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Near Coherence of Filters, I: Cofinal Equivalence of Models of Arithmetic

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Abstract We define cofinal equivalence to be the smallest equivalence relation on models of arithmetic such that every model is equivalent to all of its cofinal submodels. It is easy to classify tall models (those with no last sky) up to cofinal equivalence, but an attempt to do the same for short models leads to questions independent of the axioms of set theory. We introduce the set-theoretic principle (NCF) of near coherence of filters, whose effect is to make all short nonstandard models of arithmetic cofinally equivalent. We give several equivalent formulations and several consequences of NCF.

1 Introduction and preliminaries We are primarily concerned with models of full arithmetic, that is, with elementary extensions of the standard model whose universe is ω and whose relations and functions are all of the (finitary) relations and functions on ω. Many of the results in the early part of the paper remain true for arithmetic formulated in smaller languages, but we leave this extension to the reader.

The classification, up to isomorphism, of models of arithmetic appears hopelessly complicated, so it is reasonable to attempt a classification up to some coarser equivalence relation. We introduce in this paper one such equivalence relation, called cofinal equivalence, for which a reasonable classification may be possible.

The concept of cofinal equivalence is based on ignoring the changes that a model undergoes when new elements are added below elements already present. More precisely, a model is cofinally equivalent to each (isomorph) of its cofinal submodels, and cofinal equivalence is the smallest equivalence relation with this property.

It turns out (see Lemma 3 below) that the cofinal-equivalence classes are of two sorts, those consisting of tall models and those consisting of short models.

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It further turns out (see Theorem 4) that the classes of the former sort are easily described; they are naturally parametrized by the infinite regular cardinal numbers. The classification of short models, up to cofinal equivalence, is a much more delicate matter and indeed is nearly as hopeless as the classification up to isomorphism, provided the continuum hypothesis holds. It is, however, consistent with the usual axioms of set theory to suppose that all short nonstandard models of arithmetic are cofinally equivalent (see Theorems 8 and 9 below; the consistency proof is due to Shelah). Furthermore, this supposition is equivalent to a combinatorial principle, which we call near coherence of filters (NCF), and which turns out to have numerous interesting equivalents and consequences. Some of these are presented in the final section of this paper. Others will be discussed in a sequel.

We shall use the same notation for a relation (or function) on ω, its name in the formal language, and the denotation of this name in a nonstandard model, i.e., the canonical extension of the original relation (or function). These canonical extensions will be called the *standard* relations (and functions) of the nonstandard model. When models are considered as linearly ordered sets, it is to be understood that the ordering is the canonical extension of the usual ordering of ω.

Since our models have built-in Skolem functions, all submodels are elementary submodels. Also, since pairing functions are available, every finitely generated model is generated by a single element. Each element a in a model has a *type*, an ultrafilter $\mathcal{U}$ on ω defined by

$$\mathcal{U} = \{X \subseteq \omega | a \in \text{(the canonical extension of) } X\};$$

the submodel generated by a is isomorphic to the ultrapower $\mathcal{U}$-prod ω by an isomorphism sending an element $[f]$ of the ultrapower (where $f: \omega \to \omega$) to $f(a)$. In particular, elements of the same type generate isomorphic models. If a has type $\mathcal{U}$, then $f(a)$ has type

$$f(\mathcal{U}) = \{X \subseteq \omega | f^{-1}(X) \in \mathcal{U}\}.$$

In particular, $f(\mathcal{U})$ is the type of $[f]$ in $\mathcal{U}$-prod ω, so if two functions f and g agree when restricted to a set in $\mathcal{U}$ then $f(\mathcal{U}) = g(\mathcal{U})$.

We shall need the concept of skies introduced by Puritz [12]. Two elements $a < b$ of a model $\mathfrak{A}$ of arithmetic are said to be in the same sky if $b < f(a)$ for some standard function f or, equivalently, if the submodels generated by a and by b are cofinal in the same initial segment of $\mathfrak{A}$. The skies of $\mathfrak{A}$ are thus order-convex subsets of $\mathfrak{A}$ and constitute a partition of $\mathfrak{A}$. The ordering of $\mathfrak{A}$ induces, thanks to order-convexity, a linear ordering of the skies of $\mathfrak{A}$. An element lies in the last sky if and only if it generates a cofinal submodel of $\mathfrak{A}$. If such an element exists, i.e., if $\mathfrak{A}$ has a last sky, or equivalently if $\mathfrak{A}$ has a finitely generated, cofinal submodel, then $\mathfrak{A}$ is said to be *short*; otherwise $\mathfrak{A}$ is *tall*.

A nonprincipal ultrafilter $\mathcal{U}$ on ω is called a *P-point* if all the nonstandard elements of $\mathcal{U}$-prod ω constitute a single sky; an equivalent condition is that every $f: \omega \to \omega$ become constant or finite-to-one when restricted to a suitable set in $\mathcal{U}$. $\mathcal{U}$ is called a *Q-point* if each element of the top sky of $\mathcal{U}$-prod ω generates this whole model; an equivalent condition is that every finite-to-one $f: \omega \to \omega$ be one-to-one on some set in $\mathcal{U}$. A *selective* ultrafilter is a *P*-point that is also a Q-point; equivalent conditions are that $\mathcal{U}$-prod ω be generated by each of its nonstandard elements (i.e., that it be a minimal nonstandard model) and that every $f: \omega \to \omega$ become constant or one-to-one when restricted to some set in $\mathcal{U}$. For more information about these special sorts of ultrafilters, see [3], [12], and [13].

2 Cofinal equivalence *Cofinal equivalence* is defined to be the smallest equivalence relation, on the class of models of arithmetic, such that, whenever $\mathfrak{A}$ is isomorphic to a cofinal submodel of $\mathfrak{B}$, then $\mathfrak{A}$ is equivalent to $\mathfrak{B}$. In other words, $\mathfrak{A}$ and $\mathfrak{B}$ are cofinally equivalent if and only if there is a finite sequence of models (a "zigzag") $\mathfrak{A} = \mathfrak{A}_0, \mathfrak{A}_1, \dots, \mathfrak{A}_n = \mathfrak{B}$ such that, for each $i < n$, one of $\mathfrak{A}_i$ and $\mathfrak{A}_{i+1}$ can be cofinally embedded in the other. Our first lemma provides a slight simplification of this description of cofinal equivalence, shortening the zigzag to one zig and one zag.

Lemma 1 *If a model $\mathfrak{A}$ of arithmetic can be embedded cofinally into each of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, then there is a model $\mathfrak{C}$ into which both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be cofinally embedded.*

Proof: An easy compactness argument gives a model $\mathfrak{C}'$ into which $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be embedded with the cofinal copies of $\mathfrak{A}$ identified. (In fact, one can arrange that no other identifications occur; see [2], Theorem 1.) Then $\mathfrak{B}_1$, $\mathfrak{B}_2$ and the copies of $\mathfrak{A}$ in them are all cofinal in the same initial segment $\mathfrak{C}$ of $\mathfrak{C}'$. Finally, it is well known that such a $\mathfrak{C}$ is an elementary submodel of $\mathfrak{C}'$; see [5], Proposition 2.2.

Corollary 2 *$\mathfrak{A}$ and $\mathfrak{B}$ are cofinally equivalent if and only if there is a model $\mathfrak{C}$ into which both can be cofinally embedded.*

Proof: "If" is obvious. For "only if" note that the binary relation on models defined by "$\mathfrak{A}$ and $\mathfrak{B}$ can be cofinally embedded in the same model" is transitive, by the lemma, hence is an equivalence relation such that cofinal embeddability implies equivalence, hence includes the relation of cofinal equivalence.

Our objective is to classify the cofinal equivalence types of models. Toward this end, we note the existence of two invariants of cofinal equivalence type: cofinality and shortness.

The (ordinal) cofinality of a model, or indeed of any linearly ordered set, is the minimum cardinality of a cofinal subset. It is well known that all cofinal subsets of a linearly ordered set have the same cofinality, and it follows that cofinally equivalent models have the same cofinality.

Recall that a model is *short* if it has a finitely generated cofinal submodel; otherwise it is *tall*. The following lemma shows that these properties of a model are invariants of cofinal equivalence type.

Lemma 3 *If $\mathfrak{A}$ is cofinally embedded in $\mathfrak{B}$ and if one of $\mathfrak{A}$ and $\mathfrak{B}$ is short then so is the other.*

Proof: Without loss of generality, $\mathfrak{A}$ is a cofinal submodel of $\mathfrak{B}$. If $\mathfrak{A}$ is short, then its finitely generated cofinal submodel is also cofinal in $\mathfrak{B}$, which makes $\mathfrak{B}$ short. Conversely, suppose that $\mathfrak{B}$ has a cofinal submodel generated by finitely many elements, hence by a single element b. Since $\mathfrak{A}$ is cofinal in $\mathfrak{B}$, choose an $a \in \mathfrak{A}$ with $a \geq b$. We shall show that the submodel generated by a is cofinal in $\mathfrak{A}$. Let any $a' \in \mathfrak{A}$ be given. By choice of b, there is a standard function f with $a' \leq f(b)$ in $\mathfrak{B}$. Let g be the standard function defined by

$$g(x) = max\{f(y) | y \leq x\}.$$

Then $a' \leq f(b) \leq g(a)$, and $g(a)$ is in the submodel generated by a.

Thus, in studying cofinal equivalence types, we may treat short and tall models separately. The tall case is easier, so we handle it first.

3 Tall models For tall models, the cofinality is a complete invariant.

Theorem 4
(a) *Any two tall models of arithmetic with the same cofinality are cofinally equivalent.*
(b) *The cofinalities of tall models of arithmetic are all the regular infinite cardinal numbers.*

Proof: Recall that every model is the disjoint union of its skies, that skies are order-convex, and that a tall model has no last sky. It follows that the cofinality of a tall model $\mathfrak{A}$ is the same as the cofinality of its linearly ordered set $Sk(\mathfrak{A})$ of nonstandard skies.

Now let $\mathfrak{A}$ and $\mathfrak{B}$ be two tall models of the same cofinality. Since $Sk(\mathfrak{A})$ and $Sk(\mathfrak{B})$ have the same cofinality, it is easy to linearly order their disjoint union in such a way that each of $Sk(\mathfrak{A})$ and $Sk(\mathfrak{B})$ retains its original ordering and is cofinal in the union. By [2], Theorem 3, we can amalgamate $\mathfrak{A}$ and $\mathfrak{B}$ to a model $\mathfrak{C}$ in such a way that only the standard parts of $\mathfrak{A}$ and $\mathfrak{B}$ are identified in $\mathfrak{C}$ and the nonstandard skies of $\mathfrak{A}$ and $\mathfrak{B}$ are ordered in $\mathfrak{C}$ in the way just described. Then $\mathfrak{A}$ and $\mathfrak{B}$ are both cofinally embedded in the same model and are therefore cofinally equivalent. This proves part (a) of the theorem.

For part (b), given any infinite regular cardinal κ, we can construct a tall model of cofinality κ by the following inductive construction of length κ. Start with any model of arithmetic. At successor stages, form an elementary extension of the previous model such that some element of the extension is greater than all elements of the previous model (easy by compactness). At limit stages, take the union of the chain of previous models. Clearly, the model at stage κ is tall and has cofinality κ. (Such models can also be obtained by Gaifman's techniques [5], which yield a good deal more.)

Theorem 4 completely classifies the cofinal equivalence classes of tall models of arithmetic. We may therefore confine our attention from now on to short models.

4 Short models For short models, there is a description of cofinal equivalence dual to that in Lemma 1 and Corollary 2. (Instead of a zig and zag, we can now have the zag first and then the zig.)

Lemma 5 *If short models $\mathfrak{B}_1$ and $\mathfrak{B}_2$ can be cofinally embedded in the same model $\mathfrak{C}$, then there is a model $\mathfrak{A}$ that can be cofinally embedded into each of $\mathfrak{B}_1$ and $\mathfrak{B}_2$.*

Proof: By definition of "short", we may assume without loss of generality that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are each generated by a single element. The desired conclusion is then given by [1], Theorem 1.

Corollary 6 *Two short models are cofinally equivalent if and only if they have isomorphic cofinal submodels.*

Proof: The relation of having isomorphic cofinal submodels is, by Lemmas 3 and 5, transitive on short models, hence is an equivalence relation for which cofinal embeddability implies equivalence, hence includes the relation of cofinal equivalence. This proves "only if", and "if" is trivial.

Corollary 7 *Two minimal nonstandard models of arithmetic are cofinally equivalent if and only if they are isomorphic.*

Proof: "If" is trivial, and "only if" follows from the previous corollary, since the standard model is never cofinal in a nonstandard one and therefore a minimal nonstandard model has no cofinal submodels except itself.

This corollary shows that minimal (nonstandard) models constitute a serious obstacle to any attempt to classify models up to cofinal equivalence. Any such classification must contain a classification of minimal models up to isomorphism.

The existence of minimal nonstandard models of (full) arithmetic is independent of the usual (ZFC) axioms of set theory. The continuum hypothesis (or Martin's axiom, or any of several weaker hypotheses) implies the existence of $2^{2^{\aleph_0}}$ nonisomorphic minimal models, namely the ultrapowers $\mathfrak{U}$-prod ω for selective ultrafilters $\mathfrak{U}$, and the task of classifying them seems hopeless. On the other hand, Kunen [9] showed that, in some models of ZFC (obtained by adding many random reals to models of the continuum hypothesis), there are no selective ultrafilters, hence no minimal nonstandard models of arithmetic. In such models of ZFC, a classification of cofinal equivalence types of models of arithmetic seems conceivable.

(If, instead of working with full arithmetic, we had used a countable language, then there would always be $2^{\aleph_0}$ nonisomorphic minimal nonstandard models, by [5], Theorem 3.9, and a classification of them does not seem possible.)

Actually, there are other obstacles to the classification in Kunen's model, for in it there are many short models of arithmetic which, although not minimal, have no proper *cofinal* submodels (namely ultrapowers of the standard model with respect to Q-points). Corollary 6 implies that such models are cofinally equivalent only if they are isomorphic, and again the task of classifying them seems hopeless.

Miller [10] has shown that, in certain models of ZFC (obtained by iteratively adding $\aleph_2$ Laver or Mathias reals to a model of the continuum hypothesis), every short nonstandard model of arithmetic has a proper cofinal submodel, so the problem in the last paragraph does not arise.

Rather than outlining more problems that may impede the classification of cofinal equivalence types of short models of full arithmetic, we turn to a consideration of the possibility that (under suitable set-theoretic hypotheses) this classification may be trivial. (Corollary 15 below implies that, even in the models mentioned in the preceding paragraph, the classification is not trivial.)

5 Near coherence of filters

Theorem 8 *The following three statements are equivalent.*
(a) *Every two short nonstandard models of full arithmetic are cofinally equivalent.*
(b) *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are nonprincipal ultrafilters on ω, then there is a finite-to-one $f\colon \omega \to \omega$ such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$.*
(c) *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are filters on ω, each containing all cofinite sets, then there is a finite-to-one $f\colon \omega \to \omega$ such that $f(\mathfrak{F}_1) \cup f(\mathfrak{F}_2)$ has the finite intersection property.*

The equivalence of (a) and (b) is essentially implicit in the well-known connection between cofinal embeddings and finite-to-one maps of ultrafilters, while the equivalence of (b) and (c) is nearly trivial; nevertheless, we give proofs for the sake of completeness.

Proof: (a)$\to$(b). Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be nonprincipal ultrafilters on ω. Then the ultrapowers $\mathcal{U}_i$-prod ω of the standard model are each generated by one element, the equivalence class of the identity function, and are therefore nonstandard short models. By (a), they have isomorphic cofinal submodels, which can be taken to be generated by a single element, by Lemma 3. Let $[f_1]_{\mathcal{U}_1}$, the equivalence class of $f_1 : \omega \to \omega$ in $\mathcal{U}_1$-prod ω, generate one of these submodels, and let $[f_2]_{\mathcal{U}_2}$ be the corresponding (via the isomorphism) generator of the other model. Because of the isomorphism, these two generators have the same type, i.e., $f_1(\mathcal{U}_1) = f_2(\mathcal{U}_2)$. Also, since $[f_i]_{\mathcal{U}_i}$ generates a cofinal submodel of $\mathcal{U}_i$-prod ω, there is a standard function g_i such that $g_i[f_i]_{\mathcal{U}_i} \geq [id]_{\mathcal{U}_i}$, so $g_i(f_i(n)) \geq n$ for all n in some set $A_i \in \mathcal{U}_i$. This implies that f_i is finite-to-one on A_i.

We may assume that $\mathcal{U}_1 \neq \mathcal{U}_2$, as otherwise (b) is trivial. We may then also assume that A_1 and A_2 are disjoint, as we can intersect A_1 and A_2 with disjoint sets from $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. Let $f\colon \omega \to \omega$ agree with f_i on A_i and with the identity function on $\omega - (A_1 \cup A_2)$. Then f is finite-to-one, and

$$f(\mathcal{U}_1) = f_1(\mathcal{U}_1) = f_2(\mathcal{U}_2) = f(\mathcal{U}_2)$$

since f agrees with f_i on a set $A_i \in \mathcal{U}_i$.

(b)$\to$(a). Let $\mathfrak{A}$, $\mathfrak{B}$ be short nonstandard models of full arithmetic. Let $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ be generators of cofinal submodels $\mathfrak{A}'$ and $\mathfrak{B}'$ (as $\mathfrak{A}$ and $\mathfrak{B}$ are short), and let $\mathcal{U}_1$ and $\mathcal{U}_2$ be the types of these generators. By assumption (b), let $f\colon \omega \to \omega$ be a finite-to-one function such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$. This means that $f(a)$ and $f(b)$ have the same type and therefore generate isomorphic submodels $\mathfrak{A}''$ and $\mathfrak{B}''$.

Each element of $\mathfrak{A}'$ is of the form $g(a)$ for some standard g. Since f is finite-to-one on ω, we can define $h\colon \omega \to \omega$ by

$$h(x) = max\{g(y) \mid f(y) = x\}$$

(with some arbitrary convention for $max\ \varnothing$). Then it is true in the standard model, hence also in $\mathfrak{A}'$, that $h(f(y)) \geq g(y)$ for all y. In particular, the arbitrary element $g(a)$ of $\mathfrak{A}'$ is majorized by an element $h(f(a))$ of $\mathfrak{A}''$. So $\mathfrak{A}''$ is cofinal in $\mathfrak{A}'$, hence also in $\mathfrak{A}$. Similarly, $\mathfrak{B}''$ is cofinal in $\mathfrak{B}$. Since $\mathfrak{A}''$ and $\mathfrak{B}''$ are isomorphic, the proof of (a) is complete.

(b)$\rightarrow$(c). If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are filters as in (c), extend them, by Zorn's lemma, to ultrafilters $\mathfrak{U}_1$ and $\mathfrak{U}_2$ respectively. Let f be as in (b), and observe that $f(\mathfrak{F}_1) \cup f(\mathfrak{F}_2)$ is included in the ultrafilter $f(\mathfrak{U}_1) = f(\mathfrak{U}_2)$ and therefore has the finite intersection property.

(c)$\rightarrow$(b). Given ultrafilters $\mathfrak{U}_1$ and $\mathfrak{U}_2$ as in (b), apply (c) to get f such that $f(\mathfrak{U}_1) \cup f(\mathfrak{U}_2)$ has the finite intersection property. But each $f(\mathfrak{U}_i)$, being an ultrafilter, is maximal with respect to the finite intersection property, so we have $f(\mathfrak{U}_1) = f(\mathfrak{U}_2)$.

The remainder of this paper will be devoted to the study of the equivalent statements in Theorem 8. For brevity, we shall refer to these statements, in particular (c), as the principle of *near coherence of filters* (NCF).

The discussion at the end of the preceding section shows that NCF is false in many models of set theory. The study of NCF is not, however, vacuous:

Theorem 9 (Shelah) *NCF is consistent relative to ZFC.*

The proof of this theorem will appear in [14]. Here we mention only that a model in which NCF holds can be obtained from a model of the continuum hypothesis by a countable-support forcing iteration of length $\aleph_2$ in which each stage is the rational perfect set forcing studied by Miller [11].

We close this section by pointing out that some variations on conditions (b) and (c) in Theorem 8 are equivalent to NCF. The first variation is to allow two different functions; thus, (b) would be changed to assert that $f_1(\mathfrak{U}_1) = f_2(\mathfrak{U}_2)$ for some finite-to-one f_1 and f_2, and (c) would be changed analogously. These apparently weaker versions of (b) and (c) are not really weaker, because the proof that (b) implies (a) still works with the new (b).

The second variation goes in the opposite direction by apparently strengthening (b) and (c); it requires that the finite-to-one function f be nondecreasing. Thus, the filters $f(\mathfrak{F}_i)$ can be thought of as being obtained from the $\mathfrak{F}_i$ by collapsing certain intervals $f^{-1}\{n\}$ in ω to single points. An equivalent way of stating this variation of (c) is that, given $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we can partition ω into a sequence of finite intervals such that every set from $\mathfrak{F}_1$ and every set from $\mathfrak{F}_2$ meet at least one common interval.

To show that the second variation is only an apparent strengthening, we actually show somewhat more; the additional information will be useful in the next section. Fix, for the time being, a dominating family $\mathfrak{D}$, i.e., a family of functions $h: \omega \rightarrow \omega$ such that, for every $f: \omega \rightarrow \omega$ there exists $h \in \mathfrak{D}$ such that $h(n) > f(n)$ for all but finitely many n. For example, $\mathfrak{D}$ could consist of all functions $\omega \rightarrow \omega$, but we shall be interested in smaller families later. For each $h \in \mathfrak{D}$, partition ω into intervals

$$I_0 = [0, a_1),\ I_1 = [a_1, a_2), \ldots, I_n = [a_n, a_{n+1}), \ldots$$

in such a way that $h(x)$ is at most one interval beyond x, i.e., if $x < a_n$ then $h(x) < a_{n+1}$. This is easily accomplished by defining the a_n's inductively. Let $h': \omega \rightarrow \omega$ be the function that takes the value n on the interval I_n, for each n.

Let $h^+(x)$ (respectively $h^-(x)$) be $h'(x)/2$ rounded up (respectively down) to the nearest integer. Thus, the dominating family $\mathfrak{D}$ gives rise to a family $\mathfrak{D}^\pm = \{h^+, h^- | h \in \mathfrak{D}\}$ of finite-to-one monotone functions. We shall show that these suffice in statements (b) and (c) of Theorem 8.

Lemma 10 *If $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are nonprincipal ultrafilters and $f(\mathfrak{U}_1) = f(\mathfrak{U}_2)$ for some finite-to-one f: $\omega \to \omega$, then $f(\mathfrak{U}_1) = f(\mathfrak{U}_2)$ for some $f \in \mathfrak{D}^\pm$.*

Proof: Fix a finite-to-one f such that $f(\mathfrak{U}_1) = f(\mathfrak{U}_2)$. Define g: $\omega \to \omega$ by

$$g(n) = max\{p \in \omega | \text{ for some } q \le n, f(p) = f(q)\};$$

this makes sense as f is finite-to-one. Let $h \in \mathfrak{D}$ majorize g from some n_0 on. We shall show that either $h^+(\mathfrak{U}_1) = h^+(\mathfrak{U}_2)$ or $h^-(\mathfrak{U}_1) = h^-(\mathfrak{U}_2)$. Suppose, for a contradiction, that neither of these equations holds. The failure of the first equation means that there are $A_i \in \mathfrak{U}_i$ with $h^+(A_1)$ disjoint from $h^+(A_2)$. The failure of the second equation gives, similarly, $h^-(A_1)$ disjoint from $h^-(A_2)$; we can use the same A_1 and A_2 for both h^+ and h^-, as different A_i's in the same $\mathfrak{U}_i$ could be replaced by their intersection. Now $f(A_1)$ and $f(A_2)$ meet infinitely often, since they are both in the nonprincipal ultrafilter $f(\mathfrak{U}_1) = f(\mathfrak{U}_2)$. Choose $a_i \in A_i$ such that $f(a_1) = f(a_2)$ and $a_i \ge n_0$. Then

$$\begin{aligned} h(a_1) \ge g(a_1) &= max\{p \in \omega | \text{ for some } q \le a_1, f(p) = f(q)\} \\ &\ge max\{p \in \omega | f(p) = f(a_1)\} \ge a_2, \end{aligned}$$

and, symmetrically, $h(a_2) \ge a_1$. By definition of h', $h'(a_1)$ and $h'(a_2)$ differ by at most 1, so either $h^+(a_1) = h^+(a_2)$ or $h^-(a_1) = h^-(a_2)$. In either case, we have a contradiction to the disjointness of $h^\pm(A_1)$ and $h^\pm(A_2)$.

Corollary 11 *NCF is equivalent to statements* (b) *and* (c) *of Theorem 8 with f required to be in $\mathfrak{D}^\pm$, where $\mathfrak{D}$ is any dominating family. In particular, f can be required to be monotone.*

6 The dominating number Following the notation of [16], we let d be the smallest cardinality of a dominating family of functions $\omega \to \omega$. This cardinal, which lies between $\aleph_1$ and $2^{\aleph_0}$ (inclusive) has been extensively studied; see [7]. To avoid possible confusion, we point out that d can be different from the minimum cardinality b of an undominated family, i.e., a family $\mathfrak{B}$ of functions $\omega \to \omega$ such that no single function eventually majorizes every function in $\mathfrak{B}$. In general, b is a regular cardinal and

$$\aleph_1 \le b \le cf(d) \le d \le 2^{\aleph_0}.$$

Martin's axiom implies $b = d = 2^{\aleph_0}$. If one adds many Cohen reals (respectively random reals) to a model of CH, the resulting model satisfies $\aleph_1 = b < d = 2^{\aleph_0}$ (respectively $\aleph_1 = b = d < 2^{\aleph_0}$). For other possibilities, see [6].

The cardinal d is relevant to NCF in several ways.

Theorem 12 *If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are filters on ω, containing all cofinite sets, and generated by fewer than d sets, then there exists a finite-to-one f: $\omega \to \omega$ such that $f(\mathfrak{F}_1) \cup f(\mathfrak{F}_2)$ has the finite intersection property.*

Proof: We may assume that the given small generating families $\mathcal{B}_i$ for $\mathcal{F}_i$ are closed under finite intersections, since closing them won't increase their cardinalities. Thus, every set in $\mathcal{F}_i$ includes a set in $\mathcal{B}_i$.

For each infinite $B \leq \omega$ and each $n \in \omega$, define *next* (B, n) to be the smallest element of B that is $\geq n$. Consider the functions $g = g_{B_1, B_2}$ defined by

$$g(n) = max\{next(B_1, n), next(B_2, n)\}$$

for $B_i \in \mathcal{B}_i$. Since each $\mathcal{B}_i$ has fewer than d members, there are fewer than d of these functions g. By definition of d, there is $h: \omega \to \omega$ not eventually majorized by any of these g's. Thus, for each $B_i \in \mathcal{B}_i$, there are infinitely many n such that

$$h(n) \geq max\{next(B_1, n), next(B_2, n)\},$$

which means that, for infinitely many n, the interval $[n, h(n)]$ meets both B_1 and B_2.

As in the discussion preceding Lemma 10, use h to define intervals I_k and functions h', h^+, and h^-. We shall show that either $h^+(\mathcal{F}_1) \cup h^+(\mathcal{F}_2)$ or $h^-(\mathcal{F}_1) \cup h^-(\mathcal{F}_2)$ has the finite intersection property. If not, then, as in the proof of Lemma 10, we could find $A_i \in \mathcal{F}_i$ such that $h^+(A_1) \cap h^+(A_2) = h^-(A_1) \cap h^-(A_2) = \varnothing$, and by shrinking the A's we could arrange that $A_i \in \mathcal{B}_i$. The disjointness properties of the $h^\pm(A_i)$ mean that the union of two consecutive intervals $I_k \cup I_{k+1}$ can never meet both A_1 and A_2. But we saw above that there are intervals of the form $[n, h(n)]$ meeting both A_1 and A_2, and the definition of the I_k's implies that $[n, h(n)]$ is included in $I_k \cup I_{k+1}$ for some k. This contradiction completes the proof of the theorem.

Corollary 13 *Any two nonprincipal ultrafilters on ω that are generated by fewer than d sets have a common finite-to-one image.*

Proof: See the proof of (c)$\to$(b) in Theorem 8.

It follows from results of Ketonen [8] that ultrafilters generated by fewer than d sets are P-points but are not selective. The consistency of the existence of such ultrafilters was recently proved by Shelah, but it also follows, as we shall see, from his Theorem 9 above. (His proof of Theorem 9 also shows directly that some ultrafilters in the model have fewer than d generators.) This contrasts with Solomon's result [15] that no ultrafilter can be generated by fewer than b sets.

Theorem 14 NCF *is equivalent to the statement that for every nonprincipal ultrafilter $\mathcal{U}$ on ω, there is a finite-to-one f: $\omega \to \omega$ such that $f(\mathcal{U})$ is generated by fewer than d sets.*

Proof: If the statement holds, then, given any two nonprincipal ultrafilters on ω, we can find finite-to-one images generated by fewer than d sets. These, in turn, have a common finite-to-one image by Corollary 13. Thus, we have (b) of Theorem 8, in the first of the variations that were discussed after Theorem 9 and shown to be equivalent to NCF.

Conversely, suppose $\mathcal{U}$ were an ultrafilter with no finite-to-one image generated by fewer than d sets. We shall construct another nonprincipal ultrafilter $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ have no common finite-to-one image; this will disprove

NCF. Fix a dominating family $\mathfrak{D}$ of cardinality d. By Lemma 10, it suffices to construct $\mathcal{V}$ so that $f(\mathcal{U}) \neq f(\mathcal{V})$ for all $f \in \mathfrak{D}^{\pm}$. Since $\mathfrak{D}^{\pm}$ has cardinality d, let it be enumerated as $\{f_\alpha | \alpha < d\}$. We construct $\mathcal{V}$ by an induction of length d, ensuring at the α^{th} stage that $f_\alpha(\mathcal{U}) \neq f_\alpha(\mathcal{V})$. The induction will define a sequence of filters $\mathcal{V}_\alpha (\alpha \leq d)$ such that

(i) $\mathcal{V}_0 = \{\text{cofinite sets}\}$.
(ii) If $\alpha < \beta$ then $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$.
(iii) $\mathcal{V}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{V}_\alpha$ for limit λ.
(iv) $\mathcal{V}_{\alpha+1}$ is generated by $\mathcal{V}_\alpha$ plus one set A_α.
(v) $f_\alpha(A_\alpha) \notin f_\alpha(\mathcal{U})$.

Once this is done, any ultrafilter extending $\mathcal{V}_d$ clearly serves as the desired $\mathcal{V}$. To construct the $\mathcal{V}_\alpha$'s, it suffices to show how to choose A_α when $\mathcal{V}_\alpha$ is given. We must show that there exists a set A_α such that $\mathcal{V}_\alpha \cup \{A_\alpha\}$ generates a filter, i.e., A_α meets every set in $\mathcal{V}_\alpha$, but $f_\alpha(A_\alpha)$ is not in the ultrafilter $f_\alpha(\mathcal{U})$. To do this, it suffices to find $B \in f_\alpha(\mathcal{U})$ such that $f_\alpha^{-1}(B) \notin \mathcal{V}_\alpha$, for we can then set $A_\alpha = \omega - f_\alpha^{-1}(B)$. Indeed, since $\mathcal{V}_\alpha$ is a filter not containing $f_\alpha^{-1}(B)$, it contains no subsets of $f_\alpha^{-1}(B)$, i.e., no sets disjoint from A_α. Furthermore, $f_\alpha(A_\alpha)$ is disjoint from B, hence not in $f_\alpha(\mathcal{U})$.

To complete the proof, we suppose that no B of the desired sort exists, and we derive a contradiction. The supposition means that each $B \in f_\alpha(\mathcal{U})$ also belongs to $f_\alpha(\mathcal{V}_\alpha)$, so, as $f_\alpha(\mathcal{U})$ is an ultrafilter, we must have $f_\alpha(\mathcal{U}) = f_\alpha(\mathcal{V}_\alpha)$. Inductive hypotheses (i), (iii), and (iv), for ordinals $\leq \alpha$, imply that $\mathcal{V}_\alpha$ and, therefore, $f_\alpha(\mathcal{V}_\alpha)$ are generated by fewer than d sets, since $\alpha < d$. This contradicts the assumption that no finite-to-one image of $\mathcal{U}$ (such as $f_\alpha(\mathcal{U}) = f_\alpha(\mathcal{V}_\alpha)$) is generated by fewer than d sets.

van Douwen has informed the author that J. van Mill had shown that NCF implies the existence of ultrafilters generated by fewer than $2^{\aleph_0}$ sets. Presumably, van Mill's proof was similar to the preceding.

Corollary 15 NCF *implies each of the following statements.*
(a) *There are nonprincipal ultrafilters on ω generated by fewer than d sets.*
(b) $b < d$
(c) *For every nonprincipal ultrafilter on ω, there is a finite-to-one f such that $f(\mathcal{U})$ is a P-point.*
(d) *There are no Q-points.*

Proof: (a) follows immediately from Theorem 14, and so does (c) in view of Ketonen's result [8] that every ultrafilter generated by fewer than d sets is a P-point. Ketonen also showed that such an ultrafilter is not selective; from this, we easily obtain (d) as follows: If $\mathcal{U}$ were a Q-point, then the f given by Theorem 14 would, by definition of Q-point, be one-to-one on a set in $\mathcal{U}$. So $f(\mathcal{U})$ would be isomorphic to $\mathcal{U}$, hence would be a Q-point. But the first of the cited results of Ketonen requires $f(\mathcal{U})$ to be a P-point also. Being both a P-point and a Q-point, $f(\mathcal{U})$ would be selective, contradicting the second Ketonen theorem.

Finally, (b) follows from (a) by virtue of Solomon's result [15] that no ultrafilter can be generated by fewer than b sets.

It is immediate from the definitions of b and d that the cofinality of the ultrapower $\mathcal{U}$-prod ω is in the interval from b to d (inclusive) for all nonprincipal ultrafilters on ω. By results of Canjar [4], it is consistent relative to ZFC that $b < d$ (e.g., $b = \aleph_1$ and $d = 2^{\aleph_0}$ very large) and every regular cardinal in that interval is the cofinality of some such ultrapower. If NCF holds, then the interval from b to d contains several regular cardinals, by Corollary 15(b), yet only one of these actually occurs as the cofinality of an ultrapower as above since, by the formulation of NCF given in Theorem 8(a), all these ultrapowers are cofinally equivalent. The following result, which was noticed independently by Peter Nyikos (private communication), specifies which of the cardinals in the interval is the actual cofinality. It is the second connection between NCF and d.

Theorem 16 NCF *implies that all short nonstandard models of arithmetic have cofinality d.*

Proof: Since NCF asserts that all short nonstandard models of arithmetic are cofinally equivalent, it suffices to prove that d is the cofinality of one of them, say $\mathcal{U}$-prod ω where $\mathcal{U}$ is a nonprincipal ultrafilter on ω generated by a family $\mathcal{B}$ of fewer than d sets. As mentioned above, the cofinality is at most d, since, if $\mathcal{D}$ is any dominating family, then $\{[f] \mid f \in \mathcal{D}\}$ is cofinal in $\mathcal{U}$-prod ω. Suppose that $\mathcal{U}$-prod ω had a cofinal subset $\{[f] \mid f \in \mathcal{C}\}$ where $\mathcal{C}$ has cardinality smaller than d. We shall complete the proof by deriving a contradiction.

We may, as before, assume that the generating family $\mathcal{B}$ of $\mathcal{U}$ is closed under finite intersections, as closing it will not increase its cardinality. So each set in $\mathcal{U}$ has a subset in $\mathcal{B}$. For each $B \in \mathcal{B}$ and each $f \in \mathcal{C}$, let $g = g_{B,f}$ be defined by

$$g(n) = f(next(B,n)).$$

As both $\mathcal{B}$ and $\mathcal{C}$ have cardinality smaller than d, there are fewer than d of these functions g. Yet we shall show that every $h: \omega \to \omega$ is dominated by one of these g's, thereby contradicting the definition of d.

Let $h: \omega \to \omega$ be given, and assume, by increasing h if necessary, that h is nondecreasing. As $\mathcal{C}$ is cofinal in $\mathcal{U}$-prod ω, find $f \in \mathcal{C}$ such that $[h] \leq [f]$ in the ultrapower. Thus, the set $\{n \in \omega \mid h(n) \leq f(n)\}$ is in $\mathcal{U}$, hence has a subset $B \in \mathcal{B}$. Let $g = g_{B,f}$. Then, for all n, we have, by definition of g and B and by monotonicity of h,

$$g(n) = f(next(B,n)) \geq h(next(B,n)) \geq h(n),$$

so g dominates h.

Corollary 17 *NCF implies that d is a regular cardinal.*

We close this paper with one more connection between NCF and the dominating number d.

Lemma 18 *Let $\{\mathcal{U}_i \mid i \in I\}$ be a family of fewer than d nonprincipal ultrafilters on ω, each generated by fewer than d sets. There is a finite-to-one $f: \omega \to \omega$ such that all the ultrafilters $f(\mathcal{U}_i)$ are equal.*

Proof: The proof is a minor variation of the proof of Theorem 12. For each $i \in I$, let $\mathcal{B}_i$ be a generating family for $\mathcal{U}_i$, closed under finite intersections, and having fewer than d members. For each $i, j \in I$, each $B_i \in \mathcal{B}_i$ and each $B_j \in \mathcal{B}_j$, define $g = g_{B_i,B_j}$ by

$$g(n) = max\{next(B_i, n), next(B_j, n)\}.$$

Since there are fewer than d of these g's, let $h: \omega \to \omega$ not be eventually dominated by any of them. Let h^+ and h^- be obtained from h as in the discussion preceding Lemma 10. The proof of Theorem 12 shows that, for each $i, j \in I$, either $h^+(\mathcal{U}_i) = h^+(\mathcal{U}_j)$, in which case we write $i \sim^+ j$, or $h^-(\mathcal{U}_i) = h^-(\mathcal{U}_j)$, in which case we write $i \sim^- j$. Thus, $\sim^+$ and $\sim^-$ are two equivalence relations on I and their union is all of $I \times I$. It easily follows that one of the two is all of $I \times I$. (If $i \not\sim^+ j$, then $i \sim^- j$, and we claim that the $\sim^-$ class of i and j contains all of I. Otherwise, if k is outside this class, then $i \not\sim^- k$ and $k \not\sim^- j$, so $i \sim^+ k \sim^+ j$, contrary to $i \not\sim^+ j$.) Thus, either $h^+(\mathcal{U}_i) = h^+(\mathcal{U}_j)$ for all $i, j \in I$ or $h^-(\mathcal{U}_i) = h^-(\mathcal{U}_j)$ for all $i, j \in I$.

Theorem 19 NCF *is equivalent to each of the following:*
(a) *If $\{\mathcal{U}_i | i \in I\}$ is a family of fewer than d nonprincipal ultrafilters on ω, then there are finite-to-one functions $f_i: \omega \to \omega$ $(i \in I)$ such that all the ultrafilters $f_i(\mathcal{U}_i)$ are equal.*
(b) *If $\{\mathcal{F}_i | i \in I\}$ is a family of fewer than d filters on ω, each containing all cofinite sets, then there exist finite-to-one functions $f_i: \omega \to \omega$ $(i \in I)$ such that the union of all the filters $f_i(\mathcal{F}_i)$ $(i \in I)$ generates a proper filter.*

Proof: If NCF holds and $\{\mathcal{U}_i | i \in I\}$ is a family as in (a), then apply Theorem 14 to get finite-to-one images $g_i(\mathcal{U}_i)$, each generated by fewer than d sets, and then apply Lemma 18 to get a single finite-to-one h such that all the ultrafilters $h(g_i(\mathcal{U}_i))$ are equal. Then (a) is verified with $f_i = h \circ g_i$.

The implication (a)$\to$(b) is proved exactly like Theorem 8 (b)$\to$(c), and the implication (b)$\to$NCF is clear by Theorem 8.

Corollary 20 *NCF implies that any fewer than d short nonstandard models of arithmetic have cofinal submodels that are all isomorphic.*

REFERENCES

[1] Blass, A., "The intersection of nonstandard models of arithmetic," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 103–106.

[2] Blass, A., "Amalgamation of nonstandard models of arithmetic," *The Journal of Symbolic Logic*, vol. 42 (1977), pp. 372–386.

[3] Blass, A., "A model-theoretic view of some special ultrafilters," pp. 79–90 in *Logic Colloquium '77*, eds. A. Macintyre, L. Pacholski, and J. Paris, North-Holland, 1978.

[4] Canjar, R. M., "Model-theoretic properties of countable ultraproducts without the continuum hypothesis," thesis, University of Michigan, 1982.

[5] Gaifman, H., "Models and types for Peano's arithmetic," *Annals of Mathematical Logic*, vol. 9 (1976), pp. 223–306.

[6] Hechler, S., "On the existence of certain cofinal subsets of ${}^{\omega}\omega$," pp. 155–173 in *Axiomatic Set Theory*, Proceedings of Symposia in Pure Mathematics XIII, Part 2, ed. T. Jech, American Mathematical Society, 1974.

[7] Hechler, S., "A ubiquitous cardinal," *Proceedings of the American Mathematical Society*, vol. 52 (1975), pp. 348–352.

[8] Ketonen, J., "On the existence of P-points in the Stone-Čech compactification of the integers," *Fundamenta Mathematicae*, vol. 92 (1976), pp. 91–94.

[9] Kunen, K., "Some points in βN," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 80 (1976), pp. 385–398.

[10] Miller, A., "There are no Q-points in Laver's model for the Borel conjecture," *Proceedings of the American Mathematical Society*, vol. 78 (1980), pp. 103–106.

[11] Miller, A., "Rational perfect set forcing," pp. 143–159 in *Axiomatic Set Theory*, Contemporary Mathematics 31, eds. J. Baumgartner, D. A. Martin, and S. Shelah, *American Mathematical Society*, Providence, Rhode Island, 1984.

[12] Puritz, C., "Ultrafilters and standard functions in non-standard arithmetic," *Proceedings of the London Mathematical Society* (3), vol. 22 (1971), pp. 705–733.

[13] Puritz, C., "Skies, constellations, and monads," pp. 215–243 in *Contributions to Non-Standard Analysis*, eds. W. A. J. Luxemburg and A. Robinson, North-Holland, 1973.

[14] Shelah, S., "There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$ points and the Rubin-Keisler ordering may be downward directed," to appear in *Annals of Pure and Applied Logic.*

[15] Solomon, R. C., "Families of sets and functions," *Czechoslovak Mathematical Journal*, vol. 27 (1977), pp. 556–559.

[16] van Douwen, E., "The integers and topology," pp. 111–167 in *Handbook of Set-Theoretic Topology*, eds. K. Kunen and J. E. Vaughan, North-Holland, 1984.

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