Identities and Indiscernibility

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1 Introduction   Ehrenfeucht and Mostowski introduced in [4] the method of indiscernibles to show that every consistent theory which has an infinite model has models with arbitrarily large automorphism groups. This method provides a powerful tool for the construction of models with predetermined properties. Morley in his proof of Łoś's conjecture used the notion of a set indiscernibles. By the work of Morley, Shelah, and others this notion became important in characterizing stable theories.

The above notions of indiscernibility are all special cases of a more general one. We define it in Section 2. In solving problems concerning indiscernibility it quite often turns out that the argument is essentially combinatorial in nature and only uses properties of the equivalence relation $\sim$, defined by: $x \sim y$ iff x and y satisfy the same formulas. This gives rise to the definition of so-called identities: pairs (A, E) where E is a certain equivalence relation on $\bigcup\limits_{n \in \omega} A^n$. The first one who fruitfully used identities in model-theoretic problems was Shelah. He proved a compactness theorem for pairs of cardinals [8] and gave a combinatorial proof of Vaught's two cardinal theorem [6]. Independently of Shelah, Benda in [1] introduced identities under the name modeloids as objects worthy of study in their own right.

In Sections 2 and 3 we investigate the notion of identity and we introduce a special class of identities, the complete and homogeneous identities. In Sections 4 and 5 we apply our results about identities to investigate the hierarchy of indiscernibility, e.g., we prove that there exists a collection of complete theories $\{T_X | X \subseteq \omega\}$ such that $T_X \leq T_Y$ iff $X \subseteq Y$.

In Section 5 we show by means of an example that the hierarchy of indiscernibility contains infinite chains and infinite antichains. Also here the advantage of using identities becomes apparent. We adopt all modeltheoretic notations from Chang and Keisler [3]. We specially mention the following: $\bar{a}, \bar{b}, \ldots$ denote finite sequences. The length of a sequence $\bar{a}$ is $l(\bar{a})$ and if π is a permutation of $\{0, \ldots, l(\bar{a}) - 1\}$, then $\pi\bar{a}$ is the sequence $\langle a_{\pi(0)}, \ldots, a_{\pi(l(\bar{a})-1)} \rangle$.

If $\bar{a}$ is a sequence and α is a strictly increasing sequence $i_0 < i_1 < i_2 < \ldots < i_{k-1} < l(\bar{a})$, then $\bar{a} \restriction \alpha$ is the sequence:

$$\langle a_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}} \rangle.$$

If $\bar{a}$ and $\bar{b}$ are two sequences then $\bar{a} \frown \bar{b}$ is the sequence obtained from them by concatenation.

For any collection $\{A_i | i \in I\}$ of sets $\overset{\bullet}{\underset{i \in I}{\bigcup}} A_i$ is the disjoint union.

2 Complete and homogeneous identities   A structure $\mathfrak{A}$ is indiscernible by 1-tuples (or shortly: indiscernible) in a theory T (not necessarily for the same language) if there exists a model $\mathfrak{B}$ of T and a one-to-one mapping $f: A \to B$ such that for all $n \in \omega$ and all pairs of sequences $\langle a_1, \ldots, a_n \rangle$ and $\langle a_1', \ldots, a_n' \rangle$ in A^n: if they satisfy the same quantifier-free formulas in $\mathfrak{A}$, then $\langle f(a_1), \ldots, f(a_n) \rangle$ and $\langle f(a_1'), \ldots, f(a_n') \rangle$ satisfy the same type in $\mathfrak{B}$.

We introduce a relation $\leq$ between theories as follows: $T_1 \leq T_2$ iff for all structures $\mathfrak{A}$: if $\mathfrak{A}$ is indiscernible in T_1, then $\mathfrak{A}$ is indiscernible in T_2.

As a corollary of the definition we state the following:

Proposition 2.1   *A structure $\mathfrak{A}$ is indiscernible in a theory T iff for all finite subsets X of A the following set is consistent with T:*

$\{\phi(c_{x_1}, \ldots, c_{x_n}) \rightleftarrows \phi(c_{y_1}, \ldots, c_{y_n}) | \phi \in L_T, \langle y_1, \ldots, y_n \rangle, \langle x_1, \ldots, x_n \rangle \in X^n,$
$\langle x_1, \ldots, x_n \rangle$ *and* $\langle y_1, \ldots, y_n \rangle$ *satisfy the same type in* $\mathfrak{A}\} \cup \{C_x \neq C_y | x \neq y\}.$

This leads us to the following definitions:

An *identity* is a pair (A, E) where E is an equivalence relation on $\underset{k \in \omega}{\bigcup} A^k$ such that:

(i) $E(\bar{a}, \bar{b})$ implies $l(\bar{a}) = l(\bar{b})$
(ii) if $E(\bar{a}, \bar{b})$ and π is a permutation of $\{0, \ldots, l(\bar{a}) - 1\}$, then $E(\pi\bar{a}, \pi\bar{b})$
(iii) if $E(\bar{a}, \bar{b})$ and J is a strictly increasing sequence of elements of $\{0, \ldots, l(\bar{a}) - 1\}$, then $E(\bar{a} \restriction J, \bar{b} \restriction J)$
(iv) if $a_0 = a_1$ and $E(\bar{a}, \bar{b})$, then $b_0 = b_1$
(v)   if $a_0 = a_1$ and $b_0 = b_1$ and $E(\bar{a} \restriction \langle 1, \ldots, l(\bar{a}) - 1 \rangle, \bar{b} \restriction \langle 1, \ldots, l(\bar{a}) - 1 \rangle)$ then $E(\bar{a}, \bar{b})$.

A is the universe of the identity and $|A|$ is the cardinality of the identity. $E \cap A^n$ is the equivalence relation $E \cap (A^n)^2$ on A^n.

If (A, E) is an identity and $Y \subseteq A$ then $(Y, E \restriction Y)$ is the identity defined by $E \restriction Y(\bar{a}, \bar{b})$ iff $E(\bar{a}, \bar{b})$.

(A, E) is weakly finite if for all n: $E \cap A^n$ has only finitely many equivalence classes.

(B, F) is realized in (A, E) if there exists a one-to-one function $f: B \to A$ such that for all n and all $\langle b_0, \ldots, b_{n-1} \rangle, \langle b_0', \ldots, b_{n-1}' \rangle \in B^n$: if $F(\langle b_0, \ldots, b_{n-1} \rangle, \langle b_0', \ldots, b_{n-1}' \rangle)$, then $E(\langle f(b_0), \ldots, f(b_{n-1}) \rangle, \langle f(b_0'), \ldots, f(b_{n-1}') \rangle)$.

(A, E) is an extension of (B, F) if $A = B$ and $E \supseteq F$.

Let a structure $\mathfrak{A}$ be given. Define an identity $(A, E_{\mathfrak{A}})$ as follows: $E_{\mathfrak{A}}(\bar{a}, \bar{b})$ iff $l(\bar{a}) = l(\bar{b})$ and $\bar{a}$ and $\bar{b}$ satisfy the same type in $\mathfrak{A}$. We say that (B, F) is realized in $\mathfrak{A}$ if (B, F) is realized in (A, E) and (B, F) is realized in T if it is realized in some model of T. Finally we define, $Id(A, E)$ as the set of finite

identities that are realized in (A, E). In a similar way $Id(\mathfrak{A})$ and $Id(T)$ are defined for a structure $\mathfrak{A}$ and a theory T.

Proposition 2.2   *For all theories T_1 and T_2:*

$$T_1 \leq T_2 \text{ iff } Id(T_1) \subseteq Id(T_2).$$

Proof: Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be ω-saturated models of T_1 and T_2 respectively. Surely, for all finite identities (n, F), (n,F) is realized in $\mathfrak{A}_1$ $(\mathfrak{A}_2)$ iff it is realized in T_1 (T_2). For every type $p(x_1,\ldots,x_n)$ which is realized in $\mathfrak{A}_1$ let $R_p(x_1,\ldots,x_n)$ be a relational symbol. Let $\mathfrak{A}_1^+$ be the following structure for the language $L = \{R_p(x,\ldots,x_n)|p$ realized in $\mathfrak{A}_1\}$

$$\mathfrak{A}_1^+ = \langle A_1, R_p^{\mathfrak{A}_1^+} \rangle \ p \text{ realized in } \mathfrak{A}_1$$

where $R_p^{\mathfrak{A}_1^+}(a_1,\ldots,a_n)$ iff $\langle a_1,\ldots,a_n \rangle$ realizes the type p in $\mathfrak{A}_1$.

Two sequences $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas in $\mathfrak{A}_1^+$ iff they satisfy the same type in $\mathfrak{A}_1$.

If follows that $T_1 \leq T_2$ iff for all finite $X \subseteq A$, there exists a one-to-one mapping $f: X \to A_2$ such that for all $\langle x_1,\ldots,x_n \rangle$, $\langle y_1,\ldots,y_n \rangle \in X^n$, if they satisfy the same type in $\mathfrak{A}_1$, then they satisfy the same type in $\mathfrak{A}_2$.

The following also holds: a finite identity (n, F) is realized in $\mathfrak{A}_1(\mathfrak{A}_2)$ iff there is a one-to-one mapping $g: n \to A_1$ (A_2) such that: whenever $F(\langle i_1,\ldots,i_k \rangle$, $\langle j_1,\ldots,j_k \rangle)$, then $\langle g(i_1),\ldots,g(i_k) \rangle$ and $\langle g(j_1),\ldots,g(j_k) \rangle$ satisfy the same type in $\mathfrak{A}_1(\mathfrak{A}_2)$. The proposition now follows easily.

An identity (A, E) is *complete* if for all finite identities (n, F), if (n, F) is realized in every weakly finite extension of (A, E), then it is realized in (A, E).

An identity (A, E) *is homogeneous* if for all sequences $\bar{a}$ and $\bar{b}$ and all elements c, if $E(\bar{a}, \bar{b})$, then there exists an element d such that $E(\langle c \rangle \,^\frown \bar{a}, \langle d \rangle \,^\frown \bar{b})$.

Let a structure $\mathfrak{A}$ be given and let Δ be a set of formulas in the language of $\mathfrak{A}$ with free variables among $x_0,\ldots,x_{n-1}$. Let $\Delta^* = \{\phi(x_{\pi(0)},\ldots,x_{\pi(n-1)})|\pi$ a permutation of n and $\phi \in \Delta\}$ and define the identity (A, E_Δ) as follows:

$$E_\Delta(\bar{a}, \bar{b}) \text{ if } l(\bar{a}) = l(\bar{b}) \text{ and for all } i_1 < i_2 < \ldots < i_k \text{ in}$$
$$\{0,\ldots,n-1\} \text{ and all } \phi \in (x_0,\ldots,x_{k-1}) \in \Delta^*,$$
$$\mathfrak{A} \vDash \phi[a_{i_1},\ldots,a_{i_k}] \text{ iff } \mathfrak{A} \vDash \phi[b_{i_1},\ldots,b_{i_k}].$$

Then (A, E_Δ) is weakly finite if for all n, Δ has only finitely many formulas with free variables among $x_0,\ldots,x_{n-1}$.

The following proposition follows immediately from the compactness theorem:

Proposition 2.3
(i) *For all finite identities (n, F): (n, F) is realized in $Th(\mathfrak{A})$ iff for all finite sets Δ of formulas, (n, F) is realized in (A, E_Δ).*
(ii) *If $Id(A, E) = Id(Th(\mathfrak{A}))$ then $Id(A, E_\mathfrak{A})$ is complete.*

Theorem 2.4   *Let T be a complete theory. Then there exists a complete and homogeneous identity (A, E) such that $Id(T) = Id(A, E)$.*

Proof: Let $\mathfrak{A}$ be an ω-saturated model of T. Let $(A, E) = (A, E_\mathfrak{A})$. This identity meets the requirements.

Theorem 2.5   *Let (A, E) be a complete and homogeneous identity. Then there exists a complete theory T such that $Id(A, E) = Id(T)$.*

Proof: Let for $k \in \omega$, $k \neq 0$ $\{\alpha_i | i \in I_k\}$ be the set of all equivalence classes of $E \cap A^k$.

Let R_J be a k-ary relation symbol for each $J \subseteq I_k$.
Let $L = \{R_J | J \subseteq I_k, k \in \omega\}$.
Define structure $\mathfrak{A}$ for L whose universe is A and

$$R_J^{\mathfrak{A}} = \bigcup_{i \in J} \alpha_i \text{ for } J \subseteq I_k.$$

Let $T = Th(\mathfrak{A})$.
We can make the following observations

(i) $E(\bar{a}, \bar{b})$ iff $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas in $\mathfrak{A}$.
(ii) If $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas and $c \in A$, then there exists $d \in A$ such that $\langle c \rangle ^\frown \bar{a}$ and $\langle d \rangle ^\frown \bar{b}$ satisfy the same quantifier-free formulas.
(iii) If $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas, then they satisfy the same formulas.
(iv) $E(\bar{a}, \bar{b})$ iff $\bar{a}$ and $\bar{b}$ satisfy the same type.

Let $\phi(x_1, \ldots, x_n)$ be given. Let $J \subseteq I_n$ be defined by

$$J = \{i | i \in I_n, \phi^{\mathfrak{A}} \cap \alpha_i \neq \varnothing\}.$$

By (iv) it is easy to see that $\mathfrak{A} \models \forall \bar{x}[\phi(\bar{x}) \rightleftarrows R_J(\bar{x})]$.

Now remark that for every $n \in \omega$ and every weakly finite extension (A, E^*) of (A, E) there exists a finite set of formulas Δ such that $E^* \cap A^k = E_\Delta \cap A^k$ for all $k \leq n$.

It follows that for all finite identities (n, F): (n, F) is realized in all weakly finite extensions of (A, E) if for all finite sets of formulas Δ, (n, F) is realized in (A, E_Δ).

Using Proposition 2.3 and the completeness of (A, E), we see that T satisfies the conclusion of the theorem.

The special properties of the structure $\mathfrak{A}$ occurring in the proof above will be needed later on. So we specially state the following proposition.

Proposition 2.6   *Let (A, E) be homogeneous. Then there exists a structure $\mathfrak{A}$ such that $Id(A, E) = Id(\mathfrak{A})$ and for all (n, F): (n, F) is realized in $Th(\mathfrak{A})$ iff it is realized in every weakly finite extension of (A, E).*

The language of the structure defined above may not be countable, but we easily can show:

Proposition 2.7   *For all languages L and theories T in L there is a countable $L^* \subseteq L$ such that $Id(T) = Id(T \cap L^*)$.*

Proof: Let $L_n \subseteq L$ be finite such that for all finite (n, F), (n, F) realized in T iff (n, F) realized in $T \cap L_n$. Such L_n exist because there are at most (up to isomorphism) a finite number of finite identities of cardinality n that are not realized in T. Let $L^* = \bigcup_n L_n$.

We state without proof a combinatorial version of the theorem of Ehren-feucht and Mostowski on sequences of indiscernibles (see [4]). Let, for $n \in \omega$, (n, F_0^n) be the following identity:

$$F_0^n(\bar{a}, \bar{b}) \text{ iff } l(\bar{a}) = l(\bar{b}) \text{ and for all } i < j < l(\bar{a}),$$
$$a_i < a_j \text{ iff } b_i < b_j.$$

Theorem 2.8   *Given a finite identity (m, G) then the following are equivalent*:
(i) *(m, G) is realized in (m, F_0^m).*
(ii) *(m, G) is realized in all weakly finite identities (A, E) with infinite A.*

3 A language for identities   For $n \in \omega$, $n \neq 0$, let E_n be a relational symbol with $2n$ open places. Let $L_{id} = \{E_n | n \in \omega\}$ and $L_{id}^n = \{E_k | k \leq n\}$.

ϕ is an (n, m)-formula if ϕ can be written as $\exists x_0 \ldots \exists x_{k-1} \psi$ where $k \leq n$ and ψ is the conjunction of formulas of one of the following three kinds:

(i) $x_i = x_j$
(ii) $x_i \neq x_j$
(iii) $E_1(x_{i_1}, \ldots, x_{i_{2l}})$, $1 \leq m$.

ϕ is an (n, m)-sentence if ϕ does not have free variables.

The notions of (∞, m)-formula, (n, ∞)-formula, (∞, ∞)-formula are defined in an obvious way.

For two identities (A, E) and (B, F) we define $(A, E) \leq_{(n,m)} (B, F)$ if for all (n, m)-sentences ϕ, if $(A, E \cap A^n)_{n \in \omega} \vDash \phi$, then $(B, F \cap B^n)_{n \in \omega} \vDash \phi$.

$(A, E) \equiv_{(n,m)} (B, F)$ if $(A, E) \leq_{(n,m)} (B, F)$ and $(B, F) \leq_{(n,m)} (A, E)$. The relations $\leq_{(\infty,m)'}$, $\equiv_{(\infty,m)'}$, $\leq_{(n,\infty)'}$, $\equiv_{(n,\infty)'}$, $\leq_{(\infty,\infty)}$ and $\equiv_{(\infty,\infty)}$ are defined in a similar way.

Proposition 3.1   *$(A, E) \leq_{(n,n)} (B, F)$ iff for all (n, G), if (n, G) is realized in (A, E), then (n, G) is realized in (B, F).*

Proposition 3.2   *$(A, E) \leq_{(\infty,n)} (B, F)$ iff there is a collection of mappings J such that*
(i) *for all $f \in J$: f is one-to-one, has a finite domain, and is a homomorphism of the structure $(A, E \cap A^k)_{k \leq n}$ to the structure $(B, F \cap B^k)_{k \leq n'}$*
(ii) *for all finite $X \subseteq A$ there is an $f \in J$ such that $X \subseteq$ dom f.*

Let $n \in \omega$ and (A, E) be given. (A, E) is n-complete if for all (n, F): if (n, F) is realized in all weakly finite extensions of (A, E), then (n, F) is realized in (A, E).

Remark the following facts, which follow immediately from the definition of identity:

(i) if (A, E) and (A, E^*) are such that $E \cap A^n = E^* \cap A^n$, then they realize the same identities (n, F)
(ii) if $E \cap A^n \subseteq E^* \cap A^n$, then for all $k \leq n$: $E \cap A^k \subseteq E^* \cap A^k$
(iii) if $E \cap A^n$ has finitely many equivalence classes, then for all $k \leq n$: $E \cap A^k$ has finitely many equivalence classes.

It follows that (A, E) is n-complete if for all (n, F), (n, F) is realized in (A, E) if it is realized in all identities (A, E^*) such that $E^* \cap A^n \supseteq E \cap A^n$ and $E^* \cap A^n$ has finitely many equivalence classes.

Theorem 3.3   *Let* $(A, E) \equiv_{(n, n)} (B, F)$ *and* $(B, F) \leq_{(\infty, n)} (A, E)$. *If* (A, E) *is n-complete, then* (B, F) *is n-complete.*

Proof: Let (n, G) be realized in all identities (B, F^*) such that $F^* \cap B^n \supseteq F \cap B^n$ and has finitely many equivalence classes. Let (A, E^*) be a weakly finite extension of (A, E). We will show that (n, G) is realized in (A, E^*). Then the theorem follows from the assumptions and from Proposition 1.

Because $(B, F) \leq_{(\infty, n)} (A, E)$ we can apply Proposition 3.2. Let J be a collection of mappings as in Proposition 3.2. For every finite $X \subseteq B$, let $J_x = \{f | f \in J, X \subseteq dom\, f\}$. The collection $\{J_X | X \subseteq B, X \text{ finite}\}$ has the finite intersection property. So let $\mathbf{H}$ be an ultrafilter on J such that for all finite $X \subseteq B$, $J_X \in \mathbf{H}$.

Define an identity (B, F^*) as follows: If $l(\bar{a}) = l(\bar{b}) = k$, then $F^*(\bar{a}, \bar{b})$ if there exists $J_0 \in \mathbf{H}$ such that for all $f \in J_0$
(i) $rng(\bar{a}) \cup rng(\bar{b}) \subseteq dom\, f$
(ii) $E^*(\langle f(a_0), \ldots, f(a_{k-1}) \rangle, \langle f(b_0), \ldots, f(b_{k-1}) \rangle)$.
(A, E^*) is an identity, hence (B, F^*) is an identity. $F^* \cap B^n \supseteq F \cap B^n$. Indeed, let $l(\bar{a}) = l(\bar{b}) = n$ and $F(\bar{a}, \bar{b})$. There is $J_0 \in \mathbf{H}$ such that $rng(\bar{a}) \cup rng(\bar{b}) \subseteq dom\, f$ for all $f \in J_0$. Because of the properties of J: for all $f \in J_0$

$$E(f(a_0), \ldots, f(a_{k-1}) \rangle, \langle f(b_0), \ldots, f(b_{k-1}) \rangle)$$

and hence

$$E^*(\langle f(a_0), \ldots, f(a_{k-1}) \rangle, \langle f(b_0), \ldots, f(b_{k-1}) \rangle).$$

So by definition of F^*: $F^*(\bar{a}, \bar{b})$.

$F^* \cap B^n$ has finitely many equivalence classes. Indeed, if $E^* \cap A^n$ has k equivalence classes, then $F^* \cap B^n$ has at most k equivalence classes. Suppose this were not so. Then there are $\bar{a}_0, \ldots, \bar{a}_k \in B^n$ such that for no $i < j \leq k$, $F^*(\bar{a}_i, \bar{a}_j)$. Let $X_0, \ldots, X_{k-1}$ be the equivalence classes of $E^* \cap A^n$. From the properties of an ultrafilter it follows that for all $i \leq k$ there exists $j < k$ such that for $\mathbf{H}$-almost all $f \in J$: $f(\bar{a}_i) \in X_j$. Hence there are $i < j \leq k$ and $l < k$ such that for $\mathbf{H}$-almost all $f \in J$: $f(\bar{a}_i) \in X_1$ and $f(\bar{a}_j) \in X_1$. It follows that $F^*(\bar{a}_i, \bar{a}_j)$ — a contradiction. By assumptions (n, G) is realized in (B, F^*). Let $\{a_0, \ldots, a_{n-1}\} \subseteq B$ be such that $F^*(\langle a_{i_1}, \ldots, a_{i_k} \rangle, \langle a_{j_1}, \ldots, a_{j_k} \rangle)$ whenever $G(\langle i_1, \ldots, i_k \rangle, \langle j_1, \ldots, j_k \rangle)$. Again by the properties of an ultrafilter it follows that for $\mathbf{H}$-almost all $f \in J$:

$$\{a_0, \ldots, a_{n-1}\} \subseteq dom\, f \text{ and } E^*(\langle f(a_{i_1}), \ldots, f(a_{i_k}) \rangle, \langle f(a_{j_1}), \ldots, f(a_{j_k}) \rangle)$$

whenever $F^*(\langle a_{i_1}, \ldots, a_{i_k} \rangle, \langle a_{j_1}, \ldots, a_{j_k} \rangle)$. Take an $f \in J$ with these properties. Then (n, G) is realized in (A, E^*) by the set $\{f(a_0), \ldots, f(a_{n-1})\}$.

Corollary 3.4   *If* (A, E) *and* (B, F) *realize the same identities and* (A, E) *is complete, then* (B, F) *is complete.*

We will use Theorem 3.3 in various ways. Let us first introduce a more refined notion of completeness. Let $n, k \in \omega$ and $n, k \neq 0$.

An identity (A, E) (finite or infinite) is (n, k)-complete if for all (n, F), (n, F) is realized in (A, E) iff it is realized in all identities (A, E^*) such that $E^* \cap A^n \supseteq E \cap A^n$ and $E^* \cap A^n$ has at most k equivalence classes. We state some facts about (n, k)-completeness without proof.

Lemma 3.5

(i) *Let for all $i \in I$, (A_i, E_i) be (n, k)-complete and let $\mathbf{D}$ be an ultrafilter on I. Then $\prod_{i \in I} (A_i, E_i)/\mathbf{D}$ is (n, k)-complete.*

(ii) *Let $\mathbf{D}$ be an ultrafilter on a set I and let (A, E) be not (n, k)-complete, then $(A, E)^I/\mathbf{D}$ is not (n, k)-complete.*

(iii) *(A, E) is n-complete iff (A, E) is (n, k)-complete for some k.*

It should be obvious how the ultraproduct and ultrapower of identities are defined.

Remark, that from (i) and (ii) it follows that the class of (n, k)-complete identities, considered as structures for L_{id}, is an elementary class (see [3]).

Proposition 3.6   *Let n and k be given. Then for all identities (A, E) with $|A| \geq n$, (A, E) is (n, k)-complete iff for all finite subsets Y of A, if $(Y, E \restriction Y)$ $\equiv_{(n,n)} (A, E)$, then $(Y, E \restriction Y)$ is (n, k)-complete.*

Proof: Let (A, E) be (n, k)-complete and let $Y \subseteq A$ be finite such that $(Y, E \restriction Y) \equiv_{(n,n)} (A, E)$.

Suppose (n, G) is not realized in $(Y, E \restriction Y)$. Then (n, G) is not realized in (A, E). Hence there is (A, E^*) such that (n, G) is not realized in (A, E^*), $E^* \cap A^n \supseteq E \cap A^n$ and $E^* \cap A^n$ has at most k equivalence classes. Then surely (n, G) is not realized in $(Y, E^* \restriction Y)$. It follows that $(Y, E \restriction Y)$ is (n, k)-complete.

Conversely, let for all $Y \subseteq A$ such that $(Y, E \restriction Y) \equiv_{(n,n)} (A, E)$, $(Y, E \restriction Y)$ will be (n, k)-complete.

Let $\mathbf{E}$ be the set of those Y. Then $\mathbf{E} \neq \varnothing$. For $Y \in \mathbf{E}$ let $H_Y = \{Y' | Y' \in \mathbf{E}, Y' \supseteq Y\}$ and let $\mathbf{D}$ be an ultrafilter on $\mathbf{E}$ containing all sets H_Y. Finally, let $(B, F) = \prod_{Y \in \mathbf{E}} (Y, E \restriction Y)/\mathbf{D}$. Then $(A, E) \equiv_{(n,n)} (B, F)$ and $(A, E) \leq_{(\infty,n)} (B, F)$. By Theorem 3.3 and Lemma 3.5(i), (A, E) is (n, k)-complete.

We will need this proposition in the proof of Theorem 3.9.

Proposition 3.7   *For all identities (A, E) and (B, F): $(A, E) \leq_{(\infty,\infty)} (B, F)$ iff there exists an identity (C, G) and a one-to-one mapping $f: A \to C$ such that $(C, G) \equiv_{(\infty,\infty)} (B, F)$ and for all $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle \in A^n$,*

$$\text{if } E(\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle), \text{ then}$$
$$G(\langle f(a_1), \ldots, f(a_n) \rangle, \langle f(b_1), \ldots, f(b_n) \rangle).$$

Proof: The existence of (C, G) and f is surely sufficient to conclude $(A, E) \leq_{(\infty,\infty)} (B, F)$. So let $(A, E) \leq_{(n,n)} (B, F)$. Let $\mathfrak{B}$ be the structure $\langle B, F \cap B^n \rangle_{n \in \omega}$ for the language L_{id}. Add to L_{id} new constants c_a for all $a \in A$. It is enough to prove that the following set of sentences is consistent:

$Th(\mathfrak{B}) \cup \{c_a \neq c_b | a, b \in A, a \neq b\}$
$\quad \cup \{E_n(c_{a_1}, \ldots, c_{a_n}, c_{b_1}, \ldots, c_{b_n}) | E(\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle)\}$

The consistency immediately follows from the compactness theorem and from Proposition 3.1.

Theorem 3.8 *Every homogeneous identity has a completion in the following sense: if (A, E) is homogeneous, then there exists an identity (A^*, E^*) such that (A^*, E^*) is homogeneous and complete, $(A, E) \leq_{(\infty,\infty)} (A^*, E^*)$ and for all homogeneous and complete (B, F) such that $(A, E) \leq_{(\infty,\infty)} (B, F)$, $(A^*, E^*) \leq_{(\infty,\infty)} (B, F)$.*

Proof: Let (A, E) be homogeneous. Let $\mathfrak{A}$ be the structure the existence of which follows from Proposition 2.6. Let $\mathfrak{A}'$ be an ω-saturated model of $Th(\mathfrak{A})$ and let $(A^*, E^*) = (A', E_{\mathfrak{A}'})$. Then (A^*, E^*) is homogeneous and complete and for all (n, F), (n, F) is realized in (A^*, E^*) iff it is realized in every weakly finite extension of (A, B). Surely $(A, E) \leq_{(\infty,\infty)} (A^*, E^*)$.

Let (B, F) be homogeneous and complete, and let $(A, E) \leq_{(\infty,\infty)} (B, F)$. By Proposition 3.7 and Corollary 3.4 we may assume $A \subseteq B$ and if $E(\bar{a}, \bar{b})$, then $F(\bar{a}, \bar{b})$. Let (n, G) not be realized in (B, F). Then it is not realized in some weakly finite extension (B, F^*) of (B, F). Then (n, G) is not realized in $(A, F^* \restriction A)$. This is a weakly finite extension of (A, E), hence (n, G) is not realized in (A^*, E^*). It follows that $(A^*, E^*) \leq_{(\infty,\infty)} (B, F)$.

We could formulate the theorem above as follows:

> *For all homogeneous identities (A, E) there exists a complete theory T such that: $Id(A, E) \subseteq Id(T)$ and for all complete theories T^*: if $Id(A, E) \subseteq Id(T^*)$ then $T \leq T^*$.*

To conclude this section we will give a characterization of those sets **E** of finite identities for which there exists a complete theory T such that $Id(T) = $ **E**.

Theorem 3.9

(A) *Let* **E** *be a class of finite identities such that the following conditions are satisfied*:

(i) *if (n, F) is realized in (m, G) and $(m, G) \in$ **E**, then $(n, F) \in$ **E** (**E** is transitive)*

(ii) *if $(n, F) \in$ **E** and $(m, G) \in$ **E**, then for some $(k, H) \in$ **E**, (n, F) and (m, G) are both realized in (k, H). (**E** has the joint embedding property)*

(iii) *for all $(n, F) \in$ **E**, $F(\bar{a}, \bar{b})$ and $c \in n$ there exists $(m, G) \in$ **E** and $d \in m$ such that $n \subseteq m$, $G \restriction n \supseteq F$ and $G(\langle c \rangle ^\frown \bar{a}, \langle d \rangle ^\frown \bar{b})$. (**E** is homogeneous.)*

(iv) *for all n there exists k such that for all $(m, G) \in$ **E**, if (m, G) realizes all (n, F) that are in **E**, then (m, G) is (n, k)-complete. (**E** is complete).*

*Then there exists a complete theory T such that $Id(T) = $ **E**.*

(B) *For all complete theories T, $Id(T)$ satisfies conditions (i)-(iv).*

Proof: Part (B) immediately follows from Theorem 2.4, Theorem 2.5, and Proposition 3.6.

So let us prove (A). By Proposition 3.6 and Theorem 2.5 it is enough to prove that there exits a homogeneous (A, E) such that $Id(A, E) = $ **E**. From (ii)

it follows that for all finite $E_0 \subseteq E$ there exists $(m, H) \in E$ such that every element of E_0 is realized in (m, H). Repeatedly applying (iii) yields: for all $(n, F) \in E$ there exists $(m, H) \in E$ such that $n \subseteq m$, $H \restriction n \supseteq F$ and for all $F(\bar{a}, \bar{b})$ and $c \in n$ there exists $d \in m$ such that $H(\langle c \rangle \frown \bar{a}, \langle d \rangle \frown \bar{b})$.

For all $n \in \omega$, $n \neq 0$ let $(k_n, K_n) \in E$ be such that if $m \leq n$ and $(m, F) \in E$, then (m, F) is realized in (k_n, K_n). Moreover let (l_n, L_n) be such that $k_n \subseteq l_n$, $L_n \restriction k_n \supseteq K_n$ and whenever $K_n(\bar{a}, \bar{b})$ and $c \in k_n$ then there exists $d \in l_n$ such that $L_n(\langle c \rangle \frown \bar{a}, \langle d \rangle \frown \bar{b})$.

Define the sequence $m_1, m_2, \ldots$ and identities $(m_1, M_1), (m_2, M_2), \ldots$ as follows: let $(m_1, M_1) = (l_1, L_1)$ and let (m_i, M_i) be defined such that $(m_i, M_i) = (l_n, L_n)$. Then let $m = max(l_n, n)$ and $(m_{i+1}, M_{i+1}) = (l_m, L_m)$. Then the following holds:

for all i, (m_i, M_i) is realized in (m_{i+1}, M_{i+1}) by a function $f_i: m_i \rightarrow m_{i+1}$ such that whenever $M_i(\bar{a}, \bar{b})$ and $c \in m_i$, then there is $d \in m_{i+1}$ such that $M_{i+1}(\langle f(c) \rangle \frown \langle f(a_0), \ldots, f(a_{l(\bar{a})-1}) \rangle, \langle d \rangle \frown \langle f(b_0), \ldots, f(b_{l(\bar{b})-1}) \rangle)$.

Indeed, for all i, (m_i, M_i) is realized in (k_m, K_m), where m is as above. Then let f_i be the function realizing (m_i, M_i). By identifying corresponding elements we may assume that, for all i, f_i is the identity mapping.

So we may assume the following:

(i) $m_1 \leq m_2 \leq m_3 \leq \ldots$

(ii) for all i, $M_{i+1} \restriction m_i \supseteq M_i$ and $(m_i, M_i) \in E$

(iii) whenever $c \in m_i$ and $M_i(\bar{a}, \bar{b})$, then there exists $d \in m_{i+1}$ such that $M_{i+1}(\langle c \rangle \frown \bar{a}, \langle d \rangle \frown \bar{b})$

(iv) for all n there exists an i such that: if $(n, F) \in E$, then (n, F) is realized in (m_i, M_i).

Finally construct (A, E) as follows: $A = \bigcup_{i \in \omega} m_i$ and $E(\bar{a}, \bar{b})$ iff there exists an N such that for all $i \geq N$, $rng(\bar{a}) \cup rng(\bar{b}) \subseteq m_i$ and $M_i(\bar{a}, \bar{b})$. It is clear that (A, E) is homogeneous and $Id(A, E) = E$. This finishes the proof.

From this and Theorem 3.8 follows a closure property:

Theorem 3.10   *Let E be a set of finite identities such that E is transitive, homogeneous, and has the joint embedding property, then there exists a complete theory T such that $E \subseteq Id(T)$ and for all T^*, if $E \subseteq Id(T^*)$, then $T \leq T^*$.*

4 The hierarchy of indiscernibility   From Proposition 2.2, Theorems 2.4 and 2.5, and Proposition 3.1 it follows that the $\leq$-relation between complete theories establishes a hierarchy which is isomorphic to the hierarchy of complete and homogeneous identities induced by the relation $\leq_{(\infty,\infty)}$. In this section we try to answer questions concerning suprema and infima of chains.

Theorem 4.1   *Let $(I, \leq)$ be a linearly ordered set and, for $i \in I$, (A_i, E_i) an identity such that for all $i, j \in I$, $(A_i, E_i) \leq_{(\infty,\infty)} (A_j, E_j)$ if $i \leq j$. Then there exists a subset $I_0 \subseteq I$ cofinal and coinitial with I such that either $(I_0, \leq) \cong (2, \leq)$, or $(I_0, \leq) \cong (\omega, \leq)$, or $(I_0, \leq) \cong (\omega^*, \leq)$, or $(I_0, \leq) \cong (\omega^* + \omega, \leq)$.*

Proof: There are four cases, depending on whether I has a smallest element or not and a greatest element or not. We will consider only one case: I has a

smallest element but not a greatest element. Let for $i \in I$ and $n \in \omega$, $n \neq 0$, $Id_n(A_i, E_i)$ be the set of identities of cardinality n, that are realized in (A_i, E_i). Then for all i and n, $Id_n(A_i, E_i)$ is finite. Hence there exists i_n such that for all $i \geq i_n$:

$$Id_n(A_i, E_i) = Id_n(A_{i_n}, E_{i_n}).$$

Now choose $j_1 \leq j_2 \leq j_3 \leq \ldots$ such that $j_1 \geq i_1$, $j_2 \geq i_2, \ldots$ then $\{j_n | n \in \omega\}$ is cofinal with I.

Theorem 4.2   *Let $(A_1, E_1) \geq_{(\infty,\infty)} (A_2, E_2) \geq_{(\infty,\infty)} (A_3, E_3) \geq_{(\infty,\infty)} \ldots$ be a descending chain of homogeneous and complete identities. Then there exists a homogeneous and complete identity (A, E) such that $Id(A, E) = \bigcap_{n \in \omega} Id(A_n, E_n)$.*

Proof: We may assume that for all $n \in \omega$ and $m \geq n$:

$$(A_m, E_m) \equiv_{(n,n)} (A_n, E_n).$$

The identity (A, E) should have the following properties
 (i) for all n, $(A, E) \equiv_{(n,n)} (A_n, E_n)$
(ii) (A, E) is homogeneous.

This is also sufficient. Indeed, from property (i) and the assumptions of the theorem it follows that $(A, E) \leq_{(\infty,n)} (A_n, E_n)$ for all n (remark that every (∞, n)-sentence is an (m, m)-sentence for some $m \geq n$). Then, by Theorem 3.3, (A, E) is n-complete for all n. The existence of an identity satisfying conditions (i) and (ii) immediately follows from the compactness theorem.

Theorem 4.3   *Let $(A_1, E_1) \leq_{(\infty,\infty)} (A_2, E_2) \leq_{(\infty,\infty)} \ldots$ be an increasing chain of homogeneous and complete identities. Then there exists a homogeneous and complete identity (A, E) such that for all n, $(A_n, E_n) \leq_{(n,n)} (A, E)$ and whenever a complete and homogeneous identity (B, F) also has this property, then $(A, E) \leq_{(\infty,\infty)} (B, F)$.*

Proof: Define (A^*, E^*) as follows: let D be any nonprincipal ultrafilter on ω. $A^* = \prod_{n \in \omega} A_n / D$ and $E^*(\langle \bar{f}_1, \ldots, \bar{f}_k \rangle, \langle \bar{g}_1, \ldots, \bar{g}_k \rangle)$ iff for D-almost all i, $E_i(\langle f_i(i), \ldots, f_k(i) \rangle, \langle g_i(i), \ldots, g_k(i) \rangle)$. Then (A^*, E^*) is a homogeneous identity and (n, F) is realized in (A^*, E^*) iff there exists $N \in \omega$ such that for $k \geq N$, (n, F) is realized in (A_k, E_k). Let (A, E) be a completion of (A^*, E^*) as defined in Theorem 3.6. This identity meets the requirements.

One might wonder whether strictly increasing or strictly decreasing chains with nontrivial upperbounds or lowerbounds exist. In the next section we will construct examples which show that such chains do exist. We did not succeed in answering the question whether infima and suprema of incomparable identities exist.

5 Examples   In this section we are going to construct for every set J of prime numbers a complete and homogeneous identity (A_J, E_J) such that (A_{J_1}, E_{J_1}) $\leq_{(\infty,\infty)} (A_{J_2}, E_{J_2})$ iff $J_1 \subseteq J_2$. This implies that nontrivial chains and infinite antichains exist and that there are $2^{\aleph_0}$ incomparable theories.

For every prime number $p \in \omega$ let (p, E_p) be the smallest identity containing the following set as an equivalence class:

$$\{\langle 0, 1, \ldots, p - 1 \rangle, \langle 1, 2, \ldots, p - 1, 0 \rangle,$$
$$\langle 2, 3, \ldots, p - 1, 0, 1 \rangle, \ldots, \langle p - 1, 0, 1, \ldots, p - 2 \rangle\}.$$

This implies: for all $\bar{a}, \bar{b}$ such that $l(\bar{a}) = l(\bar{b})$ and $a_i \neq a_j$ and $b_i \neq b_j$ for all $i < j < l(\bar{a})$,

if $l(\bar{a}) = l(\bar{b}) = p$, then
$E_p(\bar{a}, \bar{b})$ iff for some permutation π, $\pi\bar{a}$ and $\pi\bar{b}$ are in the equivalence class above;
if $l(\bar{a}) = l(\bar{b}) = k < p$, then
$E_p(\bar{a}, \bar{b})$ iff for some increasing sequence α and two sequences $\bar{c}$ and $\bar{d}$ of length p, $E_p(\bar{c}, \bar{d})$ and $\bar{a} = \bar{c} \restriction \alpha$ and $\bar{b} = \bar{d} \restriction \alpha$.

Fact 1 (p, E_p) *is homogeneous.*

This immediately follows from the definition.

Fact 2 f *is an automorphism of* (p, E_p) *(i.e., of the structure* $(p, E_p \cap p^k)_{k \in \omega}$ *for* L_{id}*) which leaves the equivalence class* $\{\langle 0, \ldots, p - 1 \rangle, \langle 1, \ldots, p - 1, 0 \rangle, \ldots, \langle p - 1, 0, \ldots, p - 1 \rangle\}$ *fixed iff for all* $k < p$, $f(k) \equiv f(0) + k \pmod{p}$.

This is trivial.

Fact 3   *Suppose* $l(\bar{a}) = l(\bar{b})$; *then the following statements are equivalent*:
(i) $E_p(\bar{a}, \bar{b})$
(ii) *There exists* $k < p$ *such that for all* $i < l(\bar{a})$, $b_i \equiv a_i + k \pmod{p}$.

Proof: It easily follows from the definition that we may assume $l(\bar{a}) = l(\bar{b}) = p$ and for all $i < j < p$: $a_i \neq a_j$ and $b_i \neq b_j$. Then the following statements are equivalent:
 (i) $E_p(\bar{a}, \bar{b})$
 (ii) For some permutation π of p and for some $r, s < p$, $\pi(\langle r, r + 1, \ldots, p - 1, 0, \ldots, r - 1 \rangle) = \bar{a}$ and $\pi(\langle s, s + 1, \ldots, p - 1, 0, \ldots, s - 1 \rangle) = \bar{b}$
 (iii) for all $k < p$, $a_k \equiv \pi(k) + r \pmod{p}$ and $b_k \equiv \pi(k) + s \pmod{p}$
 (iv) for all $k < p$, $b_k \equiv a_k - r + s \pmod{p}$.

Fact 4   *If* $p \neq q$, *then* (p, E_p) *is not realized in* (q, E_q).

Proof: Suppose not. Then for some $p < q$, (p, E_p) is realized in (q, E_q). This implies that there are $n_0, \ldots, n_{p-1}$ in q such that $E_q(\langle n_0, \ldots, n_{p-1} \rangle, \langle n_1, \ldots, n_{p-1}, n_0 \rangle)$. By Fact 3 there exists a $k < q$ such that the following equations hold:

$$n_1 \equiv n_0 + k \pmod{q}, \ n_2 \equiv n_1 + k \pmod{q}, \ldots,$$
$$n_{p-1} \equiv n_{p-2} + k \pmod{q}, \ n_0 \equiv n_{p-1} + k \pmod{q}.$$

Hence $0 \equiv p.k \pmod{q}$. This is impossible.

Before constructing (A_J, E_J) we introduce an operation on identities. Let for $i \in I$, (A_i, E_i) be an identity. Then

$$\sum_{i \in I} (A_i, E_i) = \left(\overset{\bullet}{\underset{i \in I}{\bigcup}} A_i, E \right)$$

where $E(\bar{a}, \bar{b})$ iff

(i) $l(\bar{a}) = l(\bar{b})$

(ii) for all $i \in I$ and all $k < l(\bar{a})$: $a_k \in A_i$ iff $b_k \in A_i$

(iii) for all increasing sequences α, and $i \in I$: if $rng(\bar{a} \restriction \alpha) \subseteq A_i$ then $E_i(\bar{a} \restriction \alpha, \bar{b} \restriction \alpha)$.

We write $(A_1, E_1) + (A_2, E_2)$ instead of $\sum_{i \in \{1,2\}} (A_i, E_i)$.

Remark that $\sum_{i \in I} (A_i, E_i)$ is homogeneous if for all $i \in I$, (A_i, E_i) is homogeneous and if I is finite and for all $i \in I$, (A_i, E_i) is complete then $\sum_{i \in I} (A_i, E_i)$ is complete.

Let us now define (A_J, E_J). For the time being we will assume that J is infinite. At the end of this chapter we handle the case that J is finite.

We first define (B_J, F_J): Let $(B_J, F_J) = \sum_{p \in J} (p, E_p)$.

Fact 5

(i) (p, E_p) is realized in (B_J, F_J) iff $p \in J$

(ii) $(B_J, F_J) \leq_{(\infty, \infty)} (B_K, F_K)$ iff $J \subseteq K$.

These facts follows immediately from the definition of B_J and from Fact 4.

Finally let (A_J, E_J) be the completion of (B_J, F_J) as defined in Theorem 3.6. By this theorem we can conclude:

$$\text{if } J \subseteq K, \text{ then } (A_J, E_J) \leq_{(\infty, \infty)} (A_K, E_K).$$

It results to show the converse. It is clear that it is sufficient to prove the following: if $p \notin J$, then (p, E_p) is not realized in (A_J, E_J). This will follow from the following fact:

Fact 6 *There exists a complete theory T_J such that $Id(B_J, F_J) \subseteq Id(T_J)$ and (p, E_p) is realized in T_J iff $p \in J$. Hence $(A_J, E_J) \leq_{(\infty, \infty)} (A_K, E_K)$ iff $J \subseteq K$.*

Proof: For every prime number q let R_q and S_q be q-ary relation symbols. Let, for every prime number p, $\mathfrak{A}_p$ be the following structure: $\mathfrak{A}_p = \langle p, R_q^{\mathfrak{A}_p}, S_q^{\mathfrak{A}_p}, R_p^{\mathfrak{A}_p} \rangle_{q<p, q\text{prime}}$, where

$$R_p^{\mathfrak{A}_p} = \{ \langle 0, \ldots, p-1 \rangle, \langle 1, \ldots, p-1, 0 \rangle, \ldots, \langle p-1, 0, 1, \ldots, p-1 \rangle \}$$

and for $q < p$, $\bar{a} \in R_q^{\mathfrak{A}_p}$ iff for some strictly increasing q-tuple α in p and for some $\bar{b} \in R_p^{\mathfrak{A}_p}$: $\bar{a} = \bar{b} \restriction \alpha$.

Remark that R_q is definable from R_p in $\mathfrak{A}_p$. S_q will also be definable from R_p in $\mathfrak{A}_p$. Before we define S_q we state two facts about $\mathfrak{A}_p$:

(i) f is an automorphisms of $\mathfrak{A}_p$ iff for all $k < p$, $f(k) \equiv f(0) + k \ (mod \ p)$

(ii) $\bar{a}$ and $\bar{b}$ satisfy the same type in $\mathfrak{A}_p$ iff $E_p(\bar{a}, \bar{b})$.

Fact (i) is trivial and Fact (ii) follows from Fact (i), Fact 2, and Fact 3.

Now we define S_q for every prime number $q < p$: $\mathfrak{A}_p$ is a finite structure, hence there are only finitely many inequivalent formulas with free variables

$v_0, \ldots, v_{q-1}$ in which only $=$ and R_p occur. Fix a maximal sequence $\phi_1(v_0, \ldots, v_{q-1}), \ldots, \phi_N(v_0, \ldots, v_{q-1})$ of inequivalent formulas (in which only $=$ and R_p occur). Now let $\langle n_0, \ldots, n_{q-1} \rangle \in p^q$ such that for all $i < j < q$, $n_i \neq n_j$; and let $X = \{ \langle n_0, \ldots, n_{q-1} \rangle, \langle n_1, \ldots, n_{q-1}, n_0 \rangle, \ldots, \langle n_{q-1}, n_0, \ldots, n_{q-2} \rangle \}$. From Fact 4 and Fact (ii) it follows that the elements of X do not all satisfy the same type. Let k_0 be the smallest $k \in \{1, \ldots, N\}$ such that $\phi_k^{\mathfrak{A}_p} \cap X \neq \emptyset$ and $\neg \phi_k^{\mathfrak{A}_p} \cap X \neq \emptyset$. Then let $\langle n_0, \ldots, n_{q-1} \rangle \in S_q^{\mathfrak{A}_p}$ iff $\langle n_0, \ldots, n_{q-1} \rangle \in \phi_{k_0}^{\mathfrak{A}_p}$. It is easy to see that S_q is definable from R_p in $\mathfrak{A}_p$. Now let T_J be the complete theory of the structure $\mathfrak{A}_J$ defined as follows:

$$\mathfrak{A}_J = \left\langle \overset{\bullet}{\underset{p \in J}{\bigcup}} p, \, R_q^{\mathfrak{A}_J}, \, S_q^{\mathfrak{A}_J}, \, <^{\mathfrak{A}_J} \right\rangle_{q \in \omega, \, q \, \text{prime}}$$

where

$$R_q^{\mathfrak{A}_J} = \overset{\bullet}{\underset{\substack{p \in J \\ p \geq q}}{\bigcup}} R_q^{\mathfrak{A}_p}; \quad S_q^{\mathfrak{A}_J} = \overset{\bullet}{\underset{\substack{p \in J \\ p \geq q}}{\bigcup}} S_q^p$$

and $a <^{\mathfrak{A}_J} b$ iff for the unique p, q such that

$$a \in p \text{ and } b \in q, \, p < q.$$

It is easy to see that $(B_J, F_J) = \left(\overset{\bullet}{\underset{p \in J}{\bigcup}} p, \, E_{\mathfrak{A}_J} \right)$. From this it follows that $Id(B_J, F_J) \subseteq Id(T_J)$. It remains to prove: if (p, E_p) is realized in T_J, then $p \in J$. Indeed, let us define an equivalent relation $\sim$ in every model $\mathfrak{A}$ of T_J: $a \sim b$ iff $\mathfrak{A} \vDash \neg (a < c) \wedge \neg (c < a)$.

If (p, E_p) is realized in T_J, then it must be realized by elements that are equivalent. Hence the following set is consistent with T_J

$$\{ c_i \neq c_j | i < j < p \} \cup \{ \neg c_i < c_j | i, j < p \} \cup$$
$$\{ S_p(c_0, \ldots, c_{p-1}) \rightleftarrows S_p(c_1, \ldots, c_{p-1}, c_0),$$
$$\vdots$$
$$S_p(c_0, \ldots, c_{p-1}) \rightleftarrows S_p(c_{p-1}, c_0, \ldots, c_{p-2}) \}.$$

Because this set is finite, it must be realized in $\mathfrak{A}_J$. Hence $p \in J$. This proves Fact 6.

For finite sets J of prime numbers (A_J, E_J) would be finite. A slight modification in the construction of (B_J, F_J) would give us infinite identities for finite sets J. We give only a sketch of this modification.

Remark that in the proof of Fact 6 a structure $\mathfrak{A}_J$ is defined with universe $\overset{\bullet}{\underset{p \in J}{\bigcup}} p$ such that $(B_J, F_J) = \left(\overset{\bullet}{\underset{p \in J}{\bigcup}} p, \, E_{\mathfrak{A}_J} \right)$. Instead of defining the structure $\mathfrak{A}_J$ as is done above one could define $\mathfrak{A}_J$ as follows:

$$\mathfrak{A}_J = \left\langle \overset{\bullet}{\underset{p \in J}{\bigcup}} p \times \omega, \, R_q^{\mathfrak{A}_J}, \, S_q^{\mathfrak{A}_J}, \, <^{\mathfrak{A}_J} \right\rangle_{q \in \omega, \, q \, \text{prime}}$$

where $R_q^{\mathfrak{A}_J}(\langle a_0, \ldots, a_{q-1} \rangle)$ iff for some $i \in \omega$ and p,

$$\langle a_0, \ldots, a_{q-1} \rangle = \langle \langle n_0, i \rangle, \ldots, \langle n_{q-1}, i \rangle \rangle$$
$$\text{for } \langle n_0, \ldots, n_{q-1} \rangle \in R_q^{\mathfrak{A}_p}.$$

$S_q^{\mathfrak{A}_J}$ is defined in a similar way and $(n, i) <^{\mathfrak{A}_J} (m, j)$ iff for the unique p and q such that $n \in p$ and $m \in q$:

$$p < q \text{ or } p = q \text{ and } i < j.$$

Then let $(B_J, F_J) = \left(\overset{\bullet}{\underset{p \in J}{\bigcup}} p \times \omega, E_{\mathfrak{A}_J} \right)$ and let (A_J, E_J) be the completion of (B_J, F_J). Again we have $(A_J, E_J) \leq_{(\infty, \infty)} (A_K, E_K)$ iff $J \subseteq K$. Now (A_J, E_J) is infinite for all J.

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