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On the Characterizability of the Frames for the "Unpreventability of the Present and the Past"

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Introduction In several assertions involving tense and modal notions, "it is possible" is substantially understood as "it is possible, given the present and the past", that is, "it is possible under the hypothesis that exactly one present and exactly one past exist, coinciding with the actual present and past". This can be observed, for instance, when a possible evolution of a physical situation is referred to, or, more generally, when the assertion involves concepts having a history, so that different pasts generally correspond to different concepts.[1] A trivial consequence of this point of view is that the modal operators become vacuous when their scopes do not contain any reference to the future, or, which is the same, that every assertion concerning only the present and the past is necessarily true or necessarily false.

The above conception of the modal notions is usually referred to as the *unpreventability of the present and the past* and is often connected with the (Actualist) Indeterminist point of view. In [1], p. 575, a definition of satisfaction for propositional formulas with tense and modal operators is given, which agrees with this point of view: time is represented by a tree (so that every moment has exactly one past and several possible futures) and, in order to evaluate a formula at a moment x, we must specify what branch, among those starting with x, represents the 'actual' future of x. Since every branch determines its initial point, the valuation of a formula turns out to be relative to a set of branches and "it is possible α" is assumed to hold at a branch B whenever α holds at some branch having the same initial point as B.

In general, a semantics à la Kripke for a propositional tense and modal logic consists of a set of linearly ordered sets representing different evolutions of time, together with an accessibility relation between moments, which correspond to the modal operator. For suitable choices of the linearly ordered sets and of the accessibility relation we obtain a semantics equivalent to that above.

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There are, however, several nonequivalent possible choices which render the semantics sound for the unpreventability of the present and the past; of course, every one of them corresponds to particular assumptions on the mutual connections between time and modality.

These assumptions are expressed in the metalanguage, but it is natural to ask whether they can adequately be expressed in the object language. It is well known, for instance, that several usual assumptions on the structure of the time (like, e.g., linearity, discreteness or denseness, endlessness, etc.) correspond to particular formulas of a propositional language (with tense operators); cf. [1], p. 570. This problem will be treated in the present paper, mainly in connection with the semantics considered in [3], Section 4, and the corresponding propositional language. In general, negative answers will be obtained, even if infinite sets of formulas are allowed in order to express the properties of the semantical structures.

1 Preliminaries The language $\mathcal{L}_{MT}$ can be obtained from a propositional language by adding the unary operators $\Diamond$, F, and P; these symbols are to be read as: "it is possible", "at some future time", and "at some past time", respectively. The usual definitions for the dual notions are assumed: $\Box = \neg\Diamond\neg$ (necessarily), $G = \neg F \neg$ (at every future time), $H = \neg P \neg$ (at every past time). A (Kripke) *frame* for $\mathcal{L}_{MT}$ is a 4-tuple $\langle T, <, >, R\rangle$ in which T is a set and $<$, $>$, and R are binary relations on T. A *valuation* on a frame $\mathfrak{F}$ is a function from the set $\mathcal{V} = \{p_0, p_1, \ldots\}$ of the propositional variables to $\mathcal{P}(T)$, the power set of T. The set of valuations on a frame $\mathfrak{F}$ will be denoted by $Val_{\mathfrak{F}}$ and we shall write $\bar{\mathcal{V}}$ to mean $\{\neg p\colon p \in \mathcal{V}\}$.

Every valuation V can be extended to the set of all the formulas of $\mathcal{L}_{MT}$ by means of the following rules:

$$
\begin{aligned}
&V(\neg\phi) = T \backslash V(\phi)\\
&V(\phi \wedge \psi) = V(\phi) \cap V(\psi)\\
&V(F\phi) = \{x \in T\colon \exists y(y > x \text{ and } y \in V(\phi))\}\\
&V(P\phi) = \{x \in T\colon \exists y(y < x \text{ and } y \in V(\phi))\}\\
&V(\Diamond\phi) = \{x \in T\colon \exists y(xRy \text{ and } y \in V(\phi))\}.
\end{aligned}
$$

For $V \in Val_{\mathfrak{F}}$, we write $\langle \mathfrak{F}, V\rangle \models_x \phi$ (ϕ holds at x in $\langle \mathfrak{F}, V\rangle$) to mean $x \in V(\phi)$. $\langle \mathfrak{F}, V\rangle \models \phi$ and $\mathfrak{F} \models \phi$ are abbreviations for $V(\phi) = T$ and $V(\phi) = T$ for every valuation V on $\mathfrak{F}$. If Σ is a set of formulas and $\mathfrak{F} \models \phi$ holds for every $\phi \in \Sigma$, then we write $\mathfrak{F} \models \Sigma$. This is the most general definition of frame and valuation for $\mathcal{L}_{MT}$; in order to make the definitions sound for the unpreventability of the present and the past, other assumptions are obviously needed.

The intended meaning of the tense operators leads me to assume that $<$ and $>$ are linear orders and that, for all $x, y \in T$, $x < y$ iff $y > x$. Intuitively, the elements of T are moments in some evolution through the time of the universe which are represented by maximal chains in $\langle T, <\rangle$ (or $\langle T, >\rangle$); the maximal chains in T will be referred to as *time-structures*. The linearity of $<$ (and $>$) implies that the set of the time-structures constitutes a partition of T. As for the accessibility relation R, we want it to fulfill the requirement: xRy only if x and y have the same present and past; therefore, first of all, R has to be an equivalence relation. Furthermore, if "x and y have the same present and past" is understood as "whatever happens in x also happens in y and whatever happened before x also happened before y", then we have to assume that every valuation is closed under R:

(1.1) xRy and $x \in V(p)$ implies $y \in V(p)$

and that

(1.2) xRy and $z < x$ implies zRu for some $u < y$.

From now on, by valuation on a frame we shall always mean a valuation fulfilling (1.1) and T will be referred to as $\cup T_i (i \in I)$, where I is a suitable set of indexes and every T_i is a time-structure. Following [3] we say that $\mathfrak{F}$ is a *neutral* frame (briefly, an N-frame) if the above assumptions on $<$, $>$, and R hold. An easy induction on the complexity of ϕ proves that, for every neutral frame $\mathfrak{F}$

(1.3) $\mathfrak{F} \models \Diamond\phi \equiv \Box\phi$, whenever F does not occur in ϕ

which can be considered the formal version of the unpreventability of the present and the past.

The axiom schemes Axiom 0 to Axiom 12 (together with rules R1–4) below axiomatize the concept of validity with respect to N-frames; that is, $\mathfrak{F} \models \phi$ holds for every N-frame $\mathfrak{F}$ iff ϕ is provable by means of Axiom 0 to Axiom 12 and R1–4; cf. [3], p. 150.

Axiom 0	*Propositional calculus*
Axioms 1–3	$\Box\phi \rightarrow \phi$, $\Diamond\phi \rightarrow \Box\Diamond\phi$, $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
Axioms 4, 5	$H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)$, $G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$
Axioms 6–8	$\phi \rightarrow HF\phi$, $\phi \rightarrow GP\phi$, $P\phi \rightarrow GP\phi$
Axioms 9, 10	$P\phi \rightarrow H(F\phi \vee \phi \vee P\phi)$, $F\phi \rightarrow G(F\phi \vee \phi \vee P\phi)$
Axioms 11, 12	$P\Box\phi \rightarrow \Box P\Box\phi$, $p \rightarrow \Box p$ *whenever* $p \in \mathcal{V}$

R1–4 $\quad \dfrac{\phi,\ \phi \rightarrow \psi}{\psi}, \dfrac{\phi}{\Box\phi}, \dfrac{\phi}{G\phi}, \dfrac{\phi}{H\phi}.$

2 *Characterizability*

Definition 2.1 Let F and F' be two classes of frames and let $F' \subseteq F$. We say that a set Σ of formulas of $\mathfrak{L}_{MT}$ characterizes F' in F whenever, for every $\mathfrak{F} \in F$, $\mathfrak{F} \in F'$ iff $\mathfrak{F} \models \Sigma$.

The set of all the instances of Axiom 0 to Axiom 12 is easily proved to characterize the class of N-frames in the class of irreflexive frames. Note that the irreflexivity must be required metalinguistically, since, by Theorem 2.1 below, this property cannot be expressed in $\mathfrak{L}_{MT}$.

Theorem 2.1 *The set of the irreflexive frames for $\mathfrak{L}_{MT}$ is not characterized by any set of formulas of $\mathfrak{L}_{MT}$.*

Proof: Let us consider the usual language $\mathfrak{L}_T$ for tense logic and the corresponding class F_T of frames. $\mathfrak{L}_T$ can be obtained from $\mathfrak{L}_{MT}$ by depriving it of the modal operator $\Diamond$, and the elements of F_T are triples $\langle T, <, > \rangle$ with the obvious properties. We will prove that the negation of the theorem contradicts the well-known result that no set of formulas in $\mathcal{L}_T$ characterizes the set of the irreflexive frames in F_T (cf. [2], pp. 56 and 66).

Assume that a set Σ of formulas of $\mathcal{L}_{MT}$ exists which characterizes the class of the irreflexive frames for $\mathcal{L}_{MT}$. For every $\mathfrak{F} \in F_T$ we let $\mathfrak{F}'$ be the (neutral) frame in which T, $<$, and $>$ are the same as in $\mathfrak{F}$ and R is the identity relation; of course, for every formula ϕ of $\mathcal{L}_{MT}$, if ϕ^* is the formula (of $\mathcal{L}_T$) obtained by erasing all the occurrences of $\Diamond$ in ϕ, then $\mathfrak{F}' \models \phi \equiv \phi^*$ and $\mathfrak{F}' \models \phi$ iff $\mathfrak{F} \models \phi^*$.

Now the contradiction is a consequence of the following equivalences (which hold for every $\mathfrak{F} \in F_T$): $\mathfrak{F}$ is irreflexive $\Leftrightarrow \mathfrak{F}'$ is irreflexive $\Leftrightarrow \mathfrak{F}' \models \Sigma$ $\Leftrightarrow \mathfrak{F} \models \{\phi^*: \phi \in \Sigma\}$.

In this section two classes, $F1$ and $F2$, of neutral frames are considered which correspond to plausible assumptions on the connections between different time-structures and it is proved that neither $F1$ nor $F2$ is characterizable in the class of the irreflexive frames. In the next section the same result is proved for a (proper) subclass $F3$ of $F2$, whereas it is shown that $F3$ is characterizable by means of a formula containing propositional quantifiers; this makes plausible the conjecture that $F2$ is also characterizable if these operators are allowed. The class of Kamp frames and that of the $W \times T$ frames considered in [3] can trivially be proved to coincide with $F3$ and $F1 \cap F3$, respectively; the following proof of the noncharacterizability of $F1$ can be used to prove that the class of $W \times T$ frames is not characterizable either.

In the sequel we shall often write $p \in V(x)$ instead of $x \in V(p)$, for some valuation V; it is obvious that this causes no trouble since every function from $\mathcal{V}$ into $\mathcal{P}(T)$ determines in a natural way a function from T into $\mathcal{P}(\mathcal{V})$.

The requirement A1 below substantially asserts that any two time-structures (in an N-frame) have the same inner structure or, in other words, that time is essentially unique. Let $F1$ be the class of the N-frames in which A1 holds.

A1 $\forall i, j \in I$, an order isomorphism exists between T_i and T_j.

Assume that a set Σ characterizing $F1$ in the sense above exists. Let $\mathfrak{F}$ be the N-frame in which: (1) $I = \{0, 1\}$; (2) $T_0 = \omega \times \{0\}$ and $T_1 = (\omega + \omega) \times \{1\}$ (where ω is the smallest infinite ordinal); (3) $x < y$ iff they are in the same T_i and the first element of x is smaller than that of y in the usual order relation between ordinals; (4) R relates only $\langle 0, 0\rangle$ and $\langle 0, 1\rangle$ (and every moment with itself). Then $\mathfrak{F} \notin F1$ and hence there exist $\sigma \in \Sigma$, $V \in Val_{\mathfrak{F}}$, $u \in T_0 \cup T_1$, such that $\langle \mathfrak{F}, V\rangle \models_u \neg\sigma$. Now we let $\mathfrak{F}'$ be the N-frame just like $\mathfrak{F}$ except that T_0 is replaced in it by $T_0' = (\omega + \omega) \times \{0\}$; thus $\mathfrak{F}' \in F1$. We shall prove that, for every formula α and every $x \in T_0 \cup T_1$, if $\langle \mathfrak{F}, V\rangle \models_x \alpha$, then a valuation $V' \in Val_{\mathfrak{F}'}$ exists such that $\langle \mathfrak{F}', V'\rangle \models_x \alpha$; this contradicts the existence of Σ.

Let $\langle \mathfrak{F}, V\rangle \models_x \alpha$; by the equivalences $\neg P\phi \equiv H\neg\phi$, $\neg(\phi \wedge \psi) \equiv (\neg\phi \vee \neg\psi)$, etc., we can assume that in α the scopes of $\neg$ are only propositional variables.

We can consider now a sequence $Y^0_{\alpha,x}$, $Y^1_{\alpha,x}, \ldots$ of sets, fulfilling the following conditions (Y_1) to (Y_{10}); note that in (Y_3)–(Y_6) a choice is understood.

(Y_1) $\quad Y^0_{\alpha,x} = \{\langle \alpha, x\rangle\}$,

(Y_2) $\quad$ if $\langle \phi \wedge \psi, y\rangle \in Y^n_{\alpha,x}$ then $\langle \phi, y\rangle \in Y^{n+1}_{\alpha,x}$ and $\langle \psi, y\rangle \in Y^{n+1}_{\alpha,x}$,

(Y_3) if $\langle\phi_1 \vee \phi_2, y\rangle \in Y^n_{\alpha,x}$ and $\langle\mathfrak{F}, V\rangle \models_y \phi_i$ $(i \in \{1, 2\})$ then $\langle\phi_i, y\rangle \in Y^{n+1}_{\alpha,x}$

$(Y_{4\text{-}6})$ if $\langle P\phi, y\rangle$ $[\langle F\phi, y\rangle, \langle\Diamond\phi, y\rangle] \in Y^n_{\alpha,x}$ and $\langle\mathfrak{F}, V\rangle \models_z \phi$ $(z < y)$ $[(z > y), (zRy)]$ then $\langle\phi, z\rangle \in Y^{n+1}_{\alpha,x}$

$(Y_{7,8})$ if $\langle H\phi, y\rangle$ $[\langle\Box\phi, y\rangle] \in Y^n_{\alpha,x}$ then, $\forall z < y[\forall z$ such that $zRy]$, $\langle\phi, z\rangle \in Y^{n+1}_{\alpha,x}$

$(Y_{9,9'})$ if $\langle G\phi, y\rangle \in Y^n_{\alpha,x}$ and $y \in T_0$ $[y \in T_1]$ then $\langle G\phi, y\rangle \in Y^{n+1}_{\alpha,x}$ $[\forall z > y$, $\langle\phi, z\rangle \in Y^{n+1}_{\alpha,x}]$

(Y_{10}) if $\langle\phi, y\rangle \in Y^n_{\alpha,x}$ and $\phi \in \mathcal{V} \cup \bar{\mathcal{V}}$ then $\langle\phi, y\rangle \in Y^{n+1}_{\alpha,x}$

It is obvious that, for n^* sufficiently large, $Y^{n^*}_{\alpha,x} = Y^{n^*+1}_{\alpha,x} = Y^{n^*+2}_{\alpha,x} = \ldots$; furthermore, for every natural number n the set $\{\langle\phi, y\rangle \in Y^n_{\alpha,x}: y \in T_0\}$ has finitely many elements (cf. (Y_9)). Let $X_{\alpha,x}$ be $\{\langle\phi, y\rangle \in Y^{n^*}_{\alpha,x}: y \in T_0\}$ $(=\{\langle\phi_0, x_0\rangle, \ldots, \langle\phi_m, x_m\rangle\})$; the properties (a) to (d) below (of $X_{\alpha,x}$) are trivial consequences of (Y_1)–(Y_{10}).

(a) $\{x_0, \ldots, x_m\} \subset T_0$

(b) $\forall i \leq m$, ϕ_i is a subformula of α and it belongs to $\mathcal{V} \cup \bar{\mathcal{V}}$ or it is $G\psi$ for a suitable ψ

(c) $\forall i \leq m$, $\langle\mathfrak{F}, V\rangle \models_{x_i} \phi_i$

(d) $\langle\mathfrak{F}, V_1\rangle \models_x \alpha$, for every $V_1 \in Val_{\mathfrak{F}}$ such that $V(y) = V_1(y)$ whenever $y \notin T_1$ and, $\forall i \leq m$, $\langle\mathfrak{F}, V_1\rangle \models_{x_i} \phi_i$.

By (Y_3)–(Y_6), for any formula ϕ holding at y in $\langle\mathfrak{F}, V\rangle$, the set $X_{\phi,y}$ is in general not unique. In any case, unless otherwise stated, when we refer to $X_{\phi,y}$ we mean that it has been chosen among the sets having the properties (a) to (d) above. Since every $X_{\phi,y}$ is compatible with the valuation V, the results will not depend on the choice.

For every $n \in \omega$ and $t \in T_0$, the set $X^{n,t}_{\alpha,x}$ is defined, by induction on n, as follows:

(X_1) $X^{0,t}_{\alpha,x} = X_{\alpha,x}$

(X_2) $\langle\phi, y\rangle \in X^{n,t}_{\alpha,x}$ and $\phi \in \mathcal{V} \cup \bar{\mathcal{V}} \Rightarrow \langle\phi, y\rangle \in X^{n+1,t}_{\alpha,x}$

(X_3) $\langle G\psi, y\rangle \in X^{n,t}_{\alpha,x}$ and $y \geq t \Rightarrow \langle G\psi, y\rangle \in X^{n+1,t}_{\alpha,x}$

(X_4) $\langle G\psi, y\rangle \in X^{n,t}_{\alpha,x}$ and $y < t \Rightarrow \langle G\psi, t\rangle \in X^{n+1,t}_{\alpha,x}$ and $X^{n,t}_{\psi,z} \subseteq X^{n+1,t}_{\alpha,x}$ for every $z \in \{y+1, \ldots, t\}$.

By (X_3) and (X_4), for n^* sufficiently large, $X^{n',t}_{\alpha,x} = X^{n^*,t}_{\alpha,x}$ whenever $n' \geq n^*$. Let us denote $X^{n^*,t}_{\alpha,x}$ by $X^t_{\alpha,x}$. Of course, $X^t_{\alpha,x}$ (as well as every $X^{n,t}_{\alpha,x}$) has the properties (a) to (d) of $X_{\alpha,x}$ and, in addition,

(e) $\langle G\phi, y\rangle \in X^t_{\alpha,x} \Rightarrow y \geq t$.

Let t^* be any element of T_0 such that

(2.1,2) if $G\phi[H\phi]$ is a subformula of α, holding [which does not hold] (in $\langle\mathfrak{F}, V\rangle$) at some $y \in T_0$, then $G\phi[H\phi]$ holds [does not hold] at t^*.

Of course, t^* has simply to be greater than the minimum u such that $\langle\mathfrak{F}, V\rangle \models_u G\phi$ $[\langle\mathfrak{F}, V\rangle \models_u \neg H\phi]$. Notice that (2.2) implies

(2.2′) if $H\phi$ is a subformula of α, holding (in $\langle\mathfrak{F}, V\rangle$) at some $y \geq t^*$, then ϕ (and $H\phi$) holds at every $u \in T_0$.

Let now V' be the element of $Val_{\mathfrak{F}'}$ such that: (i) $V'(y) = V(y)$ for every $y \in T_0 \cup T_1$, and (ii) $V'(\langle \omega + r, 0\rangle) = V(\langle t^* + r + 1, 0\rangle)$ for every $r \in \omega$. In what follows, since only the elements of T_0 or T_0' are considered, we shall look at them as elements of $\omega + \omega$ (instead of $(\omega + \omega) \times \{0\}$) and, for every natural number r, we shall write r' and r'' as abbreviations for $t^* + r + 1$ and $\omega + r$, respectively.

Theorem 2.2 *For every subformula ϕ of α and every natural number r*

(2.3) $\langle \mathfrak{F}, V\rangle \models_{r'} \phi \Rightarrow \langle \mathfrak{F}', V'\rangle \models_{r'} \phi$ *and* $\langle \mathfrak{F}', V'\rangle \models_{r''} \phi$.

Proof: If $\phi \in \mathcal{V} \cup \bar{\mathcal{V}}$, then the theorem follows by the definition of V'. Thus, assume as inductive hypothesis that (2.3) holds for every $r \in \omega$ and every (proper) subformula of ϕ.

If H and P do not occur in ϕ, then the truth values of ϕ in $\langle \mathfrak{F}', V'\rangle$ at r' and r'' do not depend on the truth values of the subformulas of ϕ at any $y \leq t^*$, and hence (2.3) is a straightforward consequence of the inductive hypothesis. Note that, in case ϕ is $G\psi$, (2.1) has to be used: $\langle \mathfrak{F}, V\rangle \models_{r'} G\psi$ implies $\langle \mathfrak{F}, V\rangle \models_{t^*} G\psi$ and, by the inductive hypothesis, ψ holds in $\langle \mathfrak{F}', V'\rangle$ at every $y > t^*$; in particular, ψ holds at every $y > r'$ and every $y > r''$.

Let ϕ be $H\psi$ and let $\langle \mathfrak{F}, V\rangle \models_{r'} \phi$. Then ψ holds in $\langle \mathfrak{F}, V\rangle$ at every $y \in T_0$ (cf. (2.2′)) and, by the inductive hypothesis, $\langle \mathfrak{F}', V'\rangle \models_{s'} \psi$ and $\langle \mathfrak{F}', V'\rangle \models_{s''} \psi$ *for every $s \in \omega$*. Let $y \leq t^*$ and let us consider $X^{t^*}_{\psi,y}$. If $\langle \beta, z\rangle \in X^{t^*}_{\psi,y}$ and $\beta \in \mathcal{V} \cup \bar{\mathcal{V}}$, then $\langle \mathfrak{F}', V'\rangle \models_{z} \beta$ by the definition of V'. If $\langle G\delta, z\rangle \in X^{t^*}_{\psi,y}$, then $z \geq t^*$ (cf. (e)), and hence $\langle \mathfrak{F}', V'\rangle \models_{z} G\delta$ by the inductive hypothesis. The property (d) (of $X^{t^*}_{\psi,y}$) implies $\langle \mathfrak{F}', V'\rangle \models_{y} \psi$. Therefore, ψ holds in $\langle \mathfrak{F}', V'\rangle$ at every $y \in T_0'$ and the consequent of (2.3) is true.

Let ϕ be $P\psi$ and let $\langle \mathfrak{F}, V\rangle \models_{r'} \phi$. Then ψ holds at some $z < r'$ in $\langle \mathfrak{F}, V\rangle$ and we can prove $\langle \mathfrak{F}', V'\rangle \models_{z} \psi$ as in the previous case. Thus the consequent of (2.3) holds.

Now, in order to conclude that no set of formulas exists which characterizes $F1$, we have only to note that $\langle \mathfrak{F}', V'\rangle \models_{x} \alpha$, since, for every $\langle \phi, y\rangle \in X^{t^*}_{\alpha,x}$, (*) $\langle \mathfrak{F}', V'\rangle \models_{y} \phi$ (cf. (d)): if $\phi \in \mathcal{V} \cup \bar{\mathcal{V}}$, then (*) follows from the definition of V'; if ϕ is $G\psi$ then (by (e)) $y \geq t^*$ and hence Theorem 2.2 can be applied.

Notwithstanding the meaning of A1, the result just proved cannot be considered as properly connected with $\mathcal{L}_{MT}$, but essentially concerns the frames $\langle W, <\rangle$ for ordinary tense (or modal) logics and the definability of, e.g., their order type, when $<$ is linear, transitive, and well-founded. Witnesses of this are the particular choice of the N-frames $\mathfrak{F}$ and $\mathfrak{F}'$ and the structure of the proof, in which the modalities play no relevant role.

Another plausible requirement on the structure of the N-frames is A2 below, which seems to be properly concerned with $\mathcal{L}_{MT}$ since the accessibility relation between moments is explicitly considered in it.

A2 $\forall x, y \in T$, if xRy, then an order-isomorphism f exists between $\{x' : x' \leq x\}$ and $\{y' : y' \leq y\}$ such that $\forall x' \leq x$, $x'Rf(x')$.

In other words, we accept A2 when, in order to consider two possible words x and y R-equivalent, not only do we require that whatever happened before x also happened before y, but we also require that x and y must have *exactly the same past.*

Theorem 2.3 *The class F2 of N-frames fulfilling A2 is not characterized by any set of formulas of* $\mathcal{L}_{MT}$.

Proof: The statement A2 is obviously false in the N-frame $\mathfrak{F}$ defined by: (1) $I = \{0, 1\}$; (2) $T_0 = \omega \times \{0\}$, and $T_1 = (\omega + \omega) \times \{1\}$; (3) $x < y$ iff they are in the same T_i and the first element of x is *greater* than that of y; (4) R is the smallest equivalence relation on $T_0 \cup T_1$ such that $\forall n \in \omega$, $\langle n, 0\rangle R \langle n, 1\rangle$, $\langle n, 0\rangle R \langle n+2, 0\rangle$, and $\langle n, 0\rangle R \langle \omega + n, 1\rangle$. Conversely, A2 is true in the N-frame $\mathfrak{F}'$ obtained from $\mathfrak{F}$ by substituting $T_0' = (\omega + \omega) \times \{0\}$ for T_0 and by extending R to the smallest equivalence relation (on $T_0' \cup T_1$) containing $\langle\langle n, 0\rangle, \langle \omega + n, 0\rangle\rangle$ for every $n \in \omega$. Now, a construction similar to the previous one can be considered in order to prove that, for every $V \in Val_{\mathfrak{F}}$, $x \in T_0 \cup T_1$, and formula α, if $\langle \mathfrak{F}, V\rangle \models_{\overline{x}} \alpha$, then there is a $V' \in Val_{\mathfrak{F}'}$ such that $\langle \mathfrak{F}', V'\rangle \models_{\overline{x}} \alpha$.[2]

3 Propositional quantifiers in $\mathcal{L}_{MT}$ The definitions of $\langle \mathfrak{F}, V\rangle \models_{\overline{x}} \phi$, $\langle \mathfrak{F}, V\rangle \models \phi$, etc., considered in Section 1, can be extended to the case in which ϕ contains propositional quantifiers. Let $\mathfrak{F}$ be any N-frame; for every $p \in \mathcal{V}$, $V \in Val_{\mathfrak{F}}$, and subset ξ of T, closed under R, we shall denote by $V(p/\xi)$ the element of $Val_{\mathfrak{F}}$ such that $V(p/\xi)(p) = \xi$ and $V(p/\xi)(q) = V(q)$ whenever $q \neq p$. Then we can set

$$\langle \mathfrak{F}, V\rangle \models_{\overline{x}} (\forall p)\phi \text{ iff}$$
$$\langle \mathfrak{F}, V(p/\xi)\rangle \models_{\overline{x}} \phi,\ \forall \xi \subseteq T,\ \xi \text{ closed under } R.$$

The closure property of ξ is obviously required in order to make $V(p/\xi)$ an element of $Val_{\mathfrak{F}}$. In the sequel, $\mathcal{L}^*_{MT}$ will denote the language obtained from $\mathcal{L}_{MT}$ by adding to it propositional quantifiers.

Let us consider now the class $F3$ of the N-frames fulfilling

A3 $\forall i \in I,\ \forall x, y \in T_i,\ x \neq y \Rightarrow x \not R y.$

Theorem 3.1
(a) *There exists no set* Σ *of formulas in* $\mathcal{L}_{MT}$ *which characterizes* $F3$.
(b) *A formula* ϕ^* *in* $\mathcal{L}^*_{MT}$ *exists such that, for every N-frame* $\mathfrak{F}$, $\mathfrak{F} \models \phi^*$ *iff* $\mathfrak{F} \in F3$.

Proof of (a): A simple counterexample is used. Let $\mathfrak{F}$ be the N-frame in which (1) $I = \{0, 1\}$, (2) $T_i = Z \times \{i\}$, where Z denotes the set of integers, (3) $\langle n, i\rangle < \langle m, i\rangle$ iff $n < m$ in the usual order relation, and (4) R is the smallest equivalence relation on $T_0 \cup T_1$ such that $\forall i, j \in I$, $i \neq j \Rightarrow \langle n, i\rangle R \langle n+1, j\rangle$. Observe that in $\mathfrak{F}$ the modal operators are vacuous, since R has (also) the property obtained by substituting $>$ for $<$ in (1.2). If Σ would characterize $F3$, then $\langle \mathfrak{F}, V\rangle \models_{\overline{x}} \neg\sigma$ for some $V \in Val_{\mathfrak{F}}$, $\sigma \in \Sigma$, and moment $x = \langle n, i\rangle$. Now, on the basis of the observation, it is routine to prove that $\langle \mathfrak{F}', V'\rangle \models_{\overline{n}} \neg\sigma$ where $\mathfrak{F}' = \langle\{\langle Z, <\rangle\}, =\rangle$ and, for every $m \in Z$, $V'(m) = V(\langle m, i\rangle)$. Since $\mathfrak{F}'$ is trivially an element of $F3$, this class cannot be characterized in $\mathcal{L}_{MT}$.

Proof of (b): Let ϕ^* be the formula

$$(\forall p)(Pp \rightarrow (\exists q)(q \wedge P(p \wedge \neg q)))$$

where $\exists$ is defined in the obvious way.

Assume $\mathfrak{F} \notin F3$. Then two moments x and y in $\mathfrak{F}$ exist such that $x < y$ and xRy. Let V be any element of $Val_{\mathfrak{F}}$ such that $y \in V(p)$ and $z \notin V(p)$ whenever $z\not R y$. Since $x < y$ and $x \in V(p)$, $\langle \mathfrak{F}, V \rangle \models_y Pp$. If $\langle \mathfrak{F}, V(q/\xi) \rangle \models_y (q \wedge P(p \wedge \neg q))$ for some subset ξ of T closed under R, then a moment $z < y$ should exist such that $\langle \mathfrak{F}, V(q/\xi) \rangle \models_z (p \wedge \neg q)$. But, the hypothesis on V implies zRy, whereas the closure of ξ implies $z\not R y$. Therefore, $\langle \mathfrak{F}, V \rangle \models_y (\neg(\exists q)(q \wedge P(p \wedge \neg q)))$ and ϕ^* is not true in $\mathfrak{F}$.

Conversely, assume $\mathfrak{F} \in F3$. For every $V \in Val_{\mathfrak{F}}$ and moment y, if $\langle \mathfrak{F}, V \rangle \models_y Pp$ (that is, $\langle \mathfrak{F}, V \rangle \models_x p$, for some $x < y$), then $\langle \mathfrak{F}, V(q/\xi) \rangle \models_y (q \wedge P(p \wedge \neg q))$, for every $\xi \subseteq T$ (closed under R) such that $y \in \xi$ and $x \notin \xi$. Such ξs exist since $x\not R y$ and hence $\langle \mathfrak{F}, V \rangle \models_y (\exists q)(q \wedge P(p \wedge \neg q))$. V and y were chosen arbitrarily, thus $\mathfrak{F} \models \phi^*$.

NOTES

1. Let us consider, for instance, a system $\mathcal{S}$ composed by (liquid) water, considered in any homogeneous state of volume V, centigrade temperature T, and pressure P. As is well known, the constitutive equation $V = V(T, P)$ holds. Call P_a the atmospheric pressure, and set $V_0 = V(0, P_a)$ and $V_m = V(T_m, P_a)$, where V_m is the minimum volume of $\mathcal{S}$ (for $P = P_a$) so that $T_m \simeq 4$ and T_m is uniquely determined. Assume $V_m < V_* < V_0$. Then there are (exactly) two temperatures, T_1 and T_2 ($>T_1$) for which $V(T_1, P_a) = V_* = V(T_2, P_a)$.

   In relatively recent times several works belonging to foundations of physics are concerned with the problem of reducing thermal notions, such as temperature, to mechanic ones, from the macroscopic point of view, i.e., within the theory of continuous media (disregarding the kinetic theory of gases).

   From this point of view, in connection with bodies with an anomalous behavior, such as $\mathcal{S}$, it is important to be able to distinguish T_2 from T_1 in the following mechanical way, based on present possibility. Assume that presently, i.e., in the real world w_R at the instant t_*, $V = V_*$ and $P = P_a$. Then we have $T = T_2$ or $T = T_1$ according to whether or not

   (*) it is possible for V to increase continuously from the value V_* to some value $V' > V_0$, always keeping $P = P_a$.

   Now, in order to consider (*) as a good distinguishing criterion, by "it is possible..." we have to mean that a conceivable world w exists *that coincides with* w_R *at every instant* $t \leq t_*$ and in which the volume of $\mathcal{S}$ reaches the value V' at some instant $t' > t_*$, without decreasing in the interval $[t_*, t']$.

2. In the construction, the sets corresponding to the $X_{\phi,z}$'s above contain pairs $\langle \psi, y \rangle$ in which ψ belongs to $\mathcal{V} \cup \overline{\mathcal{V}}$, or it has one of the two forms $H\delta$ and $\Box\delta$; hence the detailed proof turns out to be more complex than that concerning A1.

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