discuss possible extensions of his semantics to negation. The most obvious move is

$$V(\neg A, a) = T \text{ iff } V(A, a) = F,$$

although $(A \wedge (\neg A)) \to B$ and $B \to (A \vee (\neg A))$ are now verified, a breakdown of relevance. Since the viewpoint of Anderson and Belnap predominated at that time, there was a delay in studying this semantics, but Meyer and Routley did include it in various 'classical' systems. To avoid this breakdown, Urquhart then suggests

$$V(\bar{A}, a) = T \text{ iff } V(A, a^*) = F,$$

where $*$ is a function on W satisfying

$$O^* = O \text{ and } (a^*)^* = a.$$

This alternative, as it stands, does not verify $(A \to B) \to (\bar{B} \to \bar{A})$, but it can be rescued under the first approach above by suitable further conditions. I shall discuss these two kinds of negation in detail in Part II below but, to anticipate, the appropriate axioms are as follows. Adding the more natural $^-$ to obtain $R - W$, take

$$\vdash A \to \bar{\bar{A}} \qquad \vdash \bar{\bar{A}} \to A$$
$$\vdash (A \to B) \to (\bar{B} \to \bar{A}).$$

Adding $\neg$ to obtain $CR^+ - W$, take

$$\vdash (\neg(\neg A)) \to A$$

and the rule

$$\text{if } \vdash (A \wedge B) \to (\neg C) \text{ then } \vdash (A \wedge C) \to (\neg B),$$

from which $\vdash A \to (\neg(\neg A))$ and the rule

$$\text{if } \vdash A \to B \text{ then } \vdash (\neg \mathrm{B}) \to (\neg \mathrm{A})$$

can be derived. There is no reason for not adding both, to obtain the system $CR - W$. It also turns out that $^-$ can be added under the third approach, via proof theory without Distributivity.

A final possible extension is to individual quantifiers, a topic which proved to be very difficult under the first approach above. Again the reason for this became clear at last when Fine [4] proved the incompleteness of that semantics for the natural axiomatization of quantified logic. (It is typical of these implicational logics that the most spectacular results in the area are negatives.) It was this problem which motivated Abraham in the second approach mentioned above. Given 'classical' negation, his [2] does yield a completeness proof, which I shall discuss in Part III below.

Part I Positive Logics

1 Proof theory without Distributivity In the proof theory of Intuitionist logic, Contraction of antecedents, equivalent to its thesis $(A \to (A \to B)) \to (A \to B)$ which I am omitting, plays a major role. For example, in the usual systems with only a single consequent, it is used to derive the distributivity of $\wedge$ and $\vee$. (It seems regrettably counterintuitive that this property of $\wedge$ and $\vee$ should depend upon a structural rule. The Maehara system is to be commended for having a more natural derivation.) Further, the known proofs of Cut Elimination depend essentially upon it. However as Ono and Komori point out in [8], this dependence is circular; once Contraction is omitted, it is not needed, nor are Mixes. Therefore they consider the very natural proof theory with initial sequents $P \vdash P$ for propositional parameters P, and rules

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

$$\frac{\Gamma \vdash A \qquad \Delta, B, \Sigma \vdash C}{\Delta, A \rightarrow B, \Gamma, \Sigma \vdash C} \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$$

$$\frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma, B, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C}$$

$$\frac{\Gamma, A, \Delta \vdash C \qquad \Gamma, B, \Delta \vdash C}{\Gamma, A \vee B, \Delta \vdash C} \qquad\qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \circ B, \Delta \vdash C} \qquad\qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \circ B}$$

As usual their Cut Elimination is a twofold induction on rank and grade, and the crucial step is that in which the Cut Rule

$$\frac{\Gamma \vdash A \qquad \Delta, A, \Sigma \vdash C}{\Delta, \Gamma, \Sigma \vdash C}$$

has been applied to the principal formula at both its upper sequents. In the typical case in which that formula is $B \rightarrow D$, the Cut

$$\frac{\dfrac{\Gamma, B \vdash D}{\Gamma \vdash B \rightarrow D} \qquad \dfrac{\Sigma \vdash B \qquad \Delta, D, \Pi \vdash C}{\Delta, B \rightarrow D, \Sigma, \Pi \vdash C}}{\Delta, \Gamma, \Sigma, \Pi \vdash C}$$

is replaced by the Cuts

$$\frac{\Sigma \vdash B \qquad \dfrac{\Gamma, B \vdash D \qquad A, D, \Pi \vdash C}{\Delta, \Gamma, B, \Pi \vdash C}}{\Delta, \Gamma, \Sigma, \Pi \vdash C}$$

of lower grade. (Inspection will show how this argument breaks down if multiple consequents are permitted.)

Given that, in the canonical structure, **T** is to be the set of principal theories, what are the appropriate conditions on the structures $\mathfrak{F} = (W, O, \cdot, \cap)$? They turn out to be

$$
\begin{aligned}
a \cdot 0 &= a \\
a \cdot b &= b \cdot a \\
a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\
a \cap a &= a \\
a \cap b &= b \cap a \\
a \cap (b \cap c) &= (a \cap b) \cap c \\
a \cdot (b \cap c) &= (a \cdot b) \cap (a \cdot c).
\end{aligned}
$$

Note that a relation $\leq$ can be defined by

$$a \leq b \text{ iff } a \cap b = a,$$

and that then the last equation yields

if $a \leq b$ then $a \cdot c \leq b \cdot c$.

And given that $V(A, a) = T$ is to occur in the canonical valuation iff $A_1, \ldots, A_m \vdash A$, where $A_1, \ldots, A_m$ generate the principal theory a, what are the appropriate conditions on valuations? They turn out to be that

if $V(P, a) = T$ & $V(P, b) = T$ & $a \cap b \leq c$
then $V(P, c) = T$

for each propositional parameter P, and that

$$
\begin{aligned}
V(A \to B, c) &= T \text{ iff, for each } a, b \text{ such that } c \cdot a \leq b, \\
&\quad \text{if } V(A, a) = T \text{ then } V(B, b) = T, \\
V(A \wedge B, c) &= T \text{ iff } V(A, c) = T \text{ and } V(B, c) = T \\
V(A \vee B, c) &= T \text{ iff, for some } a, b \text{ such that } a \cap b \leq c, \\
&\quad V(A, a) = T \text{ and } V(B, b) = T, \\
V(A \circ B, c) &= T \text{ iff, for some } a, b \text{ such that } a \cdot b \leq c, \\
&\quad V(A, a) = T \text{ and } V(B, b) = T.
\end{aligned}
$$

Note how all the conditions except that for $\wedge$ have become considerably more complicated than in Urquhart semantics. As before A is verified by $(\mathfrak{F}, V)$ iff $V(A, 0) = T$.

It is now straightforward to show that each model $(\mathfrak{F}, V)$ satisfying these conditions verifies $R^+ - W$, except for the Distributive axiom, and that the canonical structure $(\mathbf{T}, 0, \cdot, \cap)$ and the canonical valuation V with

$$V(A, a) = T \text{ iff } A \in a$$

do satisfy them. Verifying that

$$
\begin{aligned}
\Gamma \vdash A \to B \text{ iff, for each } \Delta, \Sigma \text{ such that } \Gamma \cdot \Delta \subseteq \Sigma, \\
\text{if } \Delta \vdash A \text{ then } \Sigma \vdash B
\end{aligned}
$$

requires Cut Elimination. And verifying the condition that

$$
\begin{aligned}
\Gamma \vdash A \vee B \text{ iff, for some } \Delta, \Sigma \text{ such that } \Delta \cap \Sigma \subseteq \Gamma, \\
\Delta \vdash A \text{ and } \Sigma \vdash B
\end{aligned}
$$

requires, from right to left, an application of the rule for $\vee$ on the right, and from left to right the choice $\Delta = [A]$, $\Sigma = [B]$.

2 Prime theories Reversing the historical priority, the conceptual priority seems to lie with Fine [3]. Given that the set of prime filters is not closed under $\cdot$ as defined in the Introduction, let us take a set W together with a distinguished subset K. In the canonical structure W will become the set **T** of all theories and K will become the set **P** of prime theories. The conditions on W are the familiar

$$\begin{gathered} a\cdot 0 = a \\ a\cdot b = b\cdot a \\ a{:}(b\cdot c) = (a\cdot b)\cdot c \\ \text{if } a \leq b \text{ then } a\cdot c \leq b\cdot c. \end{gathered}$$

Here $\leq$ is a partial-order relation on W which will become $\subseteq$ in the canonical structure. The condition on K is

$$\begin{gathered} \text{if } c\cdot d \leq a \text{ for } a \in K \text{ then, for some } b \in K, \\ d \leq b \;\&\; c\cdot b \leq a. \end{gathered}$$

The conditions on valuations V are

$$\begin{aligned} V(P, a) = T &\text{ iff, for each } b \in K, \\ &\text{if } a \leq b \text{ then } V(P, b) = T \end{aligned}$$

for each propositional parameter P, and

$$\begin{aligned} V(A \rightarrow B, c) &= T \text{ iff, for each } a, \\ &\quad \text{if } V(A, a) = T \text{ then } V(B, a\cdot c) = T, \\ V(A \wedge B, c) &= T \text{ iff } V(A, c) = T \text{ and } V(B, c) = T, \\ V(A \vee B, c) &= T \text{ iff, for each } d \in K \text{ such that } c \leq d, \\ &\quad V(A, d) = T \text{ or } V(B, d) = T, \\ V(A \circ B, c) &= T \text{ iff, for some } a, b \text{ such that } a\cdot b = c, \\ &\quad V(A, a) = T \text{ and } V(B, b) = T. \end{aligned}$$

It can be shown that the condition above on the values of propositional parameters extends to all formulas, and hence that

$$\text{if } V(A, a) = T \;\&\; a \leq b \text{ then } V(A, b) = T,$$

and for all $c \in K$,

$$V(A \vee B, c) = T \text{ iff } V(A, c) = T \text{ or } V(B, c) = T.$$

It is straightforward to show that each model $(\mathfrak{F}, V)$ with $\mathfrak{F} = (W, K, 0, \cdot, \leq)$ verifies $R^+ - W$. In order to handle the canonical model, an Extension Lemma is needed for extending theories to prime theories. A suitable result is that if c is a set of formulas satisfying

$$\text{if } A, B \in c \text{ then } A \vee B \in c$$

and d is a theory disjoint from c, then there is a prime theory d' with $d \subseteq d'$, such that d' is also disjoint from c. The proof of this lemma involves enumerating all disjunctions as $B_0 \vee C_0$, $B_1 \vee C_1$, $B_2 \vee C_2, \ldots$, and making the step-by-step construction

$$\begin{aligned}
d_{00} &= d; \\
d_{i(j+1)} &= d_{ij} \text{ if } d_{ij} \nvdash B_j \vee C_j, \\
d_{i(j+1)} &= d_{ij} \cup \{B_j\} \text{ if } d_{ij} \vdash B_j \vee C_j \text{ and} \\
&\quad c \cap \{A\colon d_{ij}, B_j \vdash A\} = \varnothing, \\
d_{i(j+1)} &= d_{ij} \cup \{C_j\} \text{ if } d_{ij} \vdash B_j \vee C_j \text{ and} \\
&\quad c \cap \{A\colon d_{ij}, B_j \vdash A\} \neq \varnothing; \\
d_{i(j+1)} &= d_{ij} \cup \{C_j\} \text{ if } d_{ij} \vdash B_j \vee C_j \text{ and} \\
&\quad c \cap \{A\colon d_{ij}, B_j \vdash A\} \neq \varnothing; \\
d_{(i+1)0} &= \bigcup_j d_{ij}; \\
d' &= \bigcup_i d_{i0}.
\end{aligned}$$

Here $d_{ij} \vdash A$ means that there is a derivation of A from formulas in d_{ij}, and so on. It can now be shown that the canonical structure $(\mathbf{T}, \mathbf{P}, 0, \cdot, \subseteq)$ and the canonical valuation V with

$$V(A, a) = T \text{ iff } A \in a$$

do satisfy the conditions of the previous paragraph.

A similar approach is given in Ono and Komori [8]. A structure $\mathfrak{F} = (W, K, 0, \cdot, \cap)$ must satisfy the conditions

$$\begin{aligned}
a \cdot 0 &= a \\
a \cdot b &= b \cdot a \\
a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\
a \cap a &= a \\
a \cap b &= b \cap a \\
a \cap (b \cap c) &= (a \cap b) \cap c
\end{aligned}$$

(so that $\leq$ can be defined by $a \leq b$ iff $a \cap b = a$)

$$\begin{gathered}
a \cdot (b \cap c) = (a \cdot b) \cap (a \cdot c) \\
\text{if } a \cap b \leq c \text{ then, for some } a', b', \\
a \leq a' \ \&\ b \leq b' \ \&\ a' \cap b' \leq c
\end{gathered}$$

(in a lattice this condition is equivalent to distributivity)

for each $a \in K$, if $a = b \cap c$ then $a = b$ or $a = c$,
for each $a \in K$, if $b \cdot d \cdot c \leq a$ then, for some $d' \in K$,
$$d \leq d' \ \&\ b \cdot d' \cdot c \leq a.$$

A valuation V is defined on the members of K, and must satisfy the conditions that

$$\begin{aligned}
&\text{if } V(P, a) = T \ \&\ V(P, b) = T \ \&\ a \cap b \leq c \\
&\qquad \text{then } V(P, c) = T
\end{aligned}$$

for each propositional parameter P, and

$$\begin{aligned}
V(A \to B, c) &= T \text{ iff, for each } a, b \in K \text{ such that } c \cdot a \leq b, \\
&\qquad \text{if } V(A, a) = T \text{ then } V(B, b) = T, \\
V(A \wedge B, c) &= T \text{ iff } V(A, c) = T \text{ and } V(B, c) = T,
\end{aligned}$$

$$\begin{aligned} V(A \vee B, c) &= T \text{ iff } V(A, c) = T \text{ or } V(B, c) = T \\ V(A \circ B, c) &= T \text{ iff, for some } a, b \in K \text{ such that } a \cdot b \leq c, \\ &\quad V(A, a) = T \text{ and } V(B, b) = T. \end{aligned}$$

Again A is verified by $(\mathfrak{F}, v)$ iff $V(A, 0) = T$.

It can now be shown that each model satisfying these conditions verifies $R^+ - W$ including the Distributive axiom. Also it can be shown that the canonical structure $(\mathbf{T}, \mathbf{P}, 0, \cdot, \cap)$ with $\cdot$ as in the Introduction and $\cap$ as set intersection, and the canonical valuation V with

$$V(A, a) = T \text{ iff } A \in a$$

do satisfy the conditions of the previous paragraph. For structures, this involves using the Distributive axiom to prove that the theories which satisfy the first condition on K above are precisely the prime theories. For the valuations, this depends upon the fact that, given a formula A and a theory a with $A \notin a$, there is a prime theory b with $a \subseteq b$ and $A \notin b$.

Clearly there would be advantages in restricting attention to prime theories, replacing the binary function $\cdot$ on theories by a ternary relation R on prime theories. (This is suggested in Fine [3], and is the starting point of the earlier Routley and Meyer [9], extended to many logics in [10].) Thus $Rabc$ can be defined within the previous structures by

$$Rabc \text{ iff } a \cdot b \leq c;$$

also $\leq$ can be defined in terms of R with

$$a \leq b \text{ iff } ROab.$$

The conditions on structures (K, O, R) now become

$a \leq a$
if $a \leq b$ & $b \leq c$ then $a \leq c$,
if $a \leq x$ & $Rxbc$ then $Rabc$,
if $Rabx$ & $x \leq c$ then $Rabc$,
if $Rabc$ then $Rbac$,
if $Rabx$ & $Rxcd$ then, for some y, $Racy$ & $Rybd$.

The last, complicated condition corresponds to the Syllogism axioms, and provides most of the problems in this ternary relation semantics, through the need to construct the prime theory y of its conclusion. The conditions on valuations V are now

$$\text{if } V(P, a) = T \;\&\; a \leq b \text{ then } V(P, b) = T$$

for each propositional parameter P, and

$$\begin{aligned} V(A \rightarrow B, c) &= T \text{ iff, for each } a, b \text{ such that } Rcab, \\ &\quad \text{if } V(A, a) = T \text{ then } V(B, b) = T, \\ V(A \wedge B, c) &= T \text{ iff } V(A, c) = T \text{ and } V(B, c) = T, \\ V(A \vee B, c) &= T \text{ iff } V(A, c) = T \text{ or } V(B, c) = T \\ V(A \circ B, c) &= T \text{ iff, for some } a, b \text{ such that } Rabc, \\ &\quad V(A, a) = T \text{ and } V(B, b) = T. \end{aligned}$$

Again A is verified by $(\mathfrak{F}, V)$ iff $V(A, O) = T$.

It is straightforward to check that each model satisfying these conditions does verify $R^+ - W$. However the canonical construction is more difficult than it has been so far, precisely because the semantics are simpler. Firstly, the properties of $(\mathbf{T}, O, \cdot, \subseteq)$ are established as before. Then it is observed that any nonthesis A of $R^+ - W$ is excluded by some prime theory T which is regular, i.e., which includes all the theses of $R^+ - W$. A T-theory a, for regular T, is a theory in which the second condition has been replaced by the stronger

$$\text{if } A \in a \ \& \ A \to B \in T \text{ then } B \in a.$$

Attention is now restricted to structures $(W_T, O_T, \cdot, \subseteq)$, where W_T is the set of T-theories, $O_T = T$, and R_T is defined on W_T by taking

$$R_T\, abc \text{ iff } a \cdot b \subseteq c.$$

(In fact if only positive logics are being considered, this move to T-theories can be avoided. Instead the canonical model $(\mathbf{P}, O, R, V)$ on all prime theories can be used—see Section 10 of [9].) It is straightforward to show that (W_T, O_T, R_T) satisfies the conditions above on structures, but of course the condition on $V(A \vee B, c)$ would fail for it. Instead $(\mathbf{P}_T, O_T, R_T)$ must be considered, with attention restricted to prime T-theories, for which the condition on R for the Syllogism axioms becomes difficult. Zorn's Lemma is used to find a maximal T-theory y' satisfying its conclusion, which can, with some difficulty, be shown to be again prime. It is now straightforward to show that the canonical valuation V_T with

$$V_T(A, a) = T \text{ iff } A \in a$$

satisfies the conditions above.

3 The generalization of Ono and Komori The greater complication of the semantics of [8] compared with those of [3] (both discussed in Section 2), has the advantage that it permits a common generalization with the semantics discussed in Section 1. In this generalization, given in [8], the condition above which yields the distributivity of $\cdot$ and $\cap$ is omitted, while the condition

$$\text{for each } a \in K, \text{ if } a = b \cap c \text{ then } a = b \text{ or } a = c$$

is replaced by

$$\begin{aligned}&\text{for each } a \in K, \text{ if } b \cap c \leq a \text{ then either } c \leq a\\ &\quad \text{or, for some } b' \in K,\ b \leq b' \ \&\ b' \cap c \leq a,\end{aligned}$$

which can be derived from it. Now valuations V must satisfy the weak condition

$$\begin{aligned} V(A \vee B, c) = T \text{ iff, for some } a, b \in K \text{ such that } a \cap b \leq c,&\\ (V(A, a) = T \text{ or } V(B, a) = T)&\\ \text{and} \quad (V(A, b) = T \text{ or } V(B, b) = T).&\end{aligned}$$

Of course, in the presence of the first condition on K, this becomes the usual 'classical' condition for $V(A \vee B, c)$. Further, if W has a greatest element ∞ then this weak condition can be replaced by the strong condition

$$V(A \vee B, c) = T \text{ iff, for some } a, b \in K \text{ such that } a \cap b \leq c, \; V(A, a) = T \text{ and } V(B, b) = T,$$

used in Section 1. It can still be shown that both weak and strong models verify $R^+ - W$ without the Distributive axiom.

Part II Two Kinds of Negation

4 Proof theory with signed formulas One way of adding negation to the system of Section 1 is to replace formulas A by signed formulas TA, FA,[3] and to replace the function V with $V(A, a) \in \{T, F\}$ by predicates $T(A, a)$, $F(A, a)$. In the Hilbert-style formulation one wants to add the axioms

$$\vdash A \to \bar{\bar{A}} \qquad \vdash \bar{\bar{A}} \to A$$
$$\vdash (A \to B) \to (\bar{B} \to \bar{A}),$$

but not formulas such as $(A \wedge \bar{A}) \to B$ and $B \to (A \vee \bar{A})$. Note that in the resulting logic, $R - W$ without Distributivity,

$$\vdash (A \circ \bar{B}) \to \overline{(A \to B)}$$
$$\vdash \overline{(A \to B)} \to (A \circ \bar{B})$$

are derivable. To achieve the equivalent effects in the Gentzen-style formulation, first take the initial sequents to be those $SP \vdash SP$ (where P is any propositional parameter and S is a parameter for T or F), and take the rules for $\bar{\ }$ to be

$$\frac{\Gamma, FA \vdash SC}{\Gamma, T\bar{A} \vdash SC} \qquad \frac{\Gamma \vdash FA}{\Gamma \vdash T\bar{A}}$$

$$\frac{\Gamma, TA \vdash SC}{\Gamma, F\bar{A} \vdash SC} \qquad \frac{\Gamma \vdash TA}{\Gamma \vdash F\bar{A}}.$$

The modifications to the rules for $\wedge$ and $\vee$ are obvious; typically

$$\frac{\Gamma, FA, \Delta \vdash SC \qquad \Gamma, FB, \Delta \vdash SC}{\Gamma, F(A \wedge B), \Delta \vdash SC} \qquad \frac{\Gamma \vdash FA}{\Gamma \vdash F(A \wedge B)}$$

$$\frac{\Gamma \vdash FB}{\Gamma \vdash F(A \wedge B)}.$$

In choosing the rules for $T(A \to B)$, the clue is to restore symmetry between T and F, with

$$\frac{\Gamma \vdash TA \qquad \Delta, TB, \Sigma \vdash SC}{\Delta, T(A \to B), \Gamma, \Sigma \vdash SC} \qquad \frac{\Gamma, TA \vdash TB \qquad \Gamma, FB \vdash FA}{\Gamma \vdash T(A \to B)}$$

$$\frac{\Gamma \vdash FB \qquad \Delta, FA, \Sigma \vdash SC}{\Delta, T(A \to B), \Gamma, \Sigma \vdash SC}.$$

With the rules for $F(A \to B)$, remember that $\overline{A \to B}$ is provably equivalent to $A \circ \bar{B}$, and take

$$\frac{\Gamma, TA, FB, \Delta \vdash SC}{\Gamma, F(A \to B), \Delta \vdash SC} \qquad \frac{\Gamma \vdash TA \qquad \Delta \vdash FB}{\Gamma, \Delta \vdash F(A \to B)}.$$

The rules for $A \circ B$ are the duals of these, identifying $T(A \circ B)$ with $F(A \to \bar{B})$ and $F(A \circ B)$ with $T(A \to \bar{B})$. Because of the double nature of the rule for $T(A \to B)$ on the right, derivations of $\vdash TA$ for theses $\vdash A$ of $R - W$ without Distributivity become much more complex. However, while many cases are needed the axioms of $R - W$ without Distributivity can still be derived. Inspection will show that the Exchange structural rule is essential for some of these negative cases. The Cut Rule with cut signed formula SA can be eliminated just as before.

The structures $\mathfrak{F} = (W, O, \cdot, \cap)$ are just as in Section 1, the hereditary conditions on the values of the propositional parameters is replaced by similar ones on $T(P, a)$, $F(P, a)$, and the conditions for $\bar{A}$ are

$$T(\bar{A}, a) \text{ iff } F(A, a),$$
$$F(\bar{A}, a) \text{ iff } T(A, a).$$

The conditions on the values of $A \wedge B$ and $A \vee B$ are obvious. The conditions for $A \to B$, corresponding to the rules in the proof theory, become

$$\begin{aligned} T(A \to B, c) \text{ iff, } & \text{for each } a, b \text{ such that } c \cdot a \leq b, \\ & (\text{if } T(A, a) \text{ then } T(B, b)) \\ \text{and } & (\text{if } F(B, b) \text{ then } F(A, a)), \\ F(A \to B, c) \text{ iff, } & \text{for some } a, b \text{ such that } a \cdot b \leq c, \\ & T(A, a) \text{ and } F(B, b). \end{aligned}$$

Again the conditions for $A \circ B$ are the dual ones, identifying it with $\overline{A \to \bar{B}}$. A formula A is verified by a model $(\mathfrak{F}, \mathrm{T}, \mathrm{F})$ iff $T(A, 0)$. The completeness proof now proceeds just as in Ono and Komori [8].

*5 **The operation*** $*$ Both Routley and Meyer [9] and Fine [3] handle negation in $R - W$ through the function $*$ introduced in Urquhart [11]. As mentioned in the Introduction, further conditions are needed to verify the Contraposition axiom. In [3], $*$ is only defined on the distinguished set K (the set of prime theories for the canonical structure), and the conditions imposed on it are

$$(a^*)^* = a$$
$$\text{if } a \leq b \text{ then } b^* \leq a^*.$$

The condition on $V(\bar{A}, a)$ is

$$V(\bar{A}, a) = T \text{ iff, for each } b \in K \text{ such that } a \leq b,\ V(A, b^*) = F.$$

In the canonical structure $*$ is interpreted as

$$a^* = \{A : \bar{A} \notin a\},$$

and now the completeness proof proceeds without much more difficulty than before. Correspondingly, in Routley and Meyer [9], the conditions on (K, O, R) are

$$(a^*)^* = a$$
$$\text{if } Rabc \text{ then } Rac^*b^*,$$

the condition on $V(\bar{A}, a)$ is

$$V(\bar{A}, a) = T \text{ iff } V(A, a^*) = F$$

and in the canonical structure

$$a^* = \{A: \bar{A} \notin a\}.$$

In establishing the properties of the canonical structure, $*$ cannot be introduced until after the move to prime theories, but then there are no great difficulties.

6 'Classical' negation The discussion in Routley and Meyer [9], of the canonical model $(\mathbf{P}, O, R, V)$ of prime theories, is continued in Meyer and Routley [5]. A considerable amount of work, including a proof that

$$V(A \circ B, O) = T \text{ iff } V(A, O) = T \text{ and } V(B, O) = T$$

and the addition of a new element O', yields a new canonical model $(\mathbf{P}', O', R', V')$ which satisfies the following conditions. Take a positive 'classical' structure (K, O, R) to be one in which the definition

$a \leq b$ iff $ROab$

and the conditions on $\leq$ are replaced by

$ROab$ iff $a = b$.

Now add 'classical' negation to obtain $CR^+ - W$ with the axiom

$\vdash (\neg(\neg A)) \rightarrow A$

and the rule

if $\vdash (A \wedge B) \rightarrow (\neg C)$ then $\vdash (A \wedge C) \rightarrow (\neg B)$,

from which

$\vdash A \rightarrow (\neg(\neg A))$
$\vdash (A \wedge (\neg A)) \rightarrow B \vdash B \rightarrow (A \vee (\neg A))$
if $\vdash A \rightarrow B$ then $\vdash (\neg B) \rightarrow (\neg A)$

are derivable. The corresponding condition on valuations for 'classical' structures (K, O, R) (but not the ordinary ones of Section 2, in which the hereditary condition on valuations prevents this move) is

$V(\neg A, a) = T$ iff $V(A, a) = F$.

The construction of the canonical model $(\mathbf{P}_T, O_T, R_T, V_T)$ is similar in outline to that sketched in Section 2. However both the prime regular theory T and the prime T-theories in $\mathbf{P}_T$ must now also be consistent-and-complete, i.e., they must contain exactly one of A and $\neg A$ for each formula A. This requires yet more hard work, especially when the existence of such a theory must be proved. The proof of this in [5] involves its Lemma 4, which uses a shortcut through the theory of ideals in Boolean algebras and an application of the omitted formula $(A \wedge (A \rightarrow B)) \rightarrow B$ (because of the unusual second condition on theories for $R^+ - W$). However this can be avoided by a direct proof as in [2].

As [6] points out, one can include both negations $^-$ *and* $\neg$, obtaining the logic $CR - W$. Combining the arguments sketched in Sections 2, 5, and 6 completeness is proved with respect to 'classical' models $(K, O, R, *, V)$ which satisfy the conditions

$ROab$ iff $a = b$,
$(a^*)^* = a$
if $Rabc$ then Rac^*b^*,
$V(\neg A, a) = T$ iff $V(A, a) = F$,
$V(\bar{A}, a) = T$ iff $V(A, a^*) = F$,

among others. (Actually [6] introduces into the formal language a new operator $*$ with $A^* = \neg\bar{A}$, but any two of $\bar{\ }$, $\neg$, $*$ will suffice as primitives. The completeness proof for $CR - W$ is not given in [6], but it is worked out in detail in [1]. The question arises: Is $CR - W$ a conservative extension of $R - W$? Of course [5] and [6] are working with R and CR, in which the presence of $(A \to (A \to B)) \to (A \to B)$ corresponds to the condition $Raaa$ on structures. Using this extra condition it can be shown that $O^* \leq O$ holds in any ordinary structure $(K, O, R, *)$, and [5] goes on to show that R is characterized by normal structures in which $O^* = O$. Making essential use of these normal structures, [6] proves that CR is a conservative extension of R. In view of all the work which has gone before, this must be the most difficult result in the area being surveyed.)

*7 **Propositional Abraham semantics*** Abraham's [2] gives a very different approach to $CR - W$, going back to the principal theories of Urquhart semantics instead of prime theories. As in the Introduction, the obvious difficulty will lie in the treatment of $V(A \vee B, c)$, a problem that is solved in Beth's Intuitionist semantics by introducing certain sets of 'possible worlds', namely the paths and bars. With this precedent in mind, consider the following generalization of Urquhart semantics. A prestructure $\mathfrak{F} = (W, O, \cdot, \leq, *)$ satisfies the familiar conditions

$$a \cdot 0 = a$$
$$a \cdot b = b \cdot a$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$
$$\text{if } a \leq b \text{ then } a \cdot c \leq b \cdot c,$$
$$(a^*)^* = a,$$

together with the unfamiliar conditions for $*$

if $a \leq b$ then $a^* \leq b^*$,
$a \cdot a^* \leq 0$
if $a \cdot b \leq 0$ then, for some c, $a \leq c$ & $b \leq c^*$.

An ideal α of $\mathfrak{F}$ is a nonempty proper subset of W such that

if $a \in \alpha$ & $b \leq a$ then $b \in \alpha$,
if $a \in \alpha$ & $b \in \alpha$ then, for some c,
  $a \leq c$ & $b \leq c$ & $c \in \alpha$,

which is maximal iff, for each ideal β of $\mathfrak{F}$,

if $\alpha \subseteq \beta$ then $\alpha = \beta$.

A set $\mathbf{N}$ of ideals of $\mathfrak{F}$ is dense iff, for each $a \in W$, $a \in \alpha$ for some $\alpha \in \mathbf{N}$. Given an ideal β of $\mathfrak{F}$, it is convenient to write

$$a \cdot \beta = \{a \cdot b \colon b \in \beta\}.$$

A structure $(\mathfrak{F}, \mathbf{N})$ consists of a prestructure $\mathfrak{F}$ and a dense set $\mathbf{N}$ of its maximal ideals, such that for each $\gamma \in \mathbf{N}$,

if $a \cdot b \in \gamma$ then, for some β,
$b \in \beta \;\&\; \beta \in \mathbf{N} \;\&\; a \cdot \beta \subseteq \gamma$.

Later it will be seen that these normal ideals correspond to prime, consistent-and-complete theories in the canonical structure.

The valuations V are required to satisfy the conditions

V1 if $V(P, a) = T \;\&\; a \leq b$ then $V(P, b) = T$,
V2 if ($a \in \alpha \;\&\; \alpha \in N$ imply $V(P, b) = T$ for some $b \in \alpha$) then $V(P, a) = T$,
V3 if (for each $x \in \alpha$, for some y such that $x \leq y$, $V(A, y) = T$) then (for some $x \in \alpha$, $V(A, x) = T$),

for each propositional parameter P and formula A, and

$V(A \to B, c) = T$ iff, for each a,
if $V(A, a) = T$ then $V(B, a \cdot c) = T$,
$V(A \wedge B, c) = T$ iff $V(A, c) = T$ and $V(B, c) = T$,
$V(A \vee B, c) = T$ iff, for each γ such that $c \in \gamma \;\&\; \gamma \in \mathbf{N}$, for some $d \in \gamma$, $V(A, d) = T$ or $V(B, d) = T$,
$V(A \circ B, c) = T$ iff, for each γ such that $c \in \gamma \;\&\; \gamma \in \mathbf{N}$, for some a, b, $a \cdot b \in \gamma \;\&\; V(A, a) = T \;\&\; V(B, b) = T$,
$V(\neg A, a) = T$ iff, for each b such that $a \leq b$, $V(A, b) = F$,
$V(\bar{A}, a) = T$ iff, for each b such that $a^* \leq b$, $V(A, b) = F$.

As usual a formula A is verified iff $V(A, O) = T$. With these semantics the proportion of effort required for showing that theses are verified is greater than with the others. Surprisingly tricky arguments show that V1 and V2 hold for all formulas and, using this fact, that all the axioms and rules of $CR - W$ except $\vdash(\neg(\neg A)) \to A$ and $\vdash \bar{\bar{A}} \to A$ are verified by all models. The use of V3 completes the proof that all theses of $CR - W$ are verified by this semantics. (Is it sufficient to assume V3 for the propositional parameters and then derive it for all formulas? This is not done in [2], but it is shown that V3 holds in the canonical model, so that some models do indeed exist.)

Consider a regular, consistent theory T which is closed with respect to the rule for classical negation, that if $\vdash(A \wedge B) \to \neg C$ then $\vdash(A \wedge C) \to \neg B$. Note that the set of theses of $CR - W$ is such a theory. Let $\mathbf{P}_T$ denote the set of principal T-theories from now on, and consider $(\mathbf{P}_T, O_T, \cdot, \subseteq, *)$ with the usual operation $\cdot$ and the unusual

$$a^* = \{A \colon \neg\bar{A} \in a\}.$$

It can be shown that this is a prestructure, making considerable use of the primeness of T and various theses of $CR - W$, including

$\vdash(A \to F) \vee ((A \to F) \to F)$, where $F = B \wedge (\neg B)$,
$\vdash(\overline{A \to (\neg A)}) \vee (A \to B)$

$$\vdash(\neg\bar{A}) \to (\overline{\neg A}) \qquad \vdash(\overline{\neg A}) \to (\neg\bar{A}).$$

The normal ideals of the structure $(\mathfrak{F}, \mathbf{N}_T)$ are taken to be the sets

$$\{a \in \mathbf{P}_T\colon a \subseteq b\}$$

determined by all the prime, consistent-and-complete T-theories b. It can be shown that $\mathbf{N}_T$ is indeed a dense set of maximal ideals satisfying the condition stated above. Proving this uses the Extension Lemma that, given principal T-theories a, b and a prime T-theory c, if $a \cdot b \subseteq c$ then there is a prime T-theory b' with $b \subseteq b'$ and $a \cdot b' \subseteq c$. This is established by enumerating all the formulas in the language and adding them to b in succession whenever this does not spoil the property that $a \cdot b' \subseteq c$, for the resulting b'. The primeness of b' is derived from that of c. The usual canonical valuation

$$V_T(A, a) = T \text{ iff } A \in a$$

can be shown to satisfy all the conditions stated above.

The proof that V3 holds in this structure involves the consistent-and-completeness of the T-theories which determine the normal ideals. This is unfortunate, for it prevents a similar completeness proof holding for $R - W$.

Part III Quantifiers

8 A weak proof-theory for first-order logic In [7] Ono adds the natural rules for quantifiers to the Gentzen-style formulation described in Section 1, yielding a system which lacks the thesis $\vdash\forall x(A \vee B) \to (A \vee \forall xB)$ for x not free in A. This is, in effect, the appropriate 'Barcan formula' for quantified logics of implication and, although it does not hold in Intuitionist logic, it is normally taken to be an essential feature of first-order logics in the neighborhood of $R^+ - W$. Taking a domain U of individual constants, the appropriate conditions on valuations are

$$V(\forall xA(x), c) = T \text{ iff } V(A(u), c) = T \text{ for each } u \in U,$$
$$V(\exists xA(x), c) = T \text{ iff, for some } \{b_i\colon i \in I\} \text{ such that } \bigcap_{i\in I} b_i \leq c, \text{ for each } i \in I, \text{ for some } u_i \in U,\ V(A(u_i), b_i) = T.$$

Note that the same domain U is used at each element of W, as in Beth semantics. An extension of the argument sketched in Section 1 yields a completeness proof. The semantics are then generalized, by analogy with the propositional case summarized in Section 3. (It is then shown that the Beth semantics are a special case of this generalization.)

9 First-order Abraham semantics Extend $CR - W$ to $CRQ - W$ by adding the axioms

$\vdash\forall xA(x) \to A(y/x)$
$\vdash A \to \forall xA$ for x not free in A,
$\vdash\forall x(A \to B) \to (\forall xA \to \forall xB)$
$\vdash(\forall xA \wedge \forall xB) \to \forall x(A \wedge B)$
$\vdash\forall x(A \vee B) \to (A \vee \forall xB)$ for x not free in A,

and their duals in terms of $\exists x$, and the rule that if $\vdash A(x)$ then $\vdash \forall x A(x)$. The clauses for the quantifiers in the semantics of [2] are that

$$V(\forall x A(x), c) = T \text{ iff } V(A(y/x), c) = T \text{ for all variables } y,$$
$$V(\exists x A(x), c) = T \text{ iff, for each } \alpha \text{ such that } c \in \alpha \;\&\; \alpha \in \mathbf{N}, \text{ for some } b \in \alpha,\ V(A(y/x), b) = T \text{ for some variable } y.$$

It can still be shown that all theses of $CRQ - W$ are verified, using V3 in the case of the 'Barcan formula'. In constructing the canonical structure, it is now necessary to take T to be $\forall$-saturated as well, and to use $\exists$-saturated as well as prime T-theories b to determine the normal ideals $\{a \in \mathbf{P}_T\colon a \subseteq b\}$. Again the set of theses of CRQ-W satisfies all these conditions. In the Extension Lemma that, given principal T-theories a, b and a prime T-theory c, if $a \cdot b \subseteq c$ then there is a prime T-theory b' with $b \subseteq b'$ and $a \cdot b' \subseteq c$, b and b' must now be saturated as well as prime. Once this is established, the completeness proof proceeds as before.

In the step-by-step construction for this Extension Lemma, whenever a formula $\exists x A(x)$ could be added to b without spoiling $a \cdot b' \subseteq c$, an instance $A(y/x)$ is actually added. The language is assumed to be countable, so y may be chosen so that it does not occur free in the formulas already added. (The situation here could be contrasted with that in [4]. The condition on K mentioned at the beginning of Section 2 is a simpler analogue of the condition on normal ideals mentioned in Section 7. Both require Extension Lemmas in the proofs that they hold for the canonical structures, but the simpler condition of Section 2 requires the more complicated Extension Lemma also mentioned there. And, unlike the Extension Lemma of Section 7, this cannot be extended to include saturation.)

NOTES

1. The equivalences and consequences mentioned here hold in quite weak systems; the logic $R^+ - W$ about to be described suffices for all of them.
2. This logic has been considered in detail by various logicians working in Australia, especially Meyer and S. Giambrone in his Ph.D. thesis at ANU.
3. The approach I have in mind, especially the treatment of implication, was first carried out by Zaslavskiĭ in his book *Constructive Symmetric Logic* [14]. He is working with Nelson's logic of constructible falsity, so that his treatment of conjunction and negation differs from that given here. When I later discovered the modifications of Ono and Komori [8] described here, a referee introduced me to Zaslavskiĭ's work.

REFERENCES

[1] Abraham, A. Q., *The Semantics of Classical Relevance Logics*, M.Sc. Thesis, University of Canterbury, 1977.

[2] Abraham, A. Q., "Completeness of quantified classical relevant logic," forthcoming in *Studia Logica*.

[3] Fine, K., "Models for entailment," *Journal of Philosophical Logic*, vol. 3 (1974), pp. 347–372.

[4] Fine, K., "Incompleteness for quantified relevance logics," forthcoming in *Directions in Relevance Logic* (ed. R. Routley), 1985.

[5] Meyer, R. K. and R. Routley, "Classical relevant logics I, " *Studia Logica*, vol. 32 (1973), pp. 51–66.

[6] Meyer, R. K. and R. Routley, "Classical relevant logics II," *Studia Logica*, vol. 33 (1974), pp. 183–194.

[7] Ono, H., "Semantical analysis of predicate logics without the contraction rule," forthcoming.

[8] Ono, H. and Y. Komori, "Logics without the contraction rule," *The Journal of Symbolic Logic*, vol. 50 (1985), pp. 169–201.

[9] Routley, R. and R. K. Meyer, "The semantics of entailment," pp. 199–243 in *Truth, Syntax and Modality*, ed. Hughes Leblanc, North-Holland, 1973.

[10] Routley, R., and R. K. Meyer, "The semantics of entailment III," *Journal of Philosophical Logic*, vol. 1 (1972), pp. 192–208.

[11] Urquhart, A., "Semantics for relevant logics," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 159–169.

[12] Urquhart, A., "Relevant implication and projective geometry," *Logique et Analyse*, vol. 26 (1983), pp. 345–357.

[13] Urquhart, A., "The undecidability of entailment and relevant implication," *The Journal of Symbolic Logic*, vol. 49 (1984), pp. 1059–1073.

[14] Zaslavskiĭ, *Constructive Symmetric Logic* (in Russian), Erevan, 1978.

Department of Mathematics
University of Canterbury
Christchurch 1
New Zealand