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Classical Harmony*

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1 A standard metaphor used in explaining the notion of a valid argument is that the conclusion of such an argument is contained in the premises. Now if the conclusion is contained in any very direct sense then the argument will lack epistemic value; it will not be of use in persuading someone to accept the conclusion who does not already do so but does accept the premises (cf. [2], pp. 300ff). The classical logician has a fairly straightforward reply to this dilemma: the conclusion is only contained in the premises in the sense that it is true in any situation in which all of the premises are true. Since this fact may not be at all evident, deductive argument, in establishing it, is epistemically useful.

For such a logician, then, validity is a semantical notion characterizable independently of the epistemic notions, such as that of proof, which are used as tests for it. Now this classical conception has come under a great deal of attack, especially from those who subscribe to the Wittgensteinian slogan that meaning is use and interpret it as requiring that all ingredients of meaning can be made manifest in our use of sentences, especially in teaching or communicating their senses, for it is often claimed that classical bivalent semantics, in ascribing truth values to sentences regardless of whether these values are discoverable, violates this requirement. Alternative conceptions of truth are then advanced which, by tying it closely to proof, justification, or a similar epistemic concept, enable one to explicate validity in a way more in keeping with the above interpretation of the Wittgensteinian slogan.

The most radical proposal is to eschew entirely an appeal to the concept of truth in explications of validity and rely purely on proof-theoretic notions. It is clear that if one is to do so, one must discriminate among purported proofs; otherwise any and every proof-system, including trivial ones in which every formula is a theorem (such as Prior's infamous system for his connective "tonk" [7]), will be equally acceptable as explications of the concept of validity. More-

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over, if one hopes to persuade others to adopt one's own account of validity when evaluating and criticizing arguments then one must propose constraints on what is to count as a proof or inference rule which are both natural and nonarbitrary. One especially valuable desideratum on such constraints is that they make nonmetaphorical sense of the notion of containment which forms part of the intuitive notion of validity.

A number of different proposals have been made as to the form of these constraints. Perhaps the most attractive program is that which has been developed by Prawitz from some suggestions of Gentzen (see [4], p. 33; [5], and [6]; also [1], pp. 220–222; and [8]). Gentzen's idea was to treat the introduction rules, in a natural deduction system, as canonical. They provide the meanings of the logical constants and hence are valid by definition. The validity of elimination rules, on the other hand, is to be determined by their relation to the introduction rules; they must be in some sense uniquely determined by the latter and yield no more than their consequences ([3], pp. 80–81).

In making clearer sense of these ideas, the standard strategy has been to appeal to normalization procedures in natural deduction proofs. There one applies reduction steps in order to eliminate any maximal formulas, formulas which occur as conclusions of introduction rules (and perhaps some other types) and major premises of elimination rules. Taking the introduction rules as valid by definition, one tries to define the valid elimination rules as those for which there are appropriate reduction procedures. These transform any proof terminating in an instance of the elimination rule in which a major premise stands as conclusion of an introduction rule into one in which the conclusion is reached directly by means of introduction rules or other separately specifiable "canonical" rules. In this way introduction and elimination rules can be shown to "harmonize" with one another in an interesting sense.

This is the main idea behind Prawitz' "Inversion Principle" ([4], p. 33) and it may be said to provide an explication of the idea that the conclusion of a valid argument is contained in the premises: one cannot get anything out of a set of premises, by using elimination rules, that is not already in the set, in the sense of being deducible by means of definitionally valid introduction rules applied to canonical proofs of the premises.

Normalization theorems also enable us to demonstrate that our proof system satisfies interesting global constraints, such as conforming to a Subformula Principle: proofs in normal form, shorn of their maximal formulas, contain no formulas which are not related in pleasing ways to the premises or conclusion (for instance, are subformulas of a premise or the conclusion). And Subformula Principles yield, in turn, neat conservativeness properties: one can lay down that rules for a logical constant K must ensure that it is noncreative. That is, if A is not derivable from Δ in the logic of the fragment of the language without K, where A and each member of Δ belong to that fragment, then neither is it derivable from Δ in the expanded system which contains K together with its inference rules. Again one might claim that this conception of noncreativity is an explication of the idea of the containment of the conclusion in the premises in a valid argument.

Now one theme which emerges quite strongly from the work of those engaged in following up Gentzen's idea is that the conception of validity which will result from this program may well be one under which distinctively classical rules of inference turn out to be invalid. In the nature of the case it cannot be conclusively shown to the satisfaction of all contending parties that classical logic fails to pass natural constraints on the interrelations of inference rules, since constraints natural for classicists may not be natural for nonclassicists and vice versa. But it is not unreasonable for an opponent of classical logic to hope that proof-theoretic constraints of sufficient generality may gain the approval of classicists, yet turn out to be violated by classical logic and hence provide persuasive reasons for the classicist to abandon his logic. Thus classical logic fails, in its standard formulations, to satisfy the simple Subformula Principle and Conservativeness Principle mentioned above, though these principles are satisfied by intuitionist logic. Negation is proof-theoretically creative, in classical logic. Adding it to the $\rightarrow$ fragment of classical logic yields Peirce's Law:

$$(((A \rightarrow B) \rightarrow A) \rightarrow A)$$

as a new theorem, for instance. Adding it to the $\rightarrow$, $\vee$ fragment generates as a new theorem

$$(((A \rightarrow (B \vee C)) \rightarrow ((A \rightarrow B) \vee (A \rightarrow C))) \ .$$

Similarly one cannot, without some forcing, view classical logic as given by a system of introduction rules together with elimination rules justifiable in terms of the former by means of Prawitz' Inversion Principle, for the distinctively classical principles, such as double negation elimination or classical reductio, resist any simple attempt to make them conform to the Principle (cf. [4], pp. 34–35). But note also the anomolous position of *ex falso quodlibet* inside the scheme of introduction/elimination rules. Prawitz, indeed, was led to conjecture that minimal logic alone was sound and complete with respect to the type of validity concept he was after ([6], p. 246).

For a classicist, of course, the preservation of the truth of the premises in the conclusion is the only important conservation principle and the proof-theoretically creative powers of negation are hardly surprising, given the tight fit between a sentence and its negation in a bivalent language. That classical logic is too strong to satisfy purely proof-theoretic attempts to make sense of the notion of the containment of conclusion in premises is, again, unsurprising given the radically nonepistemic notion of truth used in classical semantics; nor will it worry those with a sufficiently hard-headed view of the severe limitations on human epistemic capacities.

This forthright style of response on the part of the classicist seems to me to be essentially right. On the other hand, it would be blind dogmatism to suppose that the classical conception of truth as independent of proof stands in no need of justification. The rebuttal of skeptical attempts to show that the classical notions are incompatible with plausible assumptions in the theory of meaning is a challenge which the classicist ought to try to meet (though there is no legitimate requirement that he do so without availing himself of classical logic (cf. [10], pp. 178–179)). In lieu of such a rebuttal, which would by no means be a trivial matter, it may be worthwhile for classicists to turn their attention to proof-theoretic harmony constraints between introduction and elimination rules to see if the omens in this area really do point away from classical logic.

Now the classicist will clearly not be after noncreativity constraints designed to filter out overly strong logics which generate conclusions not contained in the premises. Rather the classicist will look for "nonleakage" constraints which are designed to rule out overly weak logics which fail to generate all the conclusions contained in the premises. If the most natural "nonleakage" and noncreativity constraints point to different logics as the correct one the classicist need not be too dismayed, for he need not seek a clear superiority of classical logic over its rivals in proof-theoretic terms. If we can find principles in terms of which classical logic comes out on top, and which are at least as natural as those under which it fares badly, then the proof-theoretic arguments balance out. This is to the advantage of classical logic, since the prejudices of most logicians are in its favor and, hence, the onus is on the deviant logician to produce persuasive grounds for abandonment.

In this paper, I shall attempt to neutralize the proof-theoretic arguments against classical logic by formulating a "nonleakage" harmony constraint between introduction and elimination rules which I claim is at least as natural as Prawitz' Inversion Principle (and, indeed, more justifiably termed an inversion principle than his) and, yet, which classical logic, but not intuitionistic logic, satisfies. First, the structure of introduction and elimination rules must be laid down with sufficient generality for the task at hand.

2 Consider the following as a schematic introduction rule for an n-place sentential connective Φ:

(1)
$$\frac{\begin{array}{cc} \overrightarrow{\{A^i_1\}} & \overrightarrow{\{A^i_{r(i)}\}} \\ \vdots & \vdots \\ P^i_1 & P^i_{r(i)} \end{array}}{\Phi(C_1, \ldots C_n)}$$

The notation used here is to be understood as follows. The As, Ps and Cs are sentence schemata constructed from sentence variables by means of sentential connectives. The Cs represent constituents of the conclusion, the P_1 to $P_{r(i)}$s represent immediate premises of the rule and the set $\overrightarrow{\{A_j\}}$ is the set of assumptions on which P_j depends discharged by the application of the introduction rule. The superscript i above the discharged assumptions and immediate premises is to allow for the possibility that more than one introduction rule is provided for $\Phi(C_1, \ldots, C_n)$. (Thus the standard disjunction introduction has two forms, one with the left disjunct as immediate premise, the other with right premise.)

Where all the sentence schemata in all the rule schemata are unstructured sentence variables, then we may call the rule **independent**. Where all the assumption and premise sentence variables in each rule schemata are subformulas of the Cs then we may call the rule **proper**. (1), then, is to be read: Given a proof, for each j, $1 \leq j \leq r(i)$ of immediate premise P^i_j, from the set of assumptions $\overrightarrow{\{A^i_j\}}$, possibly together some further set of assumptions $\overrightarrow{\{S^i_j\}}$, then we may infer $\Phi(C_1, \ldots, C_n)$, this conclusion depending on the assumptions $\cup^{r(i)}_{j=1} \overrightarrow{\{S_j\}}$, that is the union of all the undischarged assumptions on which each immediate premise depends.

An instance of the rule is the result of uniform substitution of sentences for sentence variables in one of the rule schemata.

As a schematic elimination rule consider:

(2)
$$\frac{\Phi(C_1,\ldots C_n) \quad \begin{matrix}\overrightarrow{\{A_1^i\}} \\ \vdots \\ M_1^i\end{matrix} \quad \begin{matrix}\overrightarrow{\{A_{r(i)}^i\}} \\ \vdots \\ M_{r(i)}^i\end{matrix}}{E_i}$$

Here M_j is a minor premise for an application of the elimination rule to $\Phi(C_1,\ldots C_n)$ with conclusion E_i, the set $\overrightarrow{\{A_j\}}$ being the set of assumptions on which M_j depends discharged by the application of the rule. Again we allow, through the variable i, for the possibility that there is more than one elimination rule for $\Phi(C_1,\ldots C_n)$ (as, for example, with & elimination).

(2) is, therefore, to be read: Given a proof, for each j, $1 \le j \le r(i)$, of minor premise M_j^i, depending on the set of assumptions $\overrightarrow{\{A_j^i\}}$, and possibly also on the set $\overrightarrow{\{S_j^i\}}$, together with a proof of $\Phi(C_1,\ldots,C_n)$, possibly dependent on the set of assumptions $\overrightarrow{\{S_0^i\}}$, then we may infer E_i, this conclusion depending on $\bigcup_{j=0}^{r(i)}\overrightarrow{\{S_j^i\}}$.

Independent and proper rules can be defined as before, extending this time the definition to cover the minor premises and the elimination consequences, as well as major premises and assumptions.

Given rules of these forms, we can define proofhood recursively, the base clause ensuring that every sentence is a proof of itself dependent on itself, and inductive clauses extending proofs to further proofs by means of one of the above types of rules. (These proposals for introduction and elimination rules are taken largely from [5], pp. 35–37.)

3 Armed, then, with characterizations of introduction and elimination rules, what alternative proposals to Prawitz' Inversion Principle, as a condition on harmony between the two types of rules, can be made? My idea is to take Prawitz' notion of an inverse relationship between corresponding introduction and elimination rules rather more literally than he does. Take, for example, normalization of a maximal disjunct in a system with standard disjunction rules:

$$\frac{\dfrac{\begin{matrix}\Delta \\ \vdots \\ A_i\end{matrix}}{A_1 \vee A_2} \quad \begin{matrix}\overline{A_1}^{\,n} \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overline{A_2}^{\,n} \\ \vdots \\ C\end{matrix}}{C}\,n \quad \Rightarrow \quad \begin{matrix}\Delta \\ \vdots \\ A_i \\ \vdots \\ C\end{matrix} \qquad i = 1,2$$

What we have here, as in Prawitz' reduction procedures generally, is the elimination of an unnecessary detour. We now arrive at the same conclusion, dependent on the same premises, more directly. But, *pace* what Prawitz says,[1] there is nothing one might wish to call **inversion** here. For we have not restored what had already been established prior to disjunction introduction, namely, A_i. Compare functions. One might substitute for the composition of two functions f and g a function h at least as simple as the simplest of f and g, hence simpler than fOg, and such that h agrees with fOg on every argument for which fOg is defined. This would clearly represent a gain in simplicity. But such a function has nothing to do with an inverse function f^{-1} such that fOf^{-1} is the identity function. Could we treat introduction and elimination rules (I rules and E rules for short) as inverses in a sense strongly analogous with "inverse" as it applies to functions? This would seem to require that application of an I rule (E rule) for a constant K immediately followed by application of an E rule (I rule) for that same constant should leave us in precisely the same position we started from just prior to application of the first rule. This is clearly not the case with a maximal occurrence of disjunction: C need not be identical with A_i, the premise of the original application of $\vee I$, nor need the assumptions on which it depends be just those on which A_i depended at that stage. However, compare the following proofs involving conjunction:

$$\dfrac{\dfrac{\begin{matrix}\Delta & \Gamma\\ \vdots & \vdots \\ A & B\end{matrix}}{A\,\&\,B}}{A} \qquad \dfrac{\dfrac{\begin{matrix}\Delta & \Gamma\\ \vdots & \vdots\\ A & B\end{matrix}}{A\,\&\,B}}{B}$$

These illustrate a sense in which standard $\&I$ and $\&E$ rules are, in one direction, inverses of one another. Our starting point was two proofs, one of A and one of B. From this position (after reduplicating this starting point on the right) we can return to the starting point, those two conclusions dependent on the same undischarged premises, by applying $\&I$ immediately followed by $\&E$. The rules are in a direct sense inverses of one another.

What about the opposite direction: $\&E$ followed by $\&I$? Here the starting point is a proof of A & B. If the rules are inverses in this direction then by proceeding from this point (possibly reduplicated) by applications of $\&E$ followed by $\&I$ we must be able to return to a proof of A & B from the same undischarged assumptions. This is easily seen to be the case, as follows:

$$\dfrac{\dfrac{\begin{matrix}\Delta\\ \vdots\\ A\,\&\,B\end{matrix}}{A} \quad \dfrac{\begin{matrix}\Delta\\ \vdots\\ A\,\&\,B\end{matrix}}{B}}{A\,\&\,B}$$

If we turn to $\to$ we come across a more complicated case. The starting point for standard $\to I$ is a proof of some sentence B possibly dependent on another sentence A. The rule then gives us:

$$\dfrac{\begin{matrix}\Delta\,\overline{A}^{\,n}\\ \vdots\\ B\end{matrix}}{A\to B}\,{}^{n}$$

From here alone there is no route to the starting point, a proof of B from Δ and A. But in applying $\rightarrow I$ we have, of course, discharged all occurrences of A as assumption. Hence it is entirely legitimate to introduce A as a further premise. In doing so we do not add to the premises present at our starting point. And now we are easily able to show that in the introduction followed by elimination direction $\rightarrow I$ and $\rightarrow E$ are inverses[2]:

$$\dfrac{\dfrac{\begin{array}{c}\Delta\overline{A}^{n}\\ \vdots\\ B\end{array}}{A \rightarrow B}\quad \overline{A}^{n}}{B}$$

Similarly, from the condition for application of $\rightarrow E$, namely a proof of $A \rightarrow B$, one is not going to get very far. But if one allows oneself to add the requisite minor premise, thus bloating, normally, the number of undischarged assumptions, one can still return, by application of $\rightarrow E$ followed by $\rightarrow I$, to the original situation. For one can, of course, use the discharge permission in the introduction rule as follows:

$$\dfrac{\dfrac{A \rightarrow B \quad \overline{A}^{1}}{B}}{A \rightarrow B}{}^{1}$$

All this suggests a rather different two-part Inversion Principle from Prawitz', one part concerned with introduction rules followed by elimination rules, the other concerned with the opposite direction. First of all, let us say that Π, a proof or sequence of proofs, is a sufficient condition for Φ under R iff application of R to the conclusion(s) of Π yields Φ. Then, suppose that Π is a sufficient condition for $\Phi(C_1, \ldots C_n)$ under an introduction rule R for Φ. Graphically:

$$\begin{array}{c}\Pi\\ \Phi(C_1,\ldots C_n)\end{array} = \dfrac{\begin{array}{c}\overrightarrow{\{A^i_1\}}\\ \vdots\\ P^i_1\end{array}\quad \begin{array}{c}\overrightarrow{\{A^i_{r(i)}\}}\\ \vdots\\ P^i_{r(i)}\end{array}}{\Phi(C_1,\ldots C_n)}{}^{I}$$

When this holds, the Inversion Principle then requires that there be a sequence of proofs each formed by deriving $\Phi(C_1, \ldots C_n)$ from Π by R, adding the discharged assumptions $\overrightarrow{\{A^i_j\}}$, $1 \leq j \leq r(i)$, as minor premises, and applying an elimination rule for Φ, and which is such that this sequence is also a sufficient condition for Φ under R, with the same major premises and dependent on the same assumptions as Π. We can represent this graphically as:

$$\dfrac{\begin{array}{c}\Pi\\ \Phi(C_1,\ldots C_n)\end{array}\overrightarrow{\{A^i_1\}} \;\ldots\; \overrightarrow{\{A^i_{r(i)}\}}}{P^i_1}{}^{E} \;\ldots\; \dfrac{\begin{array}{c}\Pi\\ \Phi(C_1,\ldots C_n)\end{array}\overrightarrow{\{A^i_1\}} \;\ldots\; \overrightarrow{\{A^i_{r(i)}\}}}{P^i_{r(i)}}{}^{E}$$

In the elimination followed by introduction direction we will require that, when Π is a sufficient condition for an application of elimination rule R to Φ, graphically

$$\begin{array}{c}\Pi\\ E_i\end{array} = \dfrac{\Phi(C_1,\ldots C_n)\quad \begin{array}{c}\overrightarrow{\{A_1^i\}}\\ \vdots\\ M_1^i\end{array}\quad \begin{array}{c}\overrightarrow{\{A_{r(i)}^i\}}\\ \vdots\\ M_{r(i)}^i\end{array}}{E_i}\,E$$

there is a proof whose conclusion is $\Phi(C_1,\ldots C_n)$ and which consists of a sequence of instances of Π, each followed by R, which is in turn followed by an instance of an introduction rule for Φ, at which all the minor premises M_j^i, $1 \leq j \leq r(i)$, are discharged. Graphically:

$$\dfrac{\dfrac{\begin{array}{c}\overline{M_1^1}^{\,!}\quad \overline{M_{r(i)}^i}^{\,!}\\ \Pi\end{array}}{E_1}\,E \quad \begin{array}{c}\cdots\ \cdots\\ \cdots\ \cdots\end{array}\quad \dfrac{\begin{array}{c}\overline{M_1^1}^{\,!}\quad \overline{M_{r(i)}^i}^{\,!}\\ \Pi\end{array}}{E_s}\,E}{\Phi(C_1,\ldots C_n)}\,I!$$

(Here ! marks the step at which the minor premises are discharged by application of the I-rule.)

Putting the two parts together we get the following

Inversion Principle

(a) *I-E*

$$\begin{array}{c}\Pi\\ \Phi(C_1\ \ldots,C_n)\end{array} = \dfrac{\begin{array}{c}\overrightarrow{\{A_1^i\}}\\ \vdots\\ P_1^i\end{array}\qquad \begin{array}{c}\overrightarrow{\{A_{r(i)}^i\}}\\ \vdots\\ P_{r(i)}^i\end{array}}{\Phi(C_1,\ ..C_n)}\,I \qquad \Rightarrow$$

$$\dfrac{\begin{array}{c}\Pi\\ \Phi(C_1,\ ..C_n)\end{array}\ \overrightarrow{\{A_1^i\}}\ \ldots\ \overrightarrow{\{A_{r(i)}^i\}}}{P_1^i}\,E \quad \begin{array}{c}\cdots\\ \cdots\end{array}\quad \dfrac{\begin{array}{c}\Pi\\ \Phi(C_1,\ ..C_n)\end{array}\ \overrightarrow{\{A_1^i\}}\ \ldots\ \overrightarrow{\{A_{r(i)}^i\}}}{P_{r(i)}^j}\,E$$

When the sufficient condition for application of an I-rule obtains, application of that rule followed immediately by application of elimination rules for the relevant constant returns us to the sufficient condition for application of the I-rule.

(b) *E-I*

$$\underset{E_i}{\Pi} = \cfrac{\Phi C_1, \ldots C_n) \quad \begin{matrix}\{A^i_1\} \\ \vdots \\ M^i_1\end{matrix} \quad \begin{matrix}\{A^i_{r(i)}\} \\ \vdots \\ M^i_{r(i)}\end{matrix}}{E_i}\,E \quad \Rightarrow$$

$$\cfrac{\cfrac{\cfrac{}{M^1_1}! \quad \cfrac{}{M^i_{r(i)}}! \quad \Pi}{E_1}\,E \quad \cdots\cdots \quad \cfrac{\cfrac{}{M^1_1}! \quad \cfrac{}{M^i_{r(i)}}! \quad \Pi}{E_s}\,E}{\Phi(C_1, ..C_n)}\,I!$$

When the sufficient condition for application of an *E*-rule obtains, application of that rule followed immediately by application of introduction rules for the relevant constant returns us to the sufficient condition for application of the *E*-rule.

This principle seems to me to require introduction and elimination rules to be inverses of one another in a sense as close as one could get to the sense in which a function can be the inverse of another. When this principle holds for a pair of rules, then, given that one has arrived at the sufficient condition for the application of one rule, application of that rule followed immediately by that of its inverse returns one to the sufficient condition for the first. Note also that this principle is, in a sense, the inverse of Prawitz' Inversion Principle. Where the latter was concerned with whether certain types of redundancy in a proof can be eliminated, the principle above is concerned with the opposite. It holds if we can find redundant loops in which one type of rule followed by the other returns us to exactly the same place.

Of course, when concerned with the first-order business of attempting to prove object language sentences, such redundancy in proofs is not a virtue. But here we are concerned with the metatheoretic business of formulating a natural harmony constraint relating introduction and elimination rules, and we need not require that the proofs which exhibit the working of the principle be useful for any other purpose. The above principle does formulate a neat and obvious balance between *I* and *E* rules. We may say, then, that a system which is too weak to balance its *I* and *E* rules in this way, by deriving the conditions for applying one type of rule from the conclusions of the other type applied to those conditions, does not extract all that is contained in the premises—there is some "leakage". This logical system is inadequate in the way a set of mathematical axioms is inadequate if it is supposed to characterize a structure in which the inverse of every function expressible in the formalization exists yet the existence theorems in question are not provable from the axioms.

True, this explication of the notion of containment is somewhat strained. But so too, it seems to me, is the explication which invokes Prawitz' Harmony

Principle. If it seems less so, this is only because one has already decided on an approach in which a subset of rules are distinguished as canonical. Granted this, the idea of characterizing other rules as creative or not immediately suggests itself. This whole approach, therefore, sets up a severe soundness test; one might say it is weighted against strength in logics. But it is not much use as a completeness test since any logic which contains the full set of logical constants together with the canonical rules which, it is claimed, give them their meaning, will be complete according to a definition of validity in which an argument is valid if canonical rules applied to canonical proofs of the premises can generate its conclusion. If we are after a challenging proof-theoretic criterion of completeness we must choose a criterion which is biased more in the opposite direction, namely, against weakness in a logical system. We have to propose proof-theoretic powers which we think are inherent in the logical constants and which must be revealed in any complete proof system. The balance between introduction and elimination rules for a given constant required by the above Inversion Principle seems as natural a criterion of that form as any.

We have seen how the standard rules for & and $\rightarrow$ meet the constraint. Are there rules for the other connectives which also meet it? Negation can be brought into the scope of the principle, of course, by defining $\sim$A as $(A \rightarrow *)$, $*$ being some absurdity constant, and subsuming it under the case of the conditional. But it also satisfied the principle directly, when given these introduction and elimination rules:

$\sim I$:

$$\begin{array}{c} \overline{A}^{1} \\ \vdots \\ * \\ \overline{\sim A}^{1} \end{array}$$

$\sim E$:

$$\begin{array}{c} \sim A \qquad A \\ \hline * \end{array}$$

That it satisfies the principle directly can be shown as follows:

(a)

$$\begin{array}{c} \overline{A}^{1} \\ \vdots \\ * \\ \overline{\sim A}^{1} \qquad A \\ \hline * \end{array}$$

(b)

$$\begin{array}{c} \sim A \qquad \overline{A}^{1} \\ \hline * \\ \overline{\sim A}^{1} \end{array}$$

But disjunction, with its standard rules, clearly does not satisfy the principle. Having concluded $A \vee B$ from A by $\vee I$, there is no way, in general, to conclude A from the disjunction by $\vee E$. Nor is it possible, in general, to derive $A \vee B$ by $\vee I$ from a conclusion C of an application of $\vee E$.

Take, however, these rules for disjunction:

$\vee I*$:

$$\dfrac{\begin{array}{c}\overline{\sim B}^{\,1}\\ \vdots \\ A\end{array}}{A \vee B}\,1 \qquad \dfrac{\begin{array}{c}\overline{\sim A}^{\,1}\\ \vdots\\ B\end{array}}{A\vee B}\,1$$

$\vee E*$:

$$\dfrac{A\vee B \qquad \sim A}{B} \qquad\qquad \dfrac{A\vee B\qquad \sim B}{A}$$

These satisfy the Inversion Principle as follows:

(a)

$$\dfrac{\dfrac{\begin{array}{c}\overline{\sim B}^{\,1}\\ \vdots\\ A\end{array}}{A\vee B}\,1 \qquad \sim B}{A}$$

(with a symmetrical proof where B is the premise of $\vee I$).

(b)

$$\dfrac{\dfrac{A\vee B \qquad \overline{\sim A}^{\,1}}{B}}{A\vee B}\,1$$

(with a symmetrical proof where $\sim B$ is minor premise). Furthermore, these rules constitute a formulation of classical logic (I will call this formulation C^- henceforth), for excluded middle, *ex falso quodlibet*, and standard disjunction elimination are all derivable using them:

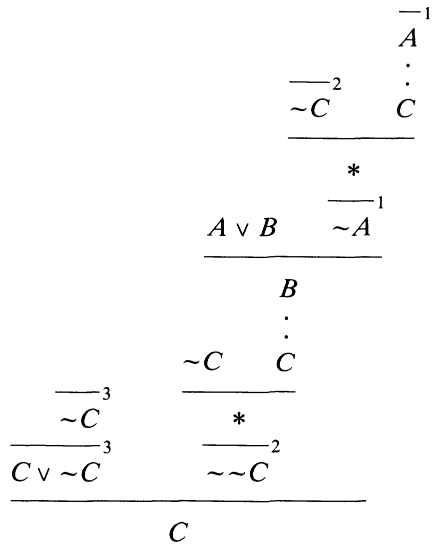
Excluded Middle:

$$\dfrac{\overline{\sim A}^{\,1}}{A\vee \sim A}\,1$$

Ex Falso Quodlibet:

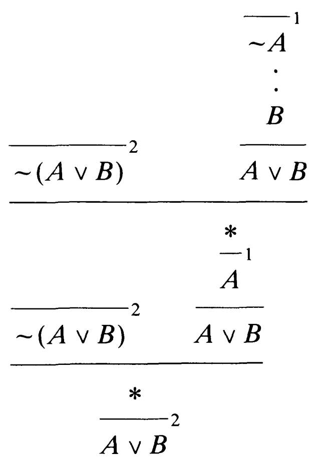
$$\dfrac{\dfrac{A}{A\vee B}\qquad \sim A}{B}$$

Standard $\vee E$:



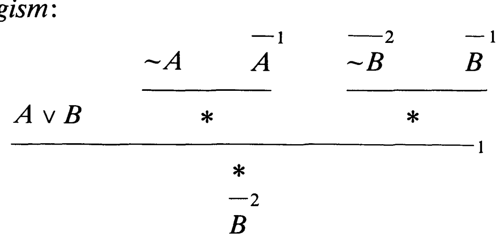
Conversely, $\vee I*$ and $\vee E*$ (disjunctive syllogism) are derivable in the standard classical systems (taking classical *reductio ad absurdum* as the classical addition to the standard intuitionist natural deduction system):

$\vee I*$:



(The proof is symmetrical where $\sim B$ is the discharged assumption in $\vee I*$.)

Disjunctive syllogism:



(The proof is symmetrical in the other case.)

Can the Inversion Principle be extended to the predicate calculus? In order to do so, we must generalize the schemata for introduction and elimination rules. This could be done by permitting the sentence variables to stand for sentences with free variables, the schemata being constructed by means of quantifiers as well as sentential connectives. We should also allow that we may specify, with respect to a subproof of a premise P_j^i, or minor premise M_j^i, a list of $\overrightarrow{\text{variables}}$ none of which must occur free in any of the undischarged premises in $\{S_j^i\}$ (see [8], p. 361). On this understanding of more general natural deduction rules it can be seen that the standard universal quantifier rules obey the Inversion Principle (in what follows *Ft* stands for the result of substituting a closed term *t*, new to *F*, for all free occurrences of *x* in *F*):

$$\cfrac{\cfrac{\begin{matrix}\Delta\\ \vdots\\ Ft\end{matrix}}{\forall xFx}}{Ft} \qquad t \text{ not in } \Delta.$$

$$\cfrac{\cfrac{\begin{matrix}\Delta\\ \vdots\\ \forall xFx\end{matrix}}{Ft}}{\forall xFx} \qquad t \text{ not in } \Delta.$$

Just as clearly, the standard existential rules do not obey the Inversion Principle. The constraints on the parameter in the existential elimination rule prevent one from deriving, by its means, a premise, *Ft*, of existential introduction from the conclusion, $\exists xFx$, of the introduction rule. This is because *t* cannot occur in the conclusion of the rule if it occurs in the discharged premise. Yet, there is no other way for *Ft* to be derived. Similarly, one cannot derive, by $\exists I$ alone, $\exists xFx$ from an arbitrary conclusion, *C*, of $\exists E$.

The amended rules for disjunction, however, suggest the following amended rules for the existential quantifier:

$\exists I*$

$$\cfrac{\begin{matrix}\overline{\forall x(\sim x = t \to \sim Fx)}^{\,n}\\ \vdots\\ Ft\end{matrix}}{\exists xFx}\,n$$

$\exists E*$

$$\frac{\exists xFx \qquad \forall x(\sim x = t \to \sim Fx)}{Ft}$$

It is easily seen that these rules obey the inversion constraint:

I-E

$$\dfrac{\dfrac{\begin{array}{c}\overline{\forall x(\sim x = t \to \sim Fx)}^{\,1}\\ \vdots \\ Ft\end{array}}{\exists xFx}\,{}^{1} \qquad \forall x(\sim x = t \to \sim Fx)}{Ft}$$

E-I

$$\dfrac{\dfrac{\overline{\exists xFx \qquad \forall x(\sim x = t \to \sim Fx)}^{\,1}}{Ft}}{\exists xFx}\,{}^{1}$$

Moreover, if we permit ourselves standard identity rules in order to treat of the identity constant introduced (say, reflexivity and Leibniz' Law), then the system of logic which results is the classical predicate calculus with identity (I will call this system *C*). For, from one set of rules, the other can be generated as derived rules. To illustrate:

$\exists I \Rightarrow \exists I*$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\overline{\forall x(\sim x = t \to \sim Fx)}^{\,1}\\ \vdots \\ Ft\end{array}}{\exists xFx} \qquad \overline{\sim\exists xFx}^{\,3}}{*}}{\sim\forall x(\sim x = t \to \sim Fx)}\,{}^{1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\sim\exists xFx}^{\,3} \qquad \dfrac{\overline{Fa}^{\,2}}{\exists xFx}}{*}}{\sim Fa}\,{}^{2}}{\sim a = t \to \sim Fa}}{\forall x(\sim x = t \to \sim Fx)}}{\dfrac{*}{\exists xFx}\,{}^{3}}$$

where a is chosen so as not to occur in F.

$\exists I* \Rightarrow \exists I$

$$\dfrac{Ft}{\exists xFx}$$

$\exists E \Rightarrow \exists E*$

$$\dfrac{\exists x Fx \qquad \begin{array}{c} \dfrac{\overline{Fa}^{\,2} \qquad \dfrac{\overline{\sim a = t}^{\,1} \qquad \dfrac{\forall x(\sim x = t \rightarrow \sim Fx)}{\sim a = t \rightarrow \sim Fa}}{\sim Fa}}{} \\ * \\ \dfrac{\overline{a = t}^{\,1} \qquad \overline{Fa}^{\,2}}{Ft} \end{array}}{Ft}\,2$$

$\exists E* \Rightarrow \exists E$ Here we require a lemma on substitution in proofs. In standard natural deduction systems we have:

Substitution Lemma *If* Π *is a proof of* Φ *from* Δ *then* $\Pi\ a^t_a$ *is a proof of* Φ^t_a *from* Δ^t_a, *t and a being closed terms.*

(Δ^t_a is the set of Ψ such that Ψ is Θ^t_a, for some Θ in Δ, Θ^t_a is the result of uniform replacement of t by a in Θ, Πa is the result of uniformly replacing all occurrences of a as a parameter in a quantifier rule by some term distinct from a and new to Π, and Π^t_a is the result of substituting a for t in every formula in Π.)

A proof of this lemma can be found in [9], pp. 67–69. It can be checked that the same reasoning yields a proof of the lemma with regard to C.[3]

Hence, if we have a proof of the form:

$$\begin{array}{c} Ft\Delta \\ \Pi \\ C \end{array}$$

there exists also a proof of the form:

$$\begin{array}{c} Fa^t_a \qquad \Delta^t_a \\ \Pi\ a^t_a \\ C^t_a \end{array}$$

Furthermore, if the conditions for $\exists E$ are satisfied, then $F^t_a = Fa$, since t is not in Fx; $\Delta^t_a = \Delta$, since t is not a member of Δ; and $C^t_a = C$, since t is not in C. Hence, the following is a proof in the amended system:

$$\dfrac{\exists xFx \qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[Fa]^{1}\ \Delta \\ \Pi\ a^{t}_{a} \\ C\end{array} \qquad [\sim C]^{2}}{*}}{\sim Fa}\,{}^{1}}{\sim a = t \rightarrow \sim Fa}\ !}{\forall x(\sim x = t \rightarrow \sim Fx)}}{Ft}$$

$$\dfrac{\dfrac{\dfrac{[\sim C]^{3}}{}}{C \vee \sim C}\,{}^{3} \qquad \dfrac{\dfrac{\begin{array}{c}Ft \quad \Delta \\ \Pi \\ C\end{array} \qquad [\sim C]^{2}}{*}}{\sim\sim C}\,{}^{2}}{C}$$

(Note that the application of $\forall I$ marked ! is correct since a is not in C, or in any undischarged assumptions on which C depends.)

Classical predicate calculus with identity, then, admits of a formulation, purely in terms of introduction and elimination rules plus the rules for the identity predicate, in which the rules harmonize in the manner laid down by the above Inversion Principle. Intuitionistic logic, however, does not admit of a "standard" formulation which satisfies that principle.

To illustrate what I mean by standard, consider proofs of $A \vee B$ from A and from B, both atomic. A standard formulation of intuitionistic logic would possess a normalization property: these proofs will be transformable into normal form proofs possessing the usual intuitionistic normal form properties. For one thing, nothing will occur in the proof unless it is a subformula of the conclusion or of a premise. So only formulas built up from A and B by means of $\vee$ will occur, and, hence, only disjunction rules will feature. Moreover, every path in the proof can be divided into an elimination section (perhaps empty) followed by an introduction section (perhaps empty). All formulas in the elimination section are proper subformulas of the preceding formula (if there is one) and all formulas in the introduction segment are proper subformulas of the succeeding one (if there is one), the sections divided by a minimum formula which is a subformula of the succeeding one, unless it is the absurdity constant.[4]

Now the last step in the normal form proof of $A \vee B$ cannot be $\vee E$ else there will be a path containing only major premises of $\vee E$ and hence a complex undischarged assumption. The conclusion must be obtained, then, by $\vee I$, the premises being proper subformulas of $A \vee B$, i.e., A in one case, B in the other. In order for this to be possible, the $\vee I$ rule must be an instance of a rule with the general form:

$\vee I$:

$$\dfrac{\begin{array}{c}\overline{\overrightarrow{\{C_i\}}}^{\,n}\\ \vdots \\ A\end{array}}{A \vee B}\,n \qquad \dfrac{\begin{array}{c}\overline{\overrightarrow{\{D_j\}}}^{\,n}\\ \vdots\\ B\end{array}}{A\vee B}\,n$$

But no such $\vee I$ rules can be part of a formulation of intuitionistic logic which meets the above Inversion Principle. For, if the right-hand introduction rule met the principle, then there would be proofs of this form:

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array}\quad \begin{array}{c}\vdots\\ \overrightarrow{\{D_j\}}\end{array}}{B}\,E$$

But take the special case where $B = {\sim}A$. Since the following is a rule, primitive or derived, in intuitionistic logic:

$$\dfrac{\begin{array}{c}\Delta\ \overline{A}^{\,n}\\ \vdots\\ {\sim}A\end{array}\quad \overline{A}^{\,n}}{{\sim}A}\,n$$

then, if the Inversion Principle were to be satisfied, we would have the following proof, A atomic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{\,1}}{A\vee{\sim}A}\quad \overline{\overrightarrow{\{D_j\}}}^{\,2}}{{\sim}A}\qquad \overline{A}^{\,1}}{{\sim}A}\,1}{A\vee{\sim}A}\,2$$

But, of course, $A \vee \sim A$ is not provable in intuitionistic logic, where A is atomic (as can be shown proof-theoretically or semantically using, e.g., Kripke trees). Hence the Inversion Principle fails for any standard formulation of intuitionistic logic.

4 We have seen, then, that there is a formulation of classical logic which satisfies the Inversion Principle, though no standard formulation of intuitionistic logic does so. Intuitionistic logic, however, clearly meets Prawitz' Inversion constraint while standard formulations of classical logic do not. Neither does the above formulation C^{-}. Disjunction is the stumbling block. For disjunction, there are two possible types of occurrence of maximal formulas for each introduction rule:

(a)

$$\dfrac{\dfrac{\begin{array}{c}\overline{\sim B}^{\,1}\ \ \Delta\\ \vdots\\ A\end{array}}{A \vee B}\,{}^{1} \qquad \begin{array}{c}\Gamma\\ \vdots\\ \sim B\end{array}}{A}$$

and

(b)

$$\dfrac{\dfrac{\begin{array}{c}\overline{\sim B}^{\,1}\ \ \Delta\\ \vdots\\ A\end{array}}{A \vee B}\,{}^{1} \qquad \begin{array}{c}\Gamma\\ \vdots\\ \sim A\end{array}}{B}$$

(The case is symmetrical where B is the premise of $\vee I*$. Note that the derivation of *ex falso quodlibet* on p. 19 is abnormal.)

Case (a) is easily handled by transforming the maximal fragment to:

$$\begin{array}{c}\Gamma\\ \vdots\\ \sim B\ \ \Delta\\ \vdots\\ A\end{array}$$

where if $\sim B$ is maximal in the new proof, it is nonetheless of no greater logical complexity than $A \vee B$ and eliminable, in turn, by the $\sim$ reduction step.

But the best we can do in case (b), it seems, is:

$$\dfrac{\dfrac{\overline{\sim B}^{\,2}}{B \vee \sim B}\,{\scriptstyle 2} \qquad \dfrac{\dfrac{\begin{matrix}\overline{\sim B}^{\,1}\ \Delta \\ \vdots \\ A\end{matrix} \quad \begin{matrix}\Gamma\\ \vdots \\ \sim A\end{matrix}}{*}}{\sim\sim B}\,{\scriptstyle 1}}{B}$$

Here the occurrence of $B \vee \sim B$ is maximal of type b and may be of greater logical complexity than $A \vee B$. An attempt to apply the normalization procedure to this proof would simply lead to

$$\dfrac{\dfrac{\overline{\sim B}^{\,3}}{B \vee \sim B}\,{\scriptstyle 3} \qquad \dfrac{\dfrac{\overline{\sim B}^{\,2} \qquad \dfrac{\dfrac{\begin{matrix}\overline{\sim B}^{\,1}\ \Delta \\ \vdots \\ A\end{matrix} \quad \begin{matrix}\Gamma\\ \vdots \\ \sim A\end{matrix}}{*}}{\sim\sim B}\,{\scriptstyle 1}}{*}}{\sim\sim B}\,{\scriptstyle 2}}{B}$$

and thus to a normalization loop.

The situation can, however, be improved a little: we can carry out further transformations of the form of Prawitz "atomizing" transformations ([4], p. 40) to ensure that the only maximal formulas of the form $A \vee \sim A$ have an atomic disjunct (let us call a formula of that form a "*LEM*" and say that it is a *LEM* of A). The general transformation we are after acts on fragments of proof of the form:

$$\dfrac{\dfrac{\overline{\sim A}^{\,2}}{A \vee \sim A}\,{\scriptstyle 2} \qquad \dfrac{\begin{matrix}\sim A\ \ \Delta \\ \vdots \\ *\end{matrix}}{\sim\sim A}}{A}$$

(that is, proofs of classical reductio within the amended system) to produce new proofs of A from Δ in which the number of maximal *LEM* formulas of the same complexity as $A \vee \sim A$ is reduced (and no new such formulas of higher complexity are created). Successive applications of such transformations ensure that any maximal *LEM* formulas which occur in the final proof have an atomic disjunct. If we describe a proof in that state, with no other type of maximal formulas, as in normal form, the result is that every proof in C^- can be transformed into a normal form proof of the same conclusion, from premises among those in the original proof.

Can this normalization result lead to any interesting Subformula property? The following *is* provable:

Subformula Principle *Every formula in a normal proof in* C^- *is*
(a) *a subformula of a premise or the conclusion (***subformula**, *for brevity); or, if not, is*
(b) *the negation of a* **subformula***; or, if neither, is*
(c) *a maximal LEM with an atomic disjunct which is a* **subformula***; or, if none of these, is*
(d) *a double negation of an atomic* **subformula** *and the minor premise of an application of* $\vee E*$ *with major premise as in* (c).

But this could hardly be claimed to be a very interesting or natural subformula property.

The classicist may claim that there is nothing surprising in negations, double negations and instances of the law of excluded middle cropping up nonconservatively in classical proofs, given the law of bivalence which holds in the intended interpretation of classical logic. But when we move to *C* things become even worse. For the sentence discharged when concluding $\exists xFx$ by $\exists I*$, and also used as minor premise in $\exists E*$, with that generalization as major premise, namely, $\forall x(\sim x = t \rightarrow \sim Fx)$, bears no simple relation at all to $\exists xFx$. Even the Subformula Principle above, then, does not hold in *C*. There is a more restricted principle, however, which does hold. Let us say that $P \leq Q$ iff every predicate constant, individual constant not occurring as a parameter in $\forall I$, and atomic sentence variable which occurs in *P*, also occurs in *Q*. Then

Restricted Subformula Principle *Every formula in a normal proof in C bears* $\leq$ *to a premise or to the conclusion.*

A classicist might claim that the above Subformula Principle is as neat as needs be required, it being unsurprising that $\forall$, $\rightarrow$, $\sim$, $\vee$, and $=$ are proof-theoretically creative in classical logic given the known interdefinabilities among them (apart from identity). But there is no gainsaying the untidiness of the subformula principle. This derives from one main source: the nonindependence and impropriety of the $\vee$ and $\exists$ rules. Structured schemata involving negation come into the $\vee$ rules. The $\exists$ rules are even worse off, involving no less than $\forall$, $\sim$, $\rightarrow$, and $=$. Hence the rules are not independent. Moreover, for no $\vee$ or $\exists$ rule are all the premises or assumptions subformulas of the constituents of the conclusion (in the *I* case) or a major premise (in the *E* case). Hence they are not proper.

The introduction of $=$ into the $\exists$ rules is particularly unfortunate since not only is it not in the circle of classically interdefinable constants but it brings along with it the Laws of Identity, which fall outside the neat set up of purely introduction and elimination rules otherwise obtaining.

This last difficulty can be overcome by moving to second-order logic and defining $t = u$ as $\forall F(Ft \leftrightarrow Fu)$. Second-order $\exists$ rules could be given, analogous to the first order, except that $(\sim\forall x(Ax \leftrightarrow Fx) \rightarrow \sim QxFx)$ plays the role of $\forall x(\sim x = t \rightarrow \sim Fx)$ with A a predicate parameter and $Qx\_\_x$ a second-level concept (considering only the monadic case for simplicity). Thus the difficulty over $=$ will not arise at the second-order level for $\leftrightarrow$ can be defined in terms of the constants at hand or introduced as a primitive with introduction and elimination rules which satisfy the Inversion Principle above,[5] as well as permit of reduction procedures for maximal formulas. Unfortunately, normalization results do not hold for second-order quantification, else the consistency of arithmetic could be proven with arithmetic, contrary to Gödel's Second Incompleteness Theorem (cf. [4], pp. 71–73]. The stumbling block is that substitutions of predicates for predicate variables may increase the logical complexity of the formulas so that new maximal formulas introduced by reduction steps may be of higher complexity. This problem could be met by moving to ramified second-order logic but this, by introducing a relativity into the concept of identity, is not particularly attractive.

Nonetheless, when the balance of considerations are weighed up, it seems to me that classicists have little cause to worry. Classical logic does not admit of formulations which have neat Subformula Properties or satisfy Prawitz' Inversion Principle. Nonetheless, it does satisfy a restricted Subformula Principle and, moreover, a very natural Inversion Principle which intuitionistic logic fails to satisfy. While the considerations behind the latter principle may be said to favor strength in logics, the considerations behind the use of Prawitz' principle—the idea of the noncreativity of elimination rules—may be said with equal justice to favor weakness. Neither attitude can form a non-question-begging base for evaluating classical logic against a more constructivist alternative. Proof-theoretically, then, there seems little to choose between the two logics, at any rate for a classicist. Constructivists attempting to persuade classical logicians to abandon their logic in favor of a weaker logic, with all the crippling restrictions on classical mathematics and, therefore, on physical science that this implies, would be better advised to concentrate not on proof theory but on semantics, for they may be able to demonstrate some serious incoherence in the notions of truth and meaning appropriate to classical semantics. I doubt very much whether this can be done, but that is a different, and even more complex, story.

NOTES

1. ". . .By an application of an elimination rule one essentially only restores what had already been established if the major premise of the application was inferred by an application of an introduction rule." See [4], p. 33.

2. In a particular case involving vacuous discharge there may be no assumption A to appeal to in $\rightarrow E$. But the question of harmony requirements between introduction and elimination rules should be conducted at the general level of the rule schemata, where the discharged assumption is provided as part of the rule schema.

3. For simplicity, I have considered only noncomplex terms. Tennant's Proof depends on the observation that Φt_n^m is $(\Phi_n^m)t_n^m$, where t, m, and n are terms which are not variables. Thus

$$\cfrac{\cfrac{}{\forall x(\sim x = t \rightarrow \sim Fx)}{}^{k} \\ \Pi \\ Ft}{\exists xFx}{}^{k}$$

becomes

$$\cfrac{\cfrac{}{\forall x(\sim x = t \rightarrow \sim Fx)}{}^{k}\ {}_n^m \quad \Delta_n^m \\ \Pi n \\ Ft_n^m}{[\exists xFx]_n^m}{}^{k}$$

Observing that $\forall x(\sim x = t \rightarrow \sim Fx)_n^m$ *is* $\forall x(\sim x = t_n^m \rightarrow \sim Fx_n^m)$, $[\exists xFx]_n^m$ is $\exists x[Fx_n^m]$, and that Ft_n^m is $F_n^m t_n^m$, this last proof is:

$$\cfrac{\cfrac{}{\forall x(\sim x = t_n^m \rightarrow \sim Fx_n^m)}{}^{k} \quad \Delta_n^m \\ \Pi n \\ F_n^m t_n^m}{\exists x[Fx_n^m]}{}^{k}$$

in which the last application of $\exists I^*$ is correct. Given the inductive hypothesis that the proof of $F_n^m t_n^m$ is valid, so too is the proof of the conclusion. Similar reasoning applies to $\exists E*$. $\vee I*$ and $\vee E*$ are easily seen to satisfy the lemma since the substitutions distribute over the connectives. Reflexivity of identity is trivial, while for Leibniz's Law we have only to note that

$$\cfrac{\begin{matrix} \Delta & \Gamma \\ \Pi & \Sigma \\ a = b & Fa \end{matrix}}{Fb}$$

becomes, after substituting and observing the identities among formulas noted above:

$$\cfrac{\begin{matrix} \Delta_n^m & \Gamma_n^m \\ \Pi_n^m & \Sigma_n^m \\ \vdots & \vdots \\ a_n^m = b_n^m & Fa_n^m \end{matrix}}{Fb_n^m}$$

4. See [4], p. 52 for paths and p. 53 for the division into sections. Because of the effect of the standard ∨ and ∃ elimination rules in allowing the same formula to occur as both premise and conclusion, these properties strictly hold of segments, not formulas (see p. 49).

5. Actually the biconditional does not quite satisfy the Inversion Principle. There are the following proofs:

$$\dfrac{\dfrac{\begin{matrix}\overline{B}^{\,n} & \overline{A}^{\,n}\\ \vdots & \vdots \\ A & B\end{matrix}}{A \leftrightarrow B}\,n \qquad B}{A} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\overline{B}^{\,m} & \overline{A}^{\,m}\\ \vdots & \vdots \\ A & B\end{matrix}}{A \leftrightarrow B}\,m \qquad A}{B}$$

But here we use minor premises from different I rules in deriving the premises for $\leftrightarrow I$. Again in this proof:

$$\dfrac{\dfrac{A \leftrightarrow B \quad \overline{B}^{\,1}}{A} \qquad \dfrac{A \leftrightarrow B \quad \overline{A}^{\,1}}{B}}{A \leftrightarrow B}\,1$$

the conditions obtaining for $\leftrightarrow$ elimination on the right-hand side are not exactly the same as those on the left, but sufficient conditions for $\leftrightarrow E$ under a different E rule for $A \leftrightarrow B$ than the latter. However, a slight generalization of the Inversion Principle would enable the biconditional to satisfy the Principle.

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