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Modality and Possibility in Some Intuitionistic Modal Logics

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1 Introduction Traditionally, since the time of Aristotle, modal logic was built upon two main concepts, namely those of necessity and possibility, currently taken in an ontological sense. In the formal language they are represented by two unary operators, L for necessity and M for possibility. In classical logic, these operators are considered to be dual to each other and mutually definable through the formulas $M \leftrightarrow \neg L \neg$ and $L \leftrightarrow \neg M \neg$. However if we work on an intuitionistic nonmodal base logic, then some properties of the negation are weakened, the duality disappears, and it is commonly admitted that both equivalences cannot remain valid, because they lead to conclusions stronger than wished (see [4]). Of course one could ignore one of the two modal operators, but we think this pointless, because the dual interpretation of one of them gives natural birth to the other one.[1] On the other hand, several studies of intuitionistic modal logic have been published where neither of the two equivalences holds, the operators L and M being both primitive and independent, and linked through other indirect properties; see [19], [4]–[6], [9], [10], [18], [17] and the global studies of [20], [3], and [8].

Our choice is to try to apply Gödel's proposal for S4 (from [13]) to an intui intuitionistic base, that is, to consider L as a primitive symbol with implicative S4-type axioms and to define M as $\neg L \neg$. Here "p is possible" just means that "it is contradictory that p is necessarily contradictory"; we do not start from a philosophical analysis of any concept of possibility (as Aristotle and the Middle-Ages logicians probably did) but rather we make their properties follow from those of a primitive concept of necessity, the link between them being a formula where the "logical" negation plays an important part.[2] So we are formalizing a kind of derived or "negative" concept of possibility and it is in this sense that we would speak of a "logical" possibility rather than of a "philosophical" or "ontological" one. It should be emphasized that the remaining alternative, that of considering M as primitive and defining L as $\neg M \neg$, is not interesting because, even if we adopt very strong axioms for M, the simplest properties of L cannot

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be proved. This is shown, with the aid of algebraic models, in example 5.10 of [12].

The specific purpose of this paper is twofold: first, to analyze the behavior of our M; second, to show the use of algebraic models to obtain logical properties of the systems under consideration. Concerning the behavior of M, we focus on two points of special interest. On the one hand, the study of all different modalities, that is, of all possible combinations of the three operators L, $\neg$, and M that are nonequivalent. As is well-known, S4 has a finite and indeed small number of different modalities, and they have a relatively simple structure (see, e.g., [7]). The situation here will be much more complex, of course, but we shall also find a finite number of modalities. On the other hand, a really interesting point is the possible definition of intuitionistic modal logics analogous to S5 in the sense of [5]. Clearly, this is easy to do: it is enough to add to the basic system any one of the theses of S5 that are not theses of S4. However, due to the peculiar features of intuitionistic negation, different but classically equivalent axioms yield intuitionistically nonequivalent systems, and so it is of interest to investigate the relationships which hold between them.

We are concerned only with extensions by formulas that have already been used in classical works of modal logic to obtain S5 as an extension of S4. Moreover, the operator M does appear in almost all these formulas, and this increases the interest of the analysis. We show four logical systems of type S5, but we make no attempt to single out one of them as "the true analogue of S5", although we see that they are of increasing strength, and share more and more modal theorems with S5. Only the last one is not intuitionistically plausible, again in the sense of [5]. We hope that the results shown in this paper can constitute a basis to reflect on and to discuss the adequacy of considering $\neg L \neg$ as a genuine intuitionistic modal operator.

As we said before, we try to make an exhaustive use of algebraic models, and accordingly we will use logical formulas only when it is strictly necessary, mainly to define logical systems and to state some results such as the reduction of modalities. The algebraic models of our systems are the topological pseudo-Boolean algebras we have studied in [11] and [12]. Thus this paper will contain few proofs[3]; the reader can consult [12] for all propositions and other facts stated without proof in Section 3.

2 The basic system and its modalities The formulas of all our logical systems are built up from a (usually denumerable) set of propositional letters with the connectives L, $\neg$ (unary), and $\wedge$, $\vee$, $\rightarrow$ (binary). We use the letters $p, q, \ldots$ as metamathematical variables for formulas, and we abbreviate $\neg L \neg$ as M. Our basic system is:

Definition 1 We call *IM4* the logical system having the following axiom schemes and rules of inference: A complete basis for intuitionistic propositional calculus, and

$Lp \rightarrow p$
$L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq)$
$Lp \rightarrow LLp$
The "Rule of Necessity": $p \vdash Lp$.

From the preceding axioms and rules a syntactical consequence relation $\vdash$ is obtained in the customary finitistic way (the symbol $\vdash$ will be omitted when it is clear that we refer to theorems). It is easy to show that in IM4, $\vdash LLp \leftrightarrow Lp$, and $p \to q \vdash Lp \to Lq$; so it is a "normal" modal logic. It has appeared elsewhere under different names (see [3], [4], [17], [18], [20]), and it is a "canonical" analogue of S4, at least regarding the necessity operator. The analogy applies also to its regular unidesignated logical matrices, that is, to its algebraic models, which are a weakening of topological Boolean algebras.

Definition 2 A *topological pseudo-Boolean algebra* (*tpBa* from now on) is an algebra $(A, I, \neg, \wedge, \vee, \to)$ of type (1, 1, 2, 2, 2) such that $(A, \neg, \wedge, \vee, \to)$ is a pseudo-Boolean algebra and I is a unary operator on A satisfying:

$Ia \leq a$ for all $a \in A$
$I(a \to b) \leq Ia \to Ib$ for all $a, b \in A$
$I^2 a = Ia$ for all $a \in A$
$I1 = 1$, where 1 is the maximum of A.

It is easy to see that I is monotone, that is, if $a \leq b$ then $Ia \leq Ib$, and that it satisfies $I(a \wedge b) = Ia \wedge Ib$ for all $a, b \in A$. We say that I is a *topological interior* operator on A. An $a \in A$ such that $Ia = a$ is called *open*, and the set of all open elements of A is denoted by B; it is a sublattice of A containing 0 and 1 and being relatively supcomplete, namely we have $Ia = max\{b \in B : b \leq a\}$, for all $a \in A$. An alternative way of defining a tpBa over a given pseudo-Boolean algebra A is to give a $B \subseteq A$ satisfying all the preceding properties. This is what we are going to do in the examples at the end of Section 3.

As is well-known, algebraic semantics is the most faithful one[4] and it gives a completeness theorem under some natural assumptions, basically equivalent to the fact that the logic admits a Lindenbaum-Tarski algebra and it is the free algebra of the class of algebraic models. Since this is our case, it follows that a formula is a theorem of IM4 if and only if it is true in every tpBa, that is, the corresponding algebraic expression equals 1 in every tpBa for all allocations of values to its propositional variables.[5] This is a usually fast way for proving things, because in tpBas we have a lot of resources other than operating with the algebraic translations of logical formulas.

For instance, the properties of M are those of the operator $\delta = \neg I \neg$. Note that from the definition we always have $\delta I \neg a = \neg I \delta a$ for all $a \in A$. If $a = \delta a$ then we say that a is *closed*, and we denote the set of all closed elements by T. We quote here the most immediate and interesting properties of δ and T.

Proposition 1 *In every tpBa A the following hold:*
(1) $\delta 0 = 0$, $a \leq \delta a$, $\delta a = \delta^2 a$ *for all* $a \in A$
(2) *If* $a \leq b$ *then* $\delta a \leq \delta b$ *for all* $a, b \in A$
(3) $I\neg a \leq \neg \delta a \leq \neg a \leq \delta \neg a \leq \neg I a$ *for all* $a \in A$
(4) $\neg\neg a \leq \delta a = \delta \neg\neg a = \neg\neg \delta a$ *for all* $a \in A$
(5) T *is closed under* $\wedge$ *and contains* 0 *and* 1, *and for all* $a \in A$ *it holds that* $\delta a = min\{t \in T : a \leq t\}$
(6) $\neg I a \in T$ *for all* $a \in A$.

We remark that (1) and (2) above tell us that δ is an order-closure operator, but it is easy to see that it is not a topological closure; see for instance Example 4 at the end of Section 3. Of course all preceding properties (better: almost all) could be rewritten in their logical form as properties of M. Let us do so in what concerns the reduction of modalities:

Proposition 2 *The following formulas are theorems of IM4:*
(1) $\neg L\neg\neg Lp \leftrightarrow \neg Lp$
(2) $L\neg\neg L\neg p \leftrightarrow L\neg p$
(3) $L\neg L\neg L\neg Lp \leftrightarrow L\neg Lp.$

Proofs: For (1) put Ia for a in the right half of (3) in Proposition 1. For (2) apply I to the left half of (3) in Proposition 1. For (3), apply I to $I\neg Ia \leq \delta I\neg Ia$ to obtain $I\neg Ia \leq I\delta I\neg Ia$, and do the same to $Ia \leq \delta Ia$ to obtain $Ia \leq I\delta Ia$, and then by negation and further application of I get $I\delta I\neg Ia = I\neg I\delta Ia \leq I\neg Ia$.[6]

To achieve the reduction of modalities it is enough to consider all combinations of $\neg$ and L, since M is nothing but $\neg L\neg$. Taking into account Proposition 2 and the fact that $LLp \leftrightarrow Lp$ and that $\neg\neg\neg p \leftrightarrow \neg p$, we see that all modalities with more than three Ls reduce to shorter ones. It is obvious that a modality with at most three Ls does not admit more than six $\neg$ without reducing to a shorter one, so we see that the total number of essentially different modalities is finite. By working methodically and with the aid of suitable tpBas we can arrive at the following:

Theorem 1 *The system IM4 has 31 different modalities, 17 being affirmative and 14 being negative, satisfying the relations shown in Figure 1 (where • means the empty modality).*

This theorem can be found, in [17] (with some mistakes) and in [8] in its right form; we are giving it here for the sake of completeness of the paper, but we will not give more details of its proof.

However, the above-mentioned papers do not use M at all, that is, all modalities appear written only with L and $\neg$. In such a way they have a unique shortest form, but if we use our M then this uniqueness disappears because of the law $\neg LMp \leftrightarrow ML\neg p$. There are some noteworthy equivalences produced by this law, such as $MLMp \leftrightarrow \neg LML\neg p$ and its "dual" $\neg MLM\neg p \leftrightarrow \neg\neg LML\neg\neg p$. On the other hand, the only "real" laws of reduction of modalities in IM4 are the ones in Proposition 2 and those arising from them; besides $LLp \leftrightarrow Lp$ we quote the following ones: $MMp \leftrightarrow Mp$, $LMLMp \leftrightarrow LMp$, $MLMLp \leftrightarrow MLp$, $\neg M\neg Lp \leftrightarrow \neg\neg Lp$, and $L\neg M\neg p \leftrightarrow L\neg\neg p$. In giving the written form of most modalities we have made use of M so as to show them in their shortest form, and when this is not unique we have simply chosen the one we found more interesting.

Another outstanding feature of IM4 modalities is the fact that if we want to use only L and M and leave $\neg$ aside then we find exactly the same modalities as in classical S4, and we find them arranged following the same scheme (see Figure 2). We can also note that if ϕ is any modality built up from L and M, then we have IM4 $\vdash \phi\phi p \leftrightarrow \phi p$, that is, iteration of modalities which can be written without $\neg$ makes no sense.

It is not surprising at all that the situation turns out to be very different when we introduce negation, and that the intuitionistic base we are working with results in a quite complicated and nonsymmetric system of modalities, either

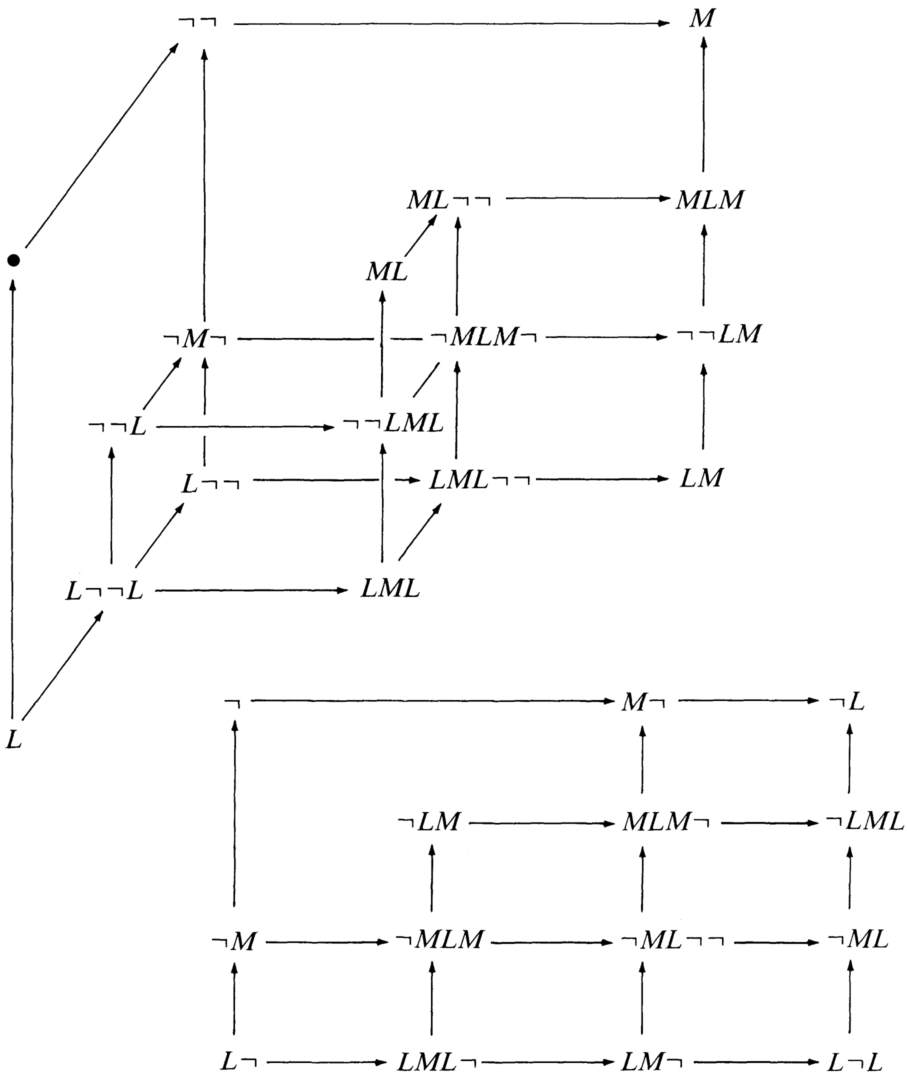

Figure 1.

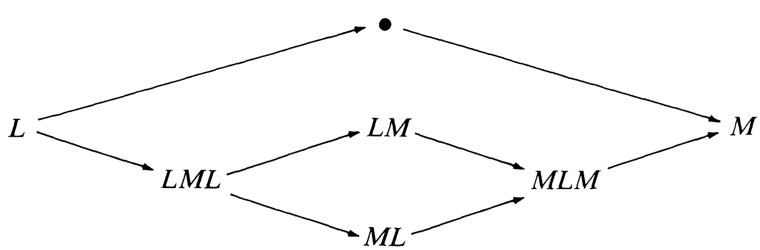

Figure 2.

affirmative or negative, as well as in the lack of symmetry (or duality) between these two groups.

3 Systems of type S5 and their modalities In this section we shall present the four extensions of IM4 we are concerned with, each one with its algebraic models, and we will state the corresponding theorem of reduction of modalities, along with some other properties. We shall complete the proofs of these theorems by going backwards from the strongest system to the weakest one, in order to reduce to a minimum the number of tpBas actually shown or the number of computations to be performed on them.

The first extension of IM4 we treat will be defined by the axiom that von Wright used in [21] to define his system M'', which is deductively equivalent to S5:

Definition 3 We call *IM4W* the extension of IM4 with the axiom $M\neg Mp \rightarrow \neg Mp$. A tpBa A will be called *weakly monadic* if and only if it satisfies $\delta\neg\delta a = \neg\delta a$ for all $a \in A$.

It is clear that weakly monadic tpBas are the algebraic models of IM4W and that we have the corresponding completeness theorem. There are some alternative definitions which use well-known conditions of classical modal logic or of its algebraic studies,[7] as the following proposition shows:

Proposition 3 *In every tpBa A the following conditions are equivalent*:
(1) $\delta\neg\delta a = \neg\delta a$ *for all* $a \in A$, *that is*, T *is closed under* $\neg$
(2) *if* $a \wedge \delta b = 0$ *then* $\delta a \wedge \delta b = 0$ *for all* $a, b \in A$
(3) $\delta(a \wedge \delta b) = \delta a \wedge \delta b$ *for all* $a, b \in A$.

Proof: See [12], Theorem 2.1.

Weakly monadic tpBas are very interesting from the algebraic point of view. For instance, in addition to (1) it can be shown that T is closed under $\rightarrow$, and, moreover, it has the structure of a Boolean algebra with a suitable supremum. As such, it is a very natural quotient of the algebra (for proofs see [12], Theorems 2.7 and 2.9).

Proposition 4 *In every weakly monadic tpBa it holds that*
(1) $\neg I\delta a = \delta I\neg a = \neg\delta a$ *for all* $a \in A$
(2) $\neg\delta Ia = \neg Ia$ *for all* $a \in A$.

Proof: (1) $\delta I\neg a = \neg I\neg I\neg a = \neg I\neg\neg\neg I\neg a = \delta\neg\delta a = \neg\delta a$ by Definition 3. (2) In a weakly monadic tpBa, (6) of Proposition 1 implies that $\neg\neg Ia \in T$ for all $a \in A$. Using (4) of the same place, we get $\neg\delta Ia = \neg\delta\neg\neg Ia = \neg\neg\neg Ia = \neg Ia$.

Theorem 2 *The logical system IM4W has 16 different modalities, 9 being affirmative and 7 negative, satisfying the relations shown in Figure 3.*

Proof: We have just explicitly seen the reductions $\neg LMp \leftrightarrow ML\neg p \leftrightarrow \neg Mp$ and $\neg MLp \leftrightarrow \neg Lp$. From them, we obtain $\neg\neg LMp \leftrightarrow Mp$, $\neg\neg Lp \leftrightarrow MLp$,

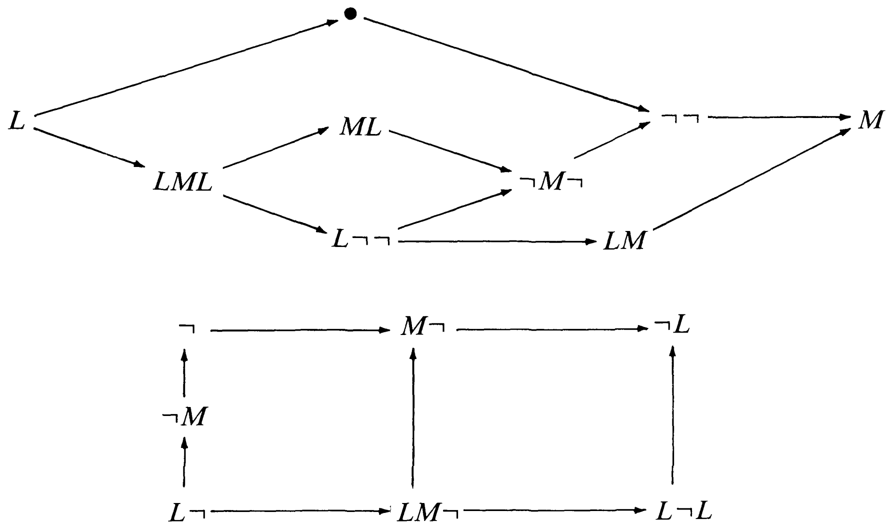

Figure 3.

$ML\neg\neg p \leftrightarrow \neg M\neg p$, and $\neg ML\neg\neg p \leftrightarrow M\neg p$ by inserting $\neg$ and taking (4) of Proposition 1 into account. Moreover, we also have $L\neg\neg Lp \leftrightarrow LMLp$ and from this and the law $L\neg M\neg p \leftrightarrow L\neg\neg p$ (which follows from (2) of Proposition 2) we find $LML\neg\neg p \leftrightarrow L\neg\neg p$ and also $LML\neg p \leftrightarrow L\neg p$. Now the diagram for IM4 becomes the one shown above. After having proved Theorems 5 and 6 we shall see that this diagram is exact, that is, that there are no implications other than those actually shown and that these are proper.

The second extension of IM4 will make use of any one of four well-known axioms and rules, originally used by Wajsberg [22], Lewis [16], and Becker [1]. The definition rests on the following:

Proposition 5 *In every tpBa A the following conditions are equivalent:*
(1) *$I\neg Ia = \neg Ia$ for all $a \in A$, that is, B is closed under $\neg$*
(2) *$I\delta a = \delta a$ for all $a \in A$, that is, $T \subseteq B$*
(3) *$a \leq I\delta a$ for all $a \in A$*
(4) *If $\delta a \leq b$ then $a \leq Ib$ for all $a, b \in A$.*

Proof: See [12], Theorem 3.1.

Definition 4 We call *IM4M* the extension of IM4 with any one of the following axioms: $\neg Lp \to L\neg Lp$, $Mp \to LMp$, $p \to LMp$, or with the rule $Mp \to q \vdash p \to Lq$.

A tpBa A will be called *monadic* if and only if it satisfies any one of the conditions in Proposition 5.

Thus monadic tpBas are the models of IM4M. It is easy to see that they are also weakly monadic (for instance, (1) of Proposition 5 implies (1) of Proposition 3, trivially), and so IM4M is actually an extension of IM4W (and Example 4 at the end of this section tells us that it is a proper one). Some of the new axioms for IM4M are themselves really classical laws of reduction of modalities. Let us see them all:

Theorem 3 *The system IM4M has 10 different modalities, 6 being affirmative and 4 negative, and they are arranged according to the scheme shown in Figure 4.*

Proof: From the very axioms we get $L\neg Lp \leftrightarrow \neg Lp$, $LM\neg p \leftrightarrow M\neg p$, $LMp \leftrightarrow Mp$, and $LMLp \leftrightarrow MLp$. Since monadic tpBas are also weakly monadic, from $a \leq \delta a$ we have $\delta\neg\delta a = \neg\delta a \leq \neg a$, and then (4) of Proposition 5 gives us $\neg\delta a \leq I\neg a$, which completes in IM4M the law $\vdash \neg Mp \leftrightarrow L\neg p$. From it one gets $\neg M\neg p \leftrightarrow L\neg\neg p$. Thus the diagram for IM4W becomes the one in Figure 4, and, as in the preceding case, we delay the complete proof a little.

$$\begin{tikzcd}
 & & \bullet \arrow[rrd] & & & \\
L \arrow[rru] \arrow[dr] & & & & \neg\neg \arrow[r] & M \\
 & ML \arrow[rr] & & L\neg\neg \arrow[ur] & &
\end{tikzcd}$$

$$L\neg \longrightarrow \neg \longrightarrow M\neg \longrightarrow \neg L$$

Figure 4

Our third extension of IM4 uses an implicative axiom without M which was used by Beth and Nieland in [2].

Definition 5 We call *IM4S* the extension of IM4 with the extra axiom $L(Lp \to q) \leftrightarrow (Lp \to Lq)$.

A tpBa A will be called *strongly monadic* if and only if it satisfies that $I(Ia \to b) = Ia \to Ib$ for all $a, b \in A$.

It is equivalent to say that B is a subalgebra of A. This makes clear that all strongly monadic tpBas are monadic, that is, IM4S is an extension of IM4M. Example 3 will show that it is a proper one. However we shall see later that IM4S has exactly the same modalities as IM4M has.

Our last extension of IM4 can be obtained with three distinct axioms. The first two are well-known modal laws whose duals have already been used; the third one appears in [4] (and in a slightly different form in [17]).

Definition 6 We call *IM5* the extension of IM4 with any one of the following axioms: $MLp \to Lp$, $MLp \to p$, and $L\neg Lp \vee Lp$.

These three axioms are equivalent on the basis of IM4 because they are true in the same class of tpBas, a very well singled out one, namely the class of all *semisimple* tpBas. The information we need is contained in the following:

Proposition 6 *In every tpBa A the following conditions are equivalent:*
(1) *A is a semisimple algebra*[8]
(2) *B is a Boolean subalgebra of A*
(3) $\delta a = min\{t \in B : a \leq t\}$ *for all* $a \in A$
(4) $\delta Ia = Ia$ *for all* $a \in A$, *that is*, $B \subseteq T$
(5) $\delta Ia \leq a$ *for all* $a \in A$
(6) $I\neg Ia \vee Ia = 1$ *for all* $a \in A$.

Proof: See [12], Theorems 4.5 and 4.6.

We see in (2) that every semisimple tpBa is strongly monadic. This tells us that IM5 is an extension of IM4S, but it also helps us to understand the structure of semisimple tpBas: they are exactly those tpBas where $T = B$ and is a subalgebra of A which is Boolean. This has an interesting logical reading: in IM5 the propositions expressing necessity and those expressing possibility are the same (indeed they express their own necessity and their own possibility) and they have a totally classical behavior. This is a characteristic property of S5, already noted by Lewis.[9] Consequently, we have the laws $\neg\neg Lp \leftrightarrow Lp$ and $\neg Lp \vee Lp$. The validity of such formulas is considered by Bull as "intuitionistically implausible" in [4] and all systems containing it are rejected as genuine intuitionistic analogues of S5 according to the criteria of [5], namely according to the one requiring that collapsing the modal operators the system must yield the intuitionistic propositional calculus. However, our IM5 is weaker than the system initially considered by Bull, because this one had the mutual interdefinability of L and M, which is not true in IM5, as we shall see later. It is easy to compare IM5 with MIPC, the system introduced by Prior in [19] and studied by Bull in [5] and [6], in spite of the difference of languages, by using the respective algebraic semantics. So we can state:

Theorem 4 *IM5 is the extension of MIPC with the extra axiom* $Mp \leftrightarrow \neg L\neg p$.

Proof: Both systems IM5 and MIPC are complete with respect to their algebraic semantics, IM5 with semisimple tpBas and MIPC with matrices $(H, K, \{1\}, \neg, \wedge, \vee, \rightarrow, I, \delta)$, where $(H, \neg, \wedge, \vee, \rightarrow)$ is a pseudo-Boolean algebra and $K \subseteq H$ is a subalgebra of H which is relatively complete, with $Ia = max\{b \in K : b \leq a\}$ and $\delta a = min\{t \in K : a \leq t\}$ for all $a \in A$. That is, MIPC is complete with respect to a special class of strongly monadic tpBas which have an additional δ not related to I. But if we extend MIPC with $Mp \leftrightarrow \neg L\neg p$ then δ becomes the usual one of all tpBas and moreover it satisfies (3) of Proposition 6, which tells us that the models of the extended system are the semisimple tpBas. So the two systems are equivalent and the theorem is proved.

It should also be noted that Ono proved in [18] that MIPC is a conservative extension of (a system equivalent to) IM4S, that is, if ψ is a formula without M then MIPC $\vdash\psi$ if and only if IM4S $\vdash\psi$. So we can say that to a certain extent MIPC is an intermediate system between IM4S and IM5.

Concerning the reduction of modalities it is quite odd that in IM5 there is only one new law of reduction, namely the one appearing in the definition, $MLp \leftrightarrow Lp$. So we have:

Theorem 5 *The logical system IM5 has 9 different modalities, 5 being affirmative and 4 negative, according to the scheme shown in Figure 5.*

Proof: It is clear that the scheme for IM4M is transformed in the one here shown for IM5. To see that all implications are proper and that no one holds between p and $L\neg\neg p$ we consider Example 1.

Example 1: On the pseudo-Boolean algebra with 12 elements $A = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$ given by the Hasse diagram[10] of Figure 6, we take $B = \{0, c, j, 1\}$ as the set of open elements. This obviously defines a tpBa, as we observed after Definition 2. We give here tables for $\neg$, I, and δ as they are the operators we use most often:

	0	a	b	c	d	e	f	g	h	i	j	1
$\neg$	1	e	i	j	b	h	c	0	e	b	c	0
I	0	0	0	c	c	c	0	c	0	c	j	1
δ	0	j	j	c	1	1	j	1	j	1	j	1

As we can see, $B = T$ and is a Boolean subalgebra of A, so this tpBa is a semisimple one, that is, a model for IM5. One can check that here $If < f < \neg\neg f$, $If < I\neg\neg f$, $I\neg\neg b < \neg\neg b$, $\neg\neg i < \delta i$, $b \nleq I\neg\neg b$, $I\neg\neg f \nleq f$, $I\neg b < \neg b < \delta\neg b$,

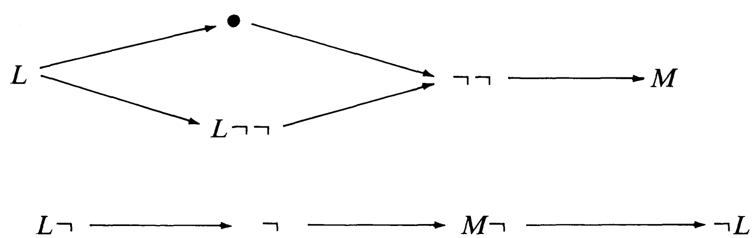

Figure 5.

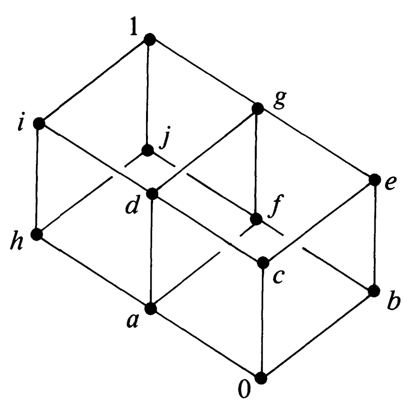

Figure 6.

and $\delta\neg f < \neg If$. Consequently no one of the implications of the scheme can be reversed nor can we add any one more: the scheme is exact.

We can also verify in the preceding example that the formula $Lp \leftrightarrow \neg M \neg p$ is not a theorem of IM5, giving p for instance the value f.

We can now complete the determination of the modalities in all systems weaker than IM5:

Theorem 6 *The logical system IM4S is weaker than IM5 and has the same modalities as IM4M.*

Proof: For the proof we are going to use Example 2.

Example 2: Let $A = \{0, a, b, c, d, e, f, g, h, 1\}$ be the pseudo-Boolean algebra given by the Hasse diagram of Figure 7, and take $B = \{0, c, e, h, 1]$. It is easy to check that B is a subalgebra of A, that is, the tpBa is strongly monadic. But B is obviously not Boolean, so the tpBa is not semisimple. We have a model for IM4S which is not a model for IM5, thus proving that the latter is stronger than the former. Concerning modalities, of course IM4S has at most those of IM4M, but in our example $h = Ih < \delta Ih = \neg\neg Ih = 1$. So the only possible new reduction (the one which holds in IM5) is not true in IM4S. Since Example 1 is also strongly monadic, we conclude that the diagram in Theorem 3 is exact for IM4S.

Note that we have already completed the proof of Theorem 3, too, because Examples 1 and 2 are monadic tpBas, and so the counterexamples found there do hold for IM4M. That this system is actually weaker than IM4S is shown by Example 3.

Example 3: Take $B = \{0, d, g, 1\}$ in the same pseudo-Boolean algebra as Example 2. Here B is closed under $\neg$ but $g \to d = f \notin B$, so this makes a monadic tpBa which is not strongly monadic, that is, a model for IM4M which is not a model for IM4S.

Now to complete the proof of Theorem 2 we will exhibit a weakly monadic tpBa where we can find counterexamples for all implications between modalities of IM4W not appearing in stronger systems. The remaining ones are proper simply because Examples 1 and 2 are, of course, weakly monadic.

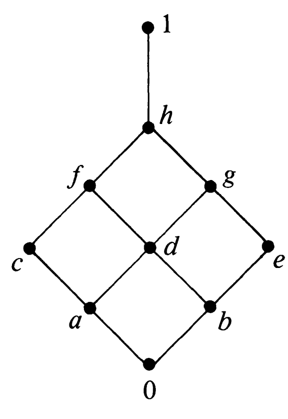

Figure 7.

Example 4: Take $B = \{0, a, b, d, 1\}$ over the same pseudo-Boolean algebra of Example 2. The resulting tpBa has the following tables for $\neg$, I, and δ:

	0	a	b	c	d	e	f	g	h	1
$\neg$	1	e	c	e	0	c	0	0	0	0
I	0	a	b	a	d	b	d	d	d	1
δ	0	c	e	c	1	e	1	1	1	1

We see that $T = \{0, c, e, 1\}$ is closed under $\neg$ but B is not. So this tpBa is weakly monadic and not monadic. This tells us that IM4M is stronger than IM4W. If we allocate p to e we see that the implications $LMLp \to MLp$, $L\neg\neg p \to \neg M\neg p$, $LMp \to Mp$, $L\neg p \to \neg Mp$, $LM\neg p \to M\neg p$, and $L\neg Lp \to \neg Lp$ cannot be reversed. So all the implications of Theorem 2 are proper. Finally there cannot be any more implications than those shown in Theorem 2 because otherwise they would appear in Theorem 3 or they are simply disproved by allocating p to a.

Note that in Example 4 $\delta c \vee \delta e = c \vee e = h \neq 1 = \delta h = \delta(c \vee e)$, which shows us that δ is not a topological closure, which we announced between Proposition 1 and Proposition 2. Actually, the condition of δ being a topological closure is true in all semisimple tpBas but it is independent of all other systems (see [12]). For the sake of completeness we should show that IM4W is really stronger than IM4:

Example 5: Take $B = \{0, c, 1\}$ in the same pseudo-Boolean algebra of the preceding examples. Now $T = \{0, e, 1\}$ which is not closed under $\neg$, so this is a tpBa which is not weakly monadic.

NOTES

1. It is worth noting that in modern studies of several modal-like logics, such as deontic, epistemic, temporal, . . . , we always find two unary operators similar to the classical ones.
2. Recall the comments of Heyting on intuitionistic negation: "In intuitionistic mathematics only falsity 'de jure' can play a part" ([15], p. 18).
3. On the other hand, this is not the first paper on reduction of modalities in intuitionistic modal logic; see [8] and [17].
4. However, this virtue can be a sin in specific circumstances, as Sotirov points out on page 160 of [20]: "(. . .) algebraic semantics is very general, but at the same time not very informative because it differs insignificantly from the logic itself."
5. Recall that in every pseudo-Boolean algebra $a \to b = 1$ iff $a \leq b$, and that $a \leftrightarrow b = 1$ iff $a = b$, iff $a \leq b$ and $b \leq a$.
6. This proposition is stated without proof in [17], where there is a mistake in (3).
7. For instance, the definition of monadic Boolean algebras by Halmos in his series of papers gathered in [14].
8. A tpBa is simple iff it has only two distinct open elements, 0 and 1 (this is equivalent to having only two distinct congruence relations, which is the original univer-

sal algebra concept of simplicity). A tpBa is semisimple iff it can be represented as a subdirect product of simple tpBas. Several properties of semisimple algebras are first proved for simple algebras and then extended to semisimple ones through this representation.

9. In [16], p. 501, we read: "The principal logical significance of the system S5 consists in the fact that it divides all propositions in two mutually exclusive classes: the intensional or modal, and the extensional or contingent. According to the principles of this system, all intensional or modal propositions are either necessarily true or necessarily false (. . .). For extensional or contingent propositions, however, possibility, truth, and necessity remain distinct". Recall that Lewis is talking about one interpretation, the one usually called the "Henle model", where the M operator only takes the values 0 and 1 (that is, simple topological Boolean algebras).

10. As is well-known, finite pseudo-Boolean algebras are the finite distributive lattices, and the operation $\rightarrow$ is characterized by $a \rightarrow b = max\{c \in A : a \wedge c \leq b\}$ for all a, $b \in A$. So the table for $\rightarrow$ can be obtained from the Hasse diagram, as can those for $\wedge$ and $\vee$; and recall that $\neg a = a \rightarrow 0$ for all $a \in A$.

REFERENCES

[1] Becker, O., "Zur Logik der Modalitäten," *Jahrbuch für Philosophie und Phänomenologische Forschung*, vol. 11 (1930), pp. 497-548.

[2] Beth, E. W. and J. F. F. Nieland, "Semantic construction of Lewis systems S4 and S5," pp. 17-24 in *The Theory of Models*, eds. J. W. Addison, L. Henkin, and A. Tarski, North-Holland, Amsterdam, 1965.

[3] Božić, M. and Došen, K., "Models for normal intuitionistic modal logics," *Studia Logica*, vol. 43 (1984), pp. 217-245.

[4] Bull, R. A., "Some modal calculi based on IC," pp. 3-7 in *Formal Systems and Recursive Functions*, eds. J. N. Crossley and M. Dummett, North-Holland, Amsterdam, 1965.

[5] Bull, R. A., "A modal extension of intuitionistic logic," *Notre Dame Journal of Formal Logic*, vol. 6 (1965), pp. 142-145.

[6] Bull, R. A., "MIPC as the formalization of an intuitionistic concept of modality," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 609-616.

[7] Chellas, B. F., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

[8] Došen, K., "Models for stronger normal intuitionistic modal logics," *Studia Logica*, vol. 44 (1985) pp. 39-70.

[9] Fischer-Servi, G., "Semantics for a class of intuitionistic modal calculi," pp. 59-72 in *Italian Studies in the Philosophy of Science*, ed. M. L. dalla Chiara, Reidel, Dordrecht, 1980.

[10] Fischer-Servi, G., "An intuitionistic analogue of S4 as a logical modelling for science," Preprint, to appear.

[11] Font, J. M., "Implication and deduction in some intuitionistic modal logics," *Reports on Mathematical Logic*, vol. 17 (1984), pp. 27-38.

[12] Font, J. M., "Monadicity in topological pseudo-Boolean algebras," pp. 169–192 in *Models and Sets, Proceedings of the Logic Colloquium '83*, Lecture Notes in Mathematics, vol. 1103, Springer-Verlag, Berlin, 1984.

[13] Gödel, K., "Eine Interpretation des Intuitionistischen Aussagenkalküls," *Ergebnisse eines Mathematischen Kolloquiums*, vol. 4 (1933), pp. 39–40.

[14] Halmos, P. R., *Algebraic Logic*, Chelsea, New York, 1962.

[15] Heyting, A., *Intuitionism. An Introduction*, North-Holland, Amsterdam, 1956.

[16] Lewis, C. I. and C. H. Langford, *Symbolic Logic*, The Century Company, New York, 1932, and Dover, New York, 1959.

[17] Mihajlova, M., "Reduction of modalities in several intuitionistic modal logics," *Comptes rendus de l'Académie Bulgaire des Sciences*, vol. 33 (1980), pp. 743–745.

[18] Ono, H., "On some intuitionistic modal logics," *Publications of the R.I.M.S. Kyoto University*, vol. 13 (1977), pp. 687–722.

[19] Prior, A. N., *Time and Modality*, Oxford University Press, 1957.

[20] Sotirov, V. H., "Modal theories with intuitionistic logic," pp. 139–172 in *Proceedings of the Conference on Mathematical Logic, Sofia, 1980*, Bulgarian Academy of Sciences, Sofia, 1984.

[21] Von Wright, G. H., *An Essay in Modal Logic*, North-Holland, Amsterdam, 1951.

[22] Wajsberg, M., "Ein erweiterter Klassenkalkül," *Monatshefte für Mathematik und Physik*, vol. 40 (1933), pp. 113–126.

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