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Model Constructions in Stationary Logic Part II: Definable Ultrapowers

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In this paper we continue our investigation of techniques for constructing new models of an arbitrary theory T in stationary logic, $L(aa)$. In Part I [2], we discussed the technique of model-theoretic forcing as a tool for building models in $L(aa)$. In this note we present a technique for constructing models by a series of definable ultrapowers.

$L(aa)$ was introduced by Shelah [3] (using the notation $L(Q_{\aleph_1}^{SS})$), and [1] contains the first explicit proofs of completeness, compactness, and omitting types for $L(aa)$. New proofs of these were given in [2] using forcing. (A sketch of the history of $L(aa)$ may be found in Section 8.1 of [1].)

One reason that ultrapowers and ultraproducts have not usually been used in the study of logics with generalized quantifiers like $L(aa)$ and $L(Q_1)$ (logic with the generalized first-order quantifier "there exist uncountably many x") is that if U is a countably incomplete ultrafilter over set I and A is countable, then $\Pi_U A$ will be uncountable. Since the main difficulty in generating models for these logics is in keeping the countable sets from growing and becoming uncountable, the usual ultrapower construction has not been helpful. Definable ultrapowers, originally introduced by Skolem, come to the rescue here. If the set I is countable and only definable functions from I into A are considered, $\pi_U A$ will still be countable. As usual in building definable ultrapowers we will need our language to contain built-in Skolem functions. With this, and by iterating the definable ultrapowers ω_1 times, we will be able to construct standard models of $L(aa)$. In particular we will give a new proof of the compactness theorem for $L(aa)$.[1]

1 Preliminaries We review very briefly the logic $L(aa)$ and the notion of a weak model for $L(aa)$. Our terminology and notation are the same as in [2]. Let $\mathcal{P}_{\omega_1}(A)$ be the set of all countable subsets of A.

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Definition Let $X \subseteq \mathcal{P}_{\omega_1}(A)$. We say that X is c.u.b. (closed and unbounded) over A iff

(i) whenever $s_0 \subseteq s_1 \subseteq s_2 \subseteq \ldots$ for $s_i \in X$, then $\bigcup s_i \in X$
(ii) if $s \in \mathcal{P}_{\omega_1}(A)$, there is an $s' \supseteq s$ with $s' \in X$.

We say $Y \subseteq \mathcal{P}_{\omega_1}(A)$ is stationary over A iff for every c.u.b. X, $X \cap Y \neq \phi$. The c.u.b. filter on A is defined to be the set $\mathfrak{F}(A) = \{X'$: there is a c.u.b. set X with $X \subseteq X'\}$.

Stationary logic is formed as an extension of first-order logic by the addition of two second-order generalized quantifiers: *aas* ("for almost all s") and *stats* ("there is a stationary set of s") where *stats* is the dual of *aas*. We can now define satisfaction for $L(aa)$. Let $\mathfrak{A}$ be a model of L, $\boldsymbol{a}$ an n-tuple of elements of A (the base set of $\mathfrak{A}$), $\boldsymbol{t}$ an m-tuple of elements of $\mathcal{P}_{\omega_1}(A)$ and ϕ a formula of $L(aa)$ whose first-order free variables are among $v_1, \ldots, v_n$, and whose second-order free variables are among $s_1, \ldots, s_m$. Satisfaction is defined as usual by induction on the complexity of formulas. The clause for *aas* is:

$$\mathfrak{A} \models aas\phi[s, \boldsymbol{t}, \boldsymbol{a}] \text{ if } \{t' \in \mathcal{P}_{\omega_1}(A)\colon \mathfrak{A} \models \phi[t', \boldsymbol{t}, \boldsymbol{a}]\} \in \mathfrak{F}(A).$$

Since *stats* abbreviates $\neg aas\neg$, it follows that $\mathfrak{A} \models stats\ \phi[s, \boldsymbol{t}, \boldsymbol{a}]$ iff $\{t' \in \mathcal{P}_{\omega_1}(A)\colon \mathfrak{A} \models \phi[t', \boldsymbol{t}, \boldsymbol{a}]\}$ is stationary. Note that the second-order variables are always interpreted as countable subsets of the base set; that is as elements of $\mathcal{P}_{\omega_1}(A)$.

Note that the second-order quantifiers $\forall s$ and $\exists s$ are *not* included in the language $L(aa)$. However, when we are dealing with countable weak models of $L(aa)$ in this paper, we will want to extend the language to $L^+(aa)$ which does include the second-order quantifiers $\forall s$ and $\exists s$, as well as *aas* and *stats*.

Our standard models of $L(aa)$ will be constructed from countable approximations, called the weak models.

Definition A weak model for $L(aa)$ is a triple $\mathfrak{A}^* = (\mathfrak{A}, \mathcal{P}, \mathfrak{F})$ where $\mathfrak{A}$ is a model for L, $\mathcal{P}$ is a collection of subsets of A, and $\mathfrak{F}$ is a collection of subsets of $\mathcal{P}$ such that $\mathcal{P} \in \mathfrak{F}$. All second-order variables range over elements of $\mathcal{P}$ and first-order variables range over elements of A. Satisfaction for these weak models is defined as usual by induction on the complexity of formulas. The *aas* clause is $(\mathfrak{A}, \mathcal{P}, \mathfrak{F}) \models aas\ \phi[s, \boldsymbol{t}, \boldsymbol{a}]$ iff $\{t' \in \mathcal{P}\colon (\mathfrak{A}, \mathcal{P}, \mathfrak{F}) \models \phi[t', \boldsymbol{t}, \boldsymbol{a}]\} \in \mathfrak{F}$. As mentioned above, we will sometimes want to use the extended logic $L^+(aa)$ which includes $\forall s$ and $\exists s$ when discussing weak models. For example $(\mathfrak{A}, \mathcal{P}, \mathfrak{F}) \models \forall s\ \phi[s, \boldsymbol{t}, \boldsymbol{a}]$ iff for all $t' \in \mathcal{P}$, $(\mathfrak{A}, \mathcal{P}, \mathfrak{F}) \models \phi[t', \boldsymbol{t}, \boldsymbol{a}]$.

We say a theory T in $L^+(aa)$ has built-in Skolem functions iff for every first- or second-order existential formula $\exists x\, \phi(x, \boldsymbol{y})$ or $\exists s\, \psi(s, \boldsymbol{y})$ there is a function symbol F_ϕ or G_ψ such that $T \models \exists x\, \phi(x, \boldsymbol{y}) \rightarrow \phi(F_\phi(\boldsymbol{y}), \boldsymbol{y})$ and $T \models \exists s\, \psi(s, \boldsymbol{y}) \rightarrow \psi(G_\psi(\boldsymbol{y}), \boldsymbol{y})$. We say model $\mathfrak{A}^*$ has built-in Skolem functions iff $Th(\mathfrak{A}^*)$ in $L^+(aa)$ does.

***2 Ultrapowers in L^+*(a)** Let $\mathfrak{A}^* = (\mathfrak{A}, \mathcal{P}^{\mathfrak{A}}, \mathfrak{F}^{\mathfrak{A}})$ be a weak model of $L(aa)$ which has built-in Skolem functions. Let U be an ultrafilter over the set $\mathcal{P}^{\mathfrak{A}}$. We define the definable ultrapower of $\mathfrak{A}^*$ over U, $\mathfrak{B}^* = (\mathfrak{B}, \mathcal{P}^{\mathfrak{A}}, \mathfrak{F}^{\mathfrak{A}}) = (\Pi_U \mathfrak{A}^*)_{def}$, as follows:

(i) $b^U \in B$ iff b: $\mathcal{P}^{\mathfrak{A}} \to A$ and there is a formula $\psi_b(s, y, \boldsymbol{z})$ and $\boldsymbol{c} \in (A \cup \mathcal{P}^{\mathfrak{A}})^{<\omega}$ such that $\mathfrak{A}^* \models \forall s\, \exists! y\, \psi_b(s, y, \boldsymbol{c})$ and such that for all $i \in \mathcal{P}^{\mathfrak{A}}$, $\mathfrak{A}^* \models \psi_b[i, b(i), \boldsymbol{c}]$. Thus b^U is the equivalence class of a definable function from $\mathcal{P}^{\mathfrak{A}}$ into A.

(ii) $s^U \in \mathcal{P}^{\mathfrak{B}}$ iff s: $\mathcal{P}^{\mathfrak{A}} \to \mathcal{P}^{\mathfrak{A}}$ and there is a formula $\phi_s(t, u, \boldsymbol{z})$ and $c \in (A \cup \mathcal{P}^{\mathfrak{A}})^{<\omega}$ such that $\mathfrak{A}^* \models \forall t\, \exists! u\, \phi_s(t, u, \boldsymbol{c})$ and for all $i \in \mathcal{P}^{\mathfrak{A}}$, $\mathfrak{A}^* \models \phi_s\, [i, s(i), \boldsymbol{c}]$. As usual $b^U \in s^U$ iff $\{i \in \mathcal{P}^{\mathfrak{A}}: b(i) \in s(i)\} \in U$.

(iii) Let $X' \subseteq \mathcal{P}^{\mathfrak{B}}$. $X' \in \mathfrak{F}^{\mathfrak{B}}$ iff there is an $X \subseteq X'$ such that for all $i \in \mathcal{P}^{\mathfrak{A}}$ there are $X_i \subseteq \mathcal{P}^{\mathfrak{A}}$ with $X = (\Pi_U X_i) \cap \mathcal{P}^{\mathfrak{B}}$ where

(a) $Z = \{i \in \mathcal{P}^{\mathfrak{A}}: X_i \in \mathfrak{F}^{\mathfrak{A}}\} \in U$, and

(b) there is a formula $\psi(t, u, \boldsymbol{x})$ and $\boldsymbol{c} \in (A \cup \mathcal{P}^{\mathfrak{A}})^{<\omega}$ such that for all $i \in Z$, $X_i = \{s \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \psi[s, i, \boldsymbol{c}]\}$.

The very careful definition of $\mathfrak{F}^{\mathfrak{B}}$ above is necessary to push through the following analog of Loś's theorem.

Theorem 1 *Let $\mathfrak{A}^*$ be a weak model of $L(aa)$ with built-in Skolem functions and let $\mathfrak{B}^* = (\Pi_U \mathfrak{A}^*)_{def}$. For all formulas $\phi(\boldsymbol{x})$ of $L(aa)$ and $\boldsymbol{b}^U \in (B \cup \mathcal{P}^{\mathfrak{B}})^{<\omega}$, $\mathfrak{B}^* \models \phi[\boldsymbol{b}]$ iff $\{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{B}^* \models \phi[\boldsymbol{b}(i)]\} \in U$.*

Proof: The proof for atomic sentences, $\wedge$, and $\neg$ is standard. The $\exists$-case goes through since we have Skolem functions.

($\Rightarrow$): Trivial.

($\Leftarrow$): Suppose $Y = \{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \exists u\, \phi[u, \boldsymbol{b}(i)]\} \in U$. Since each $b^U \in (B \cup \mathcal{P}^{\mathfrak{B}})$ is definable, there is a formula $\psi_{\boldsymbol{b}}(s, \boldsymbol{w}, \boldsymbol{z})$ and $\boldsymbol{c} \in (A \cup \mathcal{P}^{\mathfrak{A}})^{<\omega}$ such that $\mathfrak{A}^* \models \forall s\, \exists! \boldsymbol{w}\, \psi_{\boldsymbol{b}}(s, \boldsymbol{w}, \boldsymbol{c})$ and $\mathfrak{A}^* \models \psi_{\boldsymbol{b}}(i, \boldsymbol{b}(i), \boldsymbol{c})$ for all $i \in \mathcal{P}^{\mathfrak{A}}$. Thus for all $i \in Y$, $\mathfrak{A}^* \models \exists u(\exists \boldsymbol{w}(\phi(u, \boldsymbol{w}) \wedge \psi_{\boldsymbol{b}}(i, \boldsymbol{w}, \boldsymbol{c})))$. Let $f(s, \boldsymbol{z})$ be the Skolem function for the formula in parentheses. Therefore for all $i \in Y$, $\mathfrak{A}^* \models \phi(f(i, \boldsymbol{c}), \boldsymbol{b}(i))$. Let $d(i) = f(i, \boldsymbol{c})$ for all i. Therefore by induction, $\mathfrak{B}^* \models \phi[d^U, \boldsymbol{b}^U]$, and hence $\mathfrak{B}^* \models \exists u\, \phi[u, \boldsymbol{b}^U]$.

The *aa* case is a little trickier. $aa(\Leftarrow)$: Suppose $Y = \{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models aas\, \phi[s, \boldsymbol{b}(i)]\} \in U$. For $i \in Y$, let $X_i = \{s \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \phi[s, \boldsymbol{b}(i)]\}$. For $i \in \mathcal{P}^{\mathfrak{A}} - Y$, let $X_i = \mathcal{P}^{\mathfrak{A}}$. Thus $X_i \in \mathfrak{F}^{\mathfrak{A}}$ for all $i \in \mathcal{P}^{\mathfrak{A}}$. By definition $X = (\Pi_U X_i) \cap \mathcal{P}^{\mathfrak{B}} \in \mathfrak{F}^{\mathfrak{B}}$. Recall that $s^U \in X$ means $Z = \{i \in \mathcal{P}^{\mathfrak{A}}: s(i) \in X_i\} \in U$. Also for $i \in Y$, $s(i) \in X_i$ iff $\mathfrak{A}^* \models \phi[s(i), \boldsymbol{b}(i)]$. It follows that

$$\{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \phi[s(i), \boldsymbol{b}(i)]\} \supseteq Z \cap Y \in U.$$

Therefore by induction, $s^U \in X$ implies $\mathfrak{B}^* \models \phi[s^U, \boldsymbol{b}^U]$. Thus since $X \in \mathfrak{F}^{\mathfrak{B}}$, $\mathfrak{B}^* \models aas\, \phi[s, \boldsymbol{b}^U]$.

$aa(\Rightarrow)$: Suppose $\mathfrak{B}^* \models aas\, \phi[s, \boldsymbol{b}^U]$. Let $X' = \{s^U \in \mathcal{P}^{\mathfrak{B}}: \mathfrak{B}^* \models \phi[s^U, \boldsymbol{b}^U]\}$. By definition of $\mathfrak{F}^{\mathfrak{B}}$, there is an $X \subseteq X'$ and $X_i \subseteq \mathcal{P}^{\mathfrak{A}}$ for each $i \in \mathcal{P}^{\mathfrak{A}}$ such that $X = (\Pi_U X_i) \cap \mathcal{P}^{\mathfrak{B}}$, $Z = \{i \in \mathcal{P}^{\mathfrak{A}}: X_i \in \mathfrak{F}^{\mathfrak{A}}] \in U$, and there is a formula $\psi(t, u, \boldsymbol{x})$ and $\boldsymbol{c} \in (A \cup \mathcal{P}^{\mathfrak{A}})^{<\omega}$ such that for all $i \in Z$, $X_i = \{s \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \psi[s, i, \boldsymbol{c}]\}$. We need to prove that $Y = \{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models aas\, \phi[s, \boldsymbol{b}(i)]\} \in U$. This will follow if we show $W = \{i \in \mathcal{P}^{\mathfrak{A}}: X_i \subseteq \{s \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \phi[s, \boldsymbol{b}(i)]\}\} \in U$ since then $W \cap Z \in U$ and $W \cap Z \subseteq Y$.

Suppose $W \notin U$. Then $Z - W = \{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{A}^* \models \exists s(\psi[s, i, \boldsymbol{c}] \wedge \neg\phi[s, \boldsymbol{b}(i)])\} \in U$. Let $f(x, \boldsymbol{z})$ be the Skolem function for the formula in parentheses. Thus for $i \in Z - W$,

(*) $\mathfrak{A}^* \models \psi[f(i, \boldsymbol{c}), i, \boldsymbol{c}] \wedge \neg\phi[f(i, \boldsymbol{c}), \boldsymbol{b}(i)]$.

Let $s_f(i) = f(i, \boldsymbol{c})$. Thus $s_f^U \in \mathcal{P}^{\mathfrak{B}}$ and $\{i \in \mathcal{P}^{\mathfrak{A}}: s_f(i) \in X_i\} \in U$. Hence $s_f^U \in X \subseteq X'$ and thus $\mathfrak{B}^* \models \phi[s_f, \boldsymbol{b}]$. This contradicts (*) though, so we must have had $W \in U$ as desired.

Note that the proof of the above theorem does not depend on the diagonal intersection axiom.

The following lemma gives the construction which will be central to our proof of the compactness theorem. Notice the similarity between the description of the model $\mathfrak{C}^*$ below and the notion of a Φ-generic extension of $\mathfrak{B}$ with respect to *stats* ψ as given in [2].

Lemma 2 *Let $L(aa)$ be a language with built-in Skolem functions. Let $\mathfrak{B}^*$ be a countable weak model of $L(aa)$ which satisfies the Skolem function axioms as well as the closures of $\{stats\ \phi(s) \to \exists^{\geq n} s\, \phi(s): n < \omega, \phi \in L(aa)\}$. Let $b \in (B \cup \mathcal{P}^{\mathfrak{B}})^{<\omega}$ and ψ be a formula of $L(aa)$ such that $\mathfrak{B}^* \models stats\ \psi[s, \boldsymbol{b}]$.*

Then there is a countable $\mathfrak{C}^* \underset{L^+(aa)}{\succ} \mathfrak{B}^*$ such that:

(i) $\mathfrak{C}^* \models \psi[B, \boldsymbol{b}]$ *where B is the base set of $\mathfrak{B}^*$*
(ii) *whenever $\mathfrak{B}^* \models aas\ \phi[s, \boldsymbol{a}]$ then $\mathfrak{C}^* \models \phi[B, \boldsymbol{a}]$.*

Proof: We construct $\mathfrak{C}^*$ as a definable ultrapower of $\mathfrak{B}^*$. First we define $\mathcal{S}^{\mathfrak{B}}$ as follows: $X \in \mathcal{S}^{\mathfrak{B}}$ if $X \subseteq P(\mathcal{P}^{\mathfrak{B}})$ and $\mathcal{P}^{\mathfrak{B}} \setminus X \notin \mathfrak{F}^{\mathfrak{B}}$. Thus $\mathfrak{B}^* \models stats\ \phi(s)$ iff $\{s \in \mathcal{P}^{\mathfrak{B}}: \mathfrak{B}^* \models \psi[s]\} \in \mathcal{S}^{\mathfrak{B}}$.

We want to define an ultrafilter over $\mathcal{P}^{\mathfrak{B}}$ which will ensure that (i) and (ii) above both hold. Let $\phi_0(s, \boldsymbol{b}_0), \ldots, \phi_n(s, \boldsymbol{b}_n), \ldots$ be a list of all formulas $\phi_i(s, \boldsymbol{b}_i)$ with $\boldsymbol{b}_i \in (B \cup \mathcal{P}^{\mathfrak{B}})^{<\omega}$ and $\mathfrak{B}^* \models aas\ \phi_i(s, \boldsymbol{b}_i)$. Let $c_0, c_1, \ldots$ be a list of all elements of $(\Pi B)_{def}$. We will pick our ultrafilter U to contain the sets $\{U_n: n < \omega\}$ defined inductively as follows:

Let $U_0 = \{s \in \mathcal{P}^{\mathfrak{B}}: \mathfrak{B}^* \models \psi[s, \boldsymbol{b}]\} \in \mathcal{S}^{\mathfrak{B}}$. Suppose $U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots \supseteq U_{2n}$ have been defined so that for each $i < 2n$, $U_i \in \mathcal{S}^{\mathfrak{B}}$ and U_i is definable (with parameters) in $\mathfrak{B}^*$.

Stage $2n + 1$: The motivation for this stage is to ensure that s_{diag}^U, which is defined by $s_{diag}(i) = i$ for all $i \in \mathcal{P}^{\mathfrak{A}}$, will contain no elements not already in B. Let $X = \{i \in \mathcal{P}^{\mathfrak{B}}: \mathfrak{B}^* \models c_n(i) \in i\}$. Since $stats(\phi \vee \psi) \to stats\ \phi \vee stats\ \psi$, one of $U_{2n} \cap X$ and $U_{2n} \setminus X$ is in $\mathcal{S}^{\mathfrak{B}}$.

Case i. $U_{2n} \setminus X \in \mathcal{S}^{\mathfrak{B}}$. Let $U_{2n+1} = U_{2n} \setminus X$. Therefore no matter how we complete $\{U_n: n < \omega\}$ to an ultrafilter U, we will have $X \notin U$ and hence $\mathfrak{C}^* \models \neg c_n^U \in s_{diag}^U$.

Case ii. $U_{2n} \cap X \in \mathcal{S}^{\mathfrak{B}}$. Let $\theta(s, \boldsymbol{b})$ be the definition of U_{2n} in $\mathfrak{B}^*$. $U_{2n} \cap X \in \mathcal{S}^{\mathfrak{B}}$ implies $\mathfrak{B}^* \models stats\ \exists z(\theta(s, \boldsymbol{b}) \wedge c_n(s) = z \wedge z \in s)$. Therefore by the diagonal intersection schema, $\mathfrak{B}^* \models \exists z\, stats(\theta(s, \boldsymbol{b}) \wedge c_n(s) = z)$. Hence there is an $a \in B$ such that $\mathfrak{B}^* \models stats(\theta(s, \boldsymbol{b}) \wedge c_n(s) = a)$. Let U_{2n+1} be the stationary set described by the formula in parentheses. Clearly $U_{2n+1} \subseteq U_{2n}$ and for all $i \in U_{2n+1}$, $\mathfrak{B}^* \models c_n(i) = a$. Hence at the end c_n will be identified with the constant function f_a (defined by $f_a(i) = a$ for $i \in \mathcal{P}^{\mathfrak{A}}$) in the model $\mathfrak{C}^*$.

Stage $2n+2$: Let $Y = \{i \in \mathcal{P}^{\mathfrak{A}}: \mathfrak{B}^* \models \phi_n[i, \boldsymbol{b}_n]\}$. Since $Y \in \mathfrak{F}^{\mathfrak{B}}$, it follows that $U_{2n+2} = U_{2n+1} \cap X \in \mathcal{S}^{\mathfrak{B}}$ (recall the intersection of a c.u.b. set and a stationary set is itself stationary).

Let $U \supseteq \{U_n: n < \omega\}$ be a nonprincipal ultrafilter over B. We can choose U to be nonprincipal since each U_n is infinite.

Let $\mathfrak{C}^* = (\Pi_U \mathfrak{B}^*)_{def}$. We show $\mathfrak{C}^*$ satisfies the theorem. For each $b \in B$ let f_b be defined so that for all $i \in \mathcal{P}^{\mathfrak{B}}$, $f_b(i) = b$. Clearly $f_b^U \in C$ for each $b \in B$. By Loś's theorem for $L^+(aa)$ we get $\mathfrak{B}^* \underset{L^+(aa)}{\prec} \mathfrak{C}^*$ where we identify $b \in B$ with $f_b^U \in C$. Define s_{diag} so that $s_{diag}(i) = i$. Clearly $s_{diag}^U \in \mathcal{P}^{\mathfrak{C}}$. We will show $s_{diag}^U = B$ (i.e., $f^U \in s_{diag}^U$ iff $f^U = f_b^U$ for some $b \in B$). If $b \in B$, $\mathfrak{B}^* \models aas(b \in s)$ and hence in an even-numbered step we ensured that $\mathfrak{C}^* \models f_b^U \in s_{diag}^U$. On the other hand, if $\mathfrak{C}^* \models c^U \in s_{diag}^U$ then $c = c_n$ for some n. Hence at stage $2n+1$ we must have applied case ii, which would give us an $a \in B$ such that $\mathfrak{C}^* \models c = f_a^U$. Thus s_{diag}^U corresponds to B.

Since $U_0 \in U$, $\mathfrak{C}^* \models \psi[s_{diag}^U, \boldsymbol{b}]$. Similarly by the constructions of the $U_{2n} \in U$, we are assured that whenever $\mathfrak{C}^* \models aas\,\phi[s, \boldsymbol{b}]$, $\mathfrak{C}^* \models \phi[s_{diag}^U, \boldsymbol{b}]$.

The above lemma is all we need to prove the compactness theorem for $L(aa)$ (note $L^+(aa)$ is clearly *not* compact). We cannot prove the completeness theorem this way since we need the existence of Skolem functions to push through the proof.

Compactness theorem Let T be a countable set of sentences of $K(aa)$ and suppose that every finite subset of T has a standard model. Then T has a standard model.

Proof: Let $L^+(aa)$ be an extension of $K(aa)$ with Skolem functions. Clearly any standard model of $K(aa)$ can be expanded to a model of $L^+(aa)$ satisfying the Skolem axioms and the closures of $STAT = \{stats\,\phi(s) \to \exists^{\geq n} s\,\phi(s)$: $n < \omega$ and $\phi \in L^+(aa)\}$. Hence $T^* = T \cup \{\text{Skolem axioms}\} \cup STAT$ is a countable finitely satisfiable theory in $L^+(aa)$ (interpreted as a two-sorted logic where we allow existential quantification over both sorts). Thus by a Henkin-type construction there is a countable weak model $\mathfrak{A}_0^*$ satisfying T^*.

By using Lemma 2 as in the proof of the completeness theorem we can obtain a standard model $\mathfrak{A}$ of $L(aa)$ such that $\mathfrak{A} \models T$. (Note that $\mathfrak{A}$ will not necessarily preserve second-order universal statements since the union of the $\mathcal{P}^{\mathfrak{A}_\alpha}$ for $\alpha < \omega_1$ need not give all of $\mathcal{P}_{\omega_1}(A)$.)

NOTE

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