Notre Dame Journal of Formal Logic
Volume 27, Number 2, April 1986

A Story Semantics for Implication

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1 Motivation This paper results from an interest in stories and in the question of what is "in" and what is not "in" a story. Some stories have texts, some do not. A novel is a text; it is an ordered set of sentences. What is in the story of the novel is what is implied by that set of sentences.

But what kind of implication is it that we have here? If, in a story, a geometer discovers a method of trisecting the angle with ruler and compasses alone, that it rained yesterday in Detroit need not be in the story. Yet the impossibility of there being such a method together with the story's assumption that there is such a method does, classically, imply not only that it rained yesterday in Detroit, but that it didn't rain there yesterday as well.

In pondering this problem I succumb to a temptation to turn things around to see whether the notion of a story might help to shed some light on implication. It does.

What we have is the actual world and a language, or a set of beliefs or pictures, that are true or false of it. Then we have stories described in the same vehicle. Stories can be thought of as sets of these sorts of things closed under implication.[1]

The three principal features of stories I shall seize on are: (1) Some of the people, things, and places there are do not figure in some stories, even in stories which contain contradictions. (2) Stories can fail to provide answers to questions. A story in which either John killed Jim or Joe killed Jim may not contain an answer to the question of who, exactly, did kill Jim. In the story of *Moby Dick* we will not find an answer to the question of whether Hamlet's mother was or was not Antigone's sister.

Taking language as our vehicle, a sentence will be *true in a story precisely when it is in the story*, when it is a member of the set of sentences the story is. This is the way sentences are true for stories. While other aspects of the story semantics to be introduced here will vary in the course of this paper, this valuation, indeed all truth valuations, will remain firmly fixed.

I wish to thank an anonymous referee for a number of helpful suggestions.

Received October 10, 1984; revised February 25, 1985

If a story contains a conjunction, $A \wedge B$, then it should contain both A and B, and *vice versa.* I have already remarked that it is not that way with disjunctions. Neither A nor B need be in a story that contains $A \vee B$. And A should be in a story when and only when $\sim\sim A$ is in it.

Does the story of *Moby Dick* say that Moby Dick is a whale or Mae West visited Buffalo, because it says that Moby Dick is a whale?

A story, we like to think, is about a certain cast of characters and certain happenings. Mae West is not one of the characters *Moby Dick* is about.

This brings us to quantification and the range of things quantificational variables take in stories. If we make a natural and intuitive amendment to the definition of a story, so that a story is now a pair, a set of sentences and a "cast", a list of names, the names of the things the story is about, or if, in this spirit, we stipulate that no sentence in a story name an individual other than one of those the story is about, we shall have to discard the idea that if A is in a story, $A \vee B$ and $B \vee A$ are also in it.

Furthermore, letting $\rightarrow$ symbolize the connective "that... implies that... ", we shall also abandon the view that if $A \rightarrow B$ is in a story, so is $(A \wedge C) \rightarrow B$, for the sentence C may introduce new and unwanted characters. For the same reason we shall not want to hold that if $\sim A$ is in a story, $\sim(A \wedge B)$ is as well.

Some points remain untouched by the foregoing considerations. If $A \rightarrow B$ is in a story, then $A \rightarrow ((A \vee B) \wedge (B \vee A))$ can be; if $A \wedge B$ is in it, then $(A \vee B) \wedge (B \vee A)$ can be; and if $(A \rightarrow B) \wedge (A \rightarrow C)$ is in it, then $(A \wedge C) \rightarrow B$ can be.

Suppose that $(A \wedge B) \rightarrow C$ is in a story. Is $(B \wedge A) \rightarrow C$ in it as well? Of course. More generally, let X be a finite nonempty set of sentences of L, $\{A_1, \ldots, A_n\}$. We let $(\wedge X)$ denote any sentence which is, so to speak, a conjunction of the sentences of X. We can make this precise as follows: Form X^* $(= \langle X_1, \ldots, X_m, \ldots \rangle)$ as follows: Let $X_1 = \{\langle A, \{A\}\rangle : A \in X\}$. Given X_n, let $X_{n+1} = \{\langle A \wedge B, Y \cup Z\rangle : \langle A, Y\rangle \in X_i, \langle B, Z\rangle \in X_k$, and $l \leq i, k \leq n\}$. Finally, let $(\wedge X)$ denote any sentence C such that for some $x_i \in X^*$, $\langle C, X\rangle \in X_i$. $(\vee X)$ is defined similarly, using disjunction instead of conjunction. Now if $(\wedge X) \rightarrow A$ is in a story, $(\wedge X)' \rightarrow A$ certainly should be, and if $(\vee X) \rightarrow A$ is, $(\vee X)' \rightarrow A$ should be.

The following also seem reasonable requirements to have on a story semantics: if $A \wedge (A \rightarrow B)$ is in a story, B is; if $(A \rightarrow B) \wedge (B \rightarrow C)$ is, $A \rightarrow C$ is; if $(A \rightarrow B) \wedge (A \rightarrow C)$ is, $A \rightarrow (B \wedge C)$ is; if $(A \rightarrow C) \wedge (B \rightarrow C)$ is, $(A \vee B) \rightarrow C$ is; if $\sim(A \vee B)$ is, $\sim A$ is; if A is, $A \vee \sim A$ is; and if $\sim(\vee X) \rightarrow A$ is, $\sim(\vee X)' \rightarrow A$ is.[2]

What is being proposed here is a story semantics for implication. A model will have three components: an actual world, a set of atomic sentences (the ones true in the actual world), and a set of stories.

As we have already indicated, truth in stories can be defined as follows: a sentence A is *true in a story s* just when $A \in s$. What remains, indeed all that remains, is to specify when A is true in the actual world. The set of atomic sentences in the model settles it when A is an atomic sentence. The standard valuation rules for classical logic settle it when A is of the form $\sim B$, $B \wedge C$, and $B \vee C$. And last but not least, when A is $B \rightarrow C$, A will be true in the actual world just in case, for the actual world and for all stories as well, whenever B is true, C is true.

This semantics has advantages. It ties implication to the everyday and ordinary, to stories. It is intuitive and ontologically uncontroversial; stories have a clear internal structure which is evident from the requirements used in their construction, e.g., if $A \wedge B$ is in a story, so is A. In this stories are different from what are called "possible worlds", or Routley-Meyer set-ups, which, all romantic associations aside, are really featureless nodes, barren stopping places arranged in complex and puzzling ways along lines or in arrays by enigmatic accessibility relations.

Rhetoric aside, a story semantics is flexible, for those who like to tinker. To add the axiom scheme $(A \to B) \to ((C \to A) \to (C \to B))$, for example, one merely adds three conditions on stories to the semantics:

1. if $A \to B$ is in a story, so is $(C \to A) \to (C \to B)$
2. if all stories contain A only if they contain B, and $C \to A$ is in a story, so is $C \to B$
3. if all stories contain A only if they contain B, and all stories contain C only if they contain A, and C is in a story, so is B.

The following schemes can be made sound in a similar way:

$$(A \to B) \to ((B \to C) \to (A \to C))$$
$$A \to ((A \vee B) \wedge (B \vee A))$$
$$(A \to (B \to C)) \to ((A \wedge B) \to C)$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$$
$$(A \to \sim B) \to (B \to \sim A)$$
$$(A \to \sim A) \to \sim A$$
$$(A \to ((B \to D) \to C)) \to ((B \to D) \to (A \to C)).$$

While $(A \to ((B \to D) \to C)) \to ((B \to D) \to (A \to C))$ can be validated on story semantics as described thus far, it does not seem possible to validate $(A \to (B \to C)) \to (B \to (A \to C))$. But if $\Box$ is added to the language and $\Box A$ is evaluated as true in the actual world just in case A is true in the actual world and all stories as well, then $(A \to (B \to C)) \to (\Box B \to (A \to C))$ can be made sound. Of course, it is obvious that many of these schemes violate the motivational intuitions under which we have been operating.

If everything in the story of *Moby Dick* has a certain feature, does Othello have it there too? In light of the earlier discussion, it seems not. Othello is just not one of the characters in *Moby Dick*.

This calls into question the schema $(\forall x)A \to Ax/c$, where Ax/c is the usual, what results from substituting the name c for all free occurrences of the variable x in A. We might have $(\forall x)A$ in a story and yet, since c does not name anyone in the story's cast of characters, we will not have Ax/c in it, at least where x does occur free in A.

If all stories are about something, if the cast of a story cannot be empty, we shall certainly have $(\forall x)A \to (\exists x)A$. If everything in *Moby Dick* has a certain feature, then something does. And since if Ishmael has it, something has it, we will have $Ax/c \to (\exists x)A$.

Nonetheless, if everything in a story has a certain feature and a thing is in the story, then that thing has the feature. Thus we should accept $((\forall x)A \wedge B) \to Ax/c$, where c occurs in B or x does not occur free in A.

Since on story semantics $B \vee \sim B$ is true in the actual world for all sentences

B that ever occur in any story, we have sentences true in the actual world that have as constants any constants that occur in any sentence in any story. So if $(\forall x)A$ is true in the actual world and c is a name that occurs in a sentence B in some story, $(\forall x)A \wedge (B \vee \sim B)$ is true in the actual world, and hence Ax/c is true in the actual world too. If everything has a certain feature in the actual world, Othello, Ishmael, Pegasus, all have it. This shows what "$\exists$" means and does not mean, when "$\exists$" is defined as "$\sim\forall\sim$". It means "there are", as in "There are things that don't exist, viz., Othello, Ishmael, Pegasus, and Plato".[3]

This is a consequence of accepting the scheme $((\forall x)A \wedge B) \rightarrow Ax/c$. Some, however, may prefer to use $((\forall x)A \wedge Ec) \rightarrow Ax/c$, where "$E$" is an existence predicate, and, with appropriate semantic adjustments, to use "$\exists$" for "there exist", instead of for "there are".

Open sentences will not occur in stories. It seems to make no sense to include them. There is nothing that could be construed as a "cast of variables". Unlike names, which variables are used in closed sentences in stories is a matter of indifference. Let Ay/x be the result of replacing all occurrences of a variable y captured in the scope of a quantifier $(\forall y)$ in A by a variable x which does not occur in A. Surely if A is in a story Ay/x should be in it too.

Also, if a story says that there is someone x and someone y such that x hit y, it would seem to say that there is someone y and someone x such that x hit y. Let Q and Q' be strings of quantifiers and tildes such that the sentences QA and $Q'A$ are classically equivalent. If QA is in a story, $Q'A$ ought to be there too.

I conclude this section with a confession. I own up to the fact that I will not be giving a "pure" language of implication, a language for story semantics in which the sole connective is $\rightarrow$. Even if story semantics could accommodate such a language, I have no intuitions whatever about what kinds of schemes, beyond $A \rightarrow A$, it would be appropriate to impose on it.

2 The language L Let L be a sentential language with the one-place connective $\sim$ and the two-place connectives $\wedge$, $\vee$, and $\rightarrow$.

We begin with a far far simpler axiomatics than that proposed and discussed in Section 1. We say that a wff A is a *tautology in L* if

(1) $\rightarrow$ does not occur in A and A is a tautology of the propositional calculus, or
(2) for some $n \geq 1$, A is the result of substituting any wffs $B_1, \ldots, B_n$ of L for atomic wffs $p_1, \ldots, p_n$, respectively, at all their occurrences in a wff B which is a tautology of the propositional calculus.

As other primitive symbols (like $\forall$) are added to the language, the definition of a tautology will have to be suitably amended.

The axioms and rules of L are:

(A0) *All tautologies in L.*
(A0.1) $\sim(A \wedge \sim B \wedge (A \rightarrow B))$
(A1) $A \rightarrow A$
(A2) $(A \wedge B) \rightarrow A$
(A3) $((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$
(A4) $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$.

(R1) If $\vdash A$ and $\vdash A \to B$, $\vdash B$.
(R2) If $\vdash A$ and $\vdash B$, $\vdash A \wedge B$.
(R3) If $\vdash (A \wedge C) \to D$, $\vdash (B \wedge {\sim}C) \to E$, $\vdash {\sim}D$, and $\vdash {\sim}E$, $\vdash {\sim}(A \wedge B)$.
(R4) If $\vdash (\wedge X) \to A$, $\vdash (\wedge X)' \to A$.
(R5) $\vdash A$ iff $\vdash {\sim}{\sim}A$.

It should be remarked that R4 allows us to ignore parentheses in $(\wedge X)$ as regards the grouping of conjuncts out of which $(\wedge X)$ is constructed and the order in which these conjuncts appear in $(\wedge X)$. It allows us to write $\vdash (B_1 \wedge \ldots \wedge B_n) \to B$. This will be important in Section 5.

3 Semantics A model M for L is a triplet $\langle O, At, S\rangle$ where O is an item standing in for an actual world, $At \subseteq \{A: A \text{ is an atomic wff of } L\}$, and S is a set of "stories" (sets of wffs of L) such that for $s \in S$ the following conditions hold:

(S2) if $A \wedge B \in s$, $A \in s$
(S2*) if $A, B \in s$, $A \wedge B \in s$
(S3) if $(A \to B) \wedge (B \to C) \in s$, $A \to C \in s$
(S∧) if $(\wedge X) \in s$, $(\wedge X)' \in s$
(S4) if $(A \to B) \wedge (A \to C) \in s$, $A \to (B \wedge C) \in s$.

V is a valuation taking wffs of L into subsets of $\{O\} \cup S$ $(= W)$ such that for any wff A, $V(A)$ satisfies the following conditions:

(VS) for $s \in W - \{O\}$ $(= S^*)$, if $A \in s$, $s \in V(A)$; otherwise $s \notin V(A)$
(VAt) where A is an atomic wff, if $A \in At$, $O \in V(A)$ and otherwise $O \notin V(A)$
(V~) if $O \notin V(A)$, $O \in V({\sim}A)$; otherwise $O \notin V({\sim}A)$
(V∧) if $O \in V(A)$ and $O \in V(B)$, $O \in V(A \wedge B)$; otherwise $O \notin V(A \wedge B)$
(V∨) if $O \in V(A)$ or $O \in V(B)$, $O \in V(A \vee B)$; otherwise $O \notin V(A \vee B)$
(V→) if for all $w \in W$, $w \in V(A)$ only if $w \in V(B)$, $O \in V(A \to B)$; otherwise $O \notin V(A \to B)$.

$M \Vdash A$ if $O \in V(A)$. $\Vdash A$ if $M \Vdash A$ for all models M.

4 Soundness

Theorem (T4.1) *If* $\vdash A$, $\Vdash A$.

Proof: ad (A0). Let A be a tautology in L. Since (V~) – (V∨) are exactly the valuation rules for classical evaluation substituting "T" for "O" and "=" for "$\in$", A will be sound when the substitutions are reversed.

ad (A0.1). Suppose $O \notin V({\sim}(A \wedge {\sim} B \wedge (A \to B)))$. By (V~) $O \in V(A \wedge {\sim}B \wedge (A \to B))$. By (V∧) $O \in V(A)$, $V({\sim}B)$, $V(A \to B)$. By (V~) $O \notin V(B)$. By (V→) for all $w \in W$, $w \in V(A)$ only if $w \in V(B)$. $O \in W$. So $O \in V(B)$, which is impossible.

ad (A1). If $O \notin V(A \to A)$, by (V→) for some $w \in W$, $w \in V(A)$ and $w \notin V(A)$, which is impossible.

ad (A2). If $O \notin V(\text{A2})$, by (V→) for some $w \in W$, $w \in V(A \wedge B)$ and $w \notin V(A)$. If $w \in S^*$, by (VS) $A \wedge B \in w$ and $A \notin w$, which is impossible by (S2). So $w = O$. But this is impossible by (V∧).

ad (A3). If $O \notin V(\text{A3})$, by (V→) for some $w \in W$, $w \in V((A \to B) \wedge (B \to C))$ and $w \notin V(A \to C)$. If $w \in S^*$, by (VS) $(A \to B) \wedge (B \to C) \in w$ and $A \to C \notin w$, which is impossible by (S3). So $w = O$, $O \in V((A \to B) \wedge (B \to C))$, and $O \notin V(A \to C)$. By (V→), for some $w \in W$, $w \in V(A)$ and $w \notin V(C)$. By (V∧) $O \in V(A \to B)$, $V(B \to C)$. By (V→), $w \in V(B)$ and $w \in V(C)$, which is impossible.

ad (A4). If $O \notin V(\text{A4})$, by (V→) for some $w \in W$, $w \in V((A \to B) \wedge (A \to C))$ and $w \notin V(A \to (B \wedge C))$. If $w \in S^*$, by (VS) $(A \to B) \wedge (A \to C) \in w$ and $A \to (B \wedge C) \notin w$, which is impossible by (S4). So $w = O$, $O \in V((A \to B) \wedge (A \to C))$, and $O \notin V(A \to (B \wedge C))$. By (V→) for some $w \in W$, $w \in V(A)$ and $w \notin V(B \wedge C)$. By (V∧) $O \in V(A \to B)$, $V(A \to C)$. By (V→) $w \in V(B)$, $V(C)$. If $w \in S^*$, by (VS) $B, C \in w$ and $B \wedge C \notin w$, which is impossible by (S2*). So $w = O$, $O \in V(A)$, $V(B)$, $V(C)$ and $O \notin V(B \wedge C)$, which is impossible by (V∧).

ad (R1). By (V→).

ad (R2). By (V∧).

ad (R3). Suppose that $O \in V((A \wedge C) \to D)$, $V((B \wedge {\sim}C) \to E)$, $V({\sim}D)$, $V({\sim}E)$ and that $O \notin V({\sim}(A \wedge B))$. By (V~) $O \in V(A \wedge B)$ and $O \notin V(D)$, $V(E)$. By (V∧) $O \in V(A)$, $V(B)$. By (V~) $O \in V(C)$ or $O \in V({\sim}C)$. If $O \in V(C)$, by (V∧) $O \in V(A \wedge C)$, and by (V→) $O \in V(D)$, which is impossible. If $O \in V({\sim}C)$, by (V∧) $O \in V(B \wedge {\sim}C)$, and by (V→) $O \in V(E)$, which is again impossible.

ad (R4). Suppose that $O \in V((\wedge X) \to A)$ and $O \notin V((\wedge X)' \to A)$. By (V→) for some $w \in W$, $w \in V((\wedge X)')$ and $w \notin V(A)$. If $w \in S^*$, *by* (VS) $(\wedge X)' \in w$. By (S∧), $(\wedge X) \in w$. By (VS) $w \in V((\wedge X))$, and by (V→) $w \in V(A)$, which is impossible. So $w = O$, $O \in V((\wedge X)')$, and $O \notin V(A)$. By (V∧) $O \in V((\wedge X))$. By (V→) $O \in V(A)$, which is impossible.

ad (R5). By (V~).

This completes the proof of (T4.1).

5 Completeness

A few definitions:

(D1) For any set X, $X \vdash A$ if for some $n \geq i$ there are wffs $A_1, \ldots, A_n \in X$ such that $\vdash (A_1 \wedge \ldots \wedge A_n) \to A$.

(D2) X is *inconsistent* if for some wff A, $X \vdash A$ and $\vdash {\sim}A$.

(D3) X is *consistent* if X is not inconsistent.

(D4) X is *maximal* if for all wffs A of L, $A \in X$ or ${\sim}A \in X$.

Lemma (T5.1) *If* $A_1, \ldots, A_n \in X$, $X \vdash A_1 \wedge \ldots \wedge A_n$.

Proof: Immediate from (D1), (A1), and (R4).

Let A be a wff such that $\{A\}$ is consistent. We construct a maximal consistent (mc) set *Max-A* such that $A \in$ *Max-A* as usual:

Let $\langle A_1, \ldots, A_n, \ldots \rangle$ be an ordering of the wffs of L. Let $S_0 = \{A\}$. Let $S_{n+1} = S_n \cup \{A_n\}$ if $S_n \cup \{A_n\}$ is consistent, and let $S_{n+1} = S_n \cup \{{\sim}A_n\}$ otherwise. Finally, let *Max-A* be the union of the S_m as m ranges from 1 to ∞.

Lemma (T5.2) *Max-A is consistent.*

Proof: Suppose not. Then for some A_n, both $S_n \cup \{A_n\}$ and $S_n \cup \{{\sim}A_n\}$ are inconsistent. Let A_i be the first such wff. Since by hypothesis S_0 is consistent,

S_i is consistent. By (D1) and (D2), there are $B_1, \ldots, B_h$, $C_1, \ldots, C_k \in S_i$ such that $\vdash (B_1 \wedge \ldots \wedge B_h \wedge A_i) \rightarrow D$, where $\vdash {\sim} D$, and $\vdash (C_1 \wedge \ldots \wedge C_k \wedge {\sim} A_i) \rightarrow E$, where $\vdash {\sim} E$. By (R3) $\vdash {\sim}(B_1 \wedge \ldots \wedge B_h \wedge C_1 \wedge \ldots \wedge C_k)$. By (T5.1) $S_i \vdash B_1 \wedge \ldots \wedge B_h \wedge C_1 \wedge \ldots \wedge C_k$. By (D2) S_i is inconsistent, which is impossible.

Let *Max* be any mc set.

Lemma (T5.3) *If* $\vdash A$, $Max \vdash A$.

Proof: Suppose that $\vdash A$. By (T5.1) and (D4) $Max \vdash A$ or $Max \vdash {\sim} A$. By (R5) $\vdash {\sim}{\sim} A$. If $Max \vdash {\sim} A$, by (D2) *Max* is inconsistent, which is impossible. So $Max \vdash A$.

Lemma (T5.4) *If* $Max \vdash A$, $A \in Max$.

Proof: Suppose that $Max \vdash A$ and $A \notin Max$. By (D4) ${\sim} A \in Max$. By construction there are $B_1, \ldots, B_n \in Max$ such that $\vdash (B_1 \wedge \ldots \wedge B_n) \rightarrow A$. By (A2) and (R4) $\vdash (B_1 \wedge \ldots \wedge B_n \wedge {\sim} A) \rightarrow (B_1 \wedge \ldots \wedge B_n)$. By (A3), (R2), and (R1) $\vdash (B_1 \wedge \ldots \wedge B_n \wedge {\sim} A) \rightarrow A$. By (A2) and (R4) $\vdash (B_1 \wedge \ldots \wedge B_n \wedge {\sim} A) \rightarrow {\sim} A$. By (R2), (A4), and (R1) $\vdash (B_1 \wedge \ldots \wedge B_n \wedge {\sim} A) \rightarrow (A \wedge {\sim} A)$. By (D1) $Max \vdash A \wedge {\sim} A$. By (A0) $\vdash {\sim}(A \wedge {\sim} A)$. By (D2) *Max* is inconsistent, which is impossible.

Lemma (T5.5) $A \in Max$ *iff* $Max \vdash A$.

Proof: Immediate from (T5.4), (D1), and (A1).

Given an mc set *Max*, we construct a model *M* as follows:

For each wff ${\sim}(C \rightarrow D) \in Max$, let $s[{\sim}(C \rightarrow D)]$ be the intersection of all sets s that: (1) contain C; (2) for each wff $A \in s$ such that $A \rightarrow B \in Max$, $B \in s$; and (3) satisfy (S2), (S2*), (S3), (S$\wedge$), and (S4) of Section 3.

Lemma (T5.6) *For all* $A \in s[{\sim}(C \rightarrow D)]$, $C \rightarrow A \in Max$.

Proof: We note first that condition (1) above only concerns the wff C, and by (T5.5) and (A1) $C \rightarrow C \in Max$. The remainder of the proof proceeds by induction on the minimum number of applications of conditions (2) and (3) to $\{C\}$ it takes to get $A \in s[{\sim}(C \rightarrow D)]$ $(= s)$.

Inductive Hypothesis (IH): (T5.6) is satisfied for all applications of (2) and (3) less than n.

We shall show that (T5.6) holds for the nth application as well.

ad condition (2). That $E \rightarrow A \in Max$ and $E \in s$ are given, where $C \rightarrow E \in Max$ by IH. By (A3) $Max \vdash C \rightarrow A$. By (T5.5) $C \rightarrow A \in Max$.

ad (S2). $C \rightarrow (A \wedge E) \in Max$ by IH. $(A \wedge E) \rightarrow A \in Max$ by (A2), (T5.3), and (T5.5). $C \rightarrow A \in Max$ by (A3), (D1), and (T5.5).

ad (S2*). A is $E \wedge F$ and $C \rightarrow E$, $C \rightarrow F \in Max$ by IH. By (A4), (D1), and (T5.5) $C \rightarrow A \in Max$.

ad (S3). A is $E \rightarrow G$, where $C \rightarrow ((E \rightarrow F) \wedge (E \rightarrow G)) \in Max$ by IH. By (A3), (T5.3), (D1), and (T5.5) $C \rightarrow A \in Max$.

ad (S$\wedge$). A is $(\wedge X)'$, *where* $C \rightarrow (\wedge X) \in Max$ by IH. By (A1) and (R4) $\vdash (\wedge X) \rightarrow (\wedge X)'$. By (A3), (T5.3), (D1), and (T5.5) $C \rightarrow A \in Max$.

ad (S4). A is $E \rightarrow (F \wedge G)$, where $C \rightarrow ((E \rightarrow F) \wedge (E \rightarrow G)) \in Max$ by IH. By (A4), (A3), (D1), (T5.3), and (T5.5) $C \rightarrow A \in Max$.

This concludes the proof of (T5.6).

Lemma (T5.7) $D \notin s[\sim(C \rightarrow D)]$.

Proof: By (T5.6) if $D \in s[\sim(C \rightarrow D)]$, $C \rightarrow D \in Max$. By construction of $s[\sim(C \rightarrow D)]$, $\sim(C \rightarrow D) \in Max$. By (A1) and (D1) $Max \vdash (C \rightarrow D) \wedge \sim(C \rightarrow D)$. By (A0) $\vdash \sim((C \rightarrow D) \wedge \sim(C \rightarrow D))$. By (D2) Max is inconsistent, which is impossible.

We let $M = \langle O, At, S\rangle$, where $O = Max$, $At = \{B$: B is an atomic wff and $B \in Max\}$, and $S = \{s[\sim(C \rightarrow D)]: \sim(C \rightarrow D) \in Max\}$. That M is a model is immediate.

Lemma (T5.8) *If, for all $s \in S$, $A \in s$ only if $B \in s$, then $A \rightarrow B \in Max$.*

Proof: Suppose for all such s, $A \in s$ only if $B \in s$ and $A \rightarrow B \notin Max$. Since Max is mc $\sim(A \rightarrow B) \in Max$, $s[\sim(A \rightarrow B)] \in S$. $A \in s[\sim(A \rightarrow B)]$ by construction. But then $B \in s[\sim(A \rightarrow B)]$, which is impossible by (T5.7).

Lemma (T5.9) *$A \wedge B \in Max$ iff A, $B \in Max$.*

Proof: (1) Suppose $A \wedge B \in Max$. $Max \vdash A$ by (A2) and (D1), and $A \in Max$ by (T5.5). $\vdash (A \wedge B) \rightarrow (B \wedge A)$ by (A1) and (R4). $(B \wedge A) \in Max$ by (D1) and (T5.5). By (A2), (D1), and (T5.5) $B \in Max$.

(2) Suppose that A, $B \in Max$. Then by (A1), (D1), and (T5.5) $A \wedge B \in Max$.

Lemma (T5.10) *$A \vee B \in Max$ iff $A \in Max$ or $B \in Max$.*

Proof: (1) If $A \in Max$ or $B \in Max$ and $A \vee B \notin Max$, then $\sim(A \vee B) \in Max$. By (A1) and (D1) $Max \vdash A \wedge \sim(A \vee B)$ or $Max \vdash B \wedge \sim(A \vee B)$. By (A0) $\vdash \sim(A \wedge \sim(A \vee B))$ and $\vdash \sim(B \wedge \sim(A \vee B))$. By (D2) Max is inconsistent, which is impossible.

(2) If $A \vee B \in Max$ and A, $B \notin Max$, $\sim A$, $\sim B \in Max$. By (D1) and (A1) $Max \vdash (A \vee B) \wedge \sim A \wedge \sim B$. By (A0) $\vdash \sim((A \vee B) \wedge \sim A \wedge \sim B)$. By (D2) Max is inconsistent, which is impossible.

Lemma (T5.11) *If $A \rightarrow B \in Max$, then $A \in Max$ only if $B \in Max$.*

Proof: Suppose that $A \rightarrow B$, $A \in Max$ and $B \notin Max$. $\sim B \in Max$. *By* $(A1)$ *and* $(D1)$ $Max \vdash A \wedge \sim B \wedge (A \rightarrow B)$. By (A0.1) $\vdash \sim(A \wedge \sim B \wedge (A \rightarrow B))$. By (D2) Max is inconsistent, which is impossible.

Theorem (T5.12) *$B \in Max$ iff $Max \in V(B)$.*

Proof: By induction on the complexity of B.
Base Case: B is atomic. Then $B \in Max$ iff, by construction, $B \in At$ iff, by (VAt), $Max \in V(B)$.

IH: (T5.12) holds for all wffs of complexity less than n.

Where B is of complexity n we have four cases:

(1) B is $\sim C$. $\sim C \in Max$ iff, by the mc of Max, $C \notin Max$ iff, by IH, $Max \notin V(C)$ iff, by (V$\sim$), $Max \in V(\sim C)$.
(2) B is $C \wedge D$. $C \wedge D \in Max$ iff, by (T5.9), C, $D \in Max$ iff, by IH, $Max \in V(C)$, $V(D)$ iff, by (V$\wedge$), $Max \in V(C \wedge D)$.
(3) B is $C \vee D$. $C \vee D \in Max$ iff, by (T5.10), $C \in Max$ or $D \in Max$ iff, by IH, $Max \in V(C)$ or $Max \in V(D)$ iff, by (V$\vee$), $Max \in V(C \vee D)$.

(4) B is $C \rightarrow D$. $C \rightarrow D \in Max$ iff, by construction of S, (T5.11), and (T5.8), for all $w \in \{Max\} \cup S$ $C \in w$ only if $D \in w$ iff, by IH and (VS), $w \in V(C)$ only if $w \in V(D)$ iff, by (V→), $Max \in V(C \rightarrow D)$.

This completes the proof of (T5.12).

Theorem (T5.13) (Completeness) $\Vdash A$ *iff* $\vdash A$.

Proof: $\nVdash A$ iff for some model M, $M \nVdash A$ iff for some model M, $O \notin v(A)$ iff, by (V~), for some model M, $O \in V(\sim A)$ iff, by soundness, for some model Max, $Max \in V(\sim A)$ iff, by (T5.12), for some model Max, $\sim A \in Max$ iff, by mc, $\nvdash \sim\sim A$ iff, by (R3), $\nvdash A$.

6 Implicational extensions of L L is an awkward system owing to the presence in it of rules (R3), (R4), and (R5). These rules can be eliminated in favor of axioms (although not vice versa). We can replace (R3) by the axiom pair:

(A5) $(\sim B \wedge (A \rightarrow B)) \rightarrow \sim A$
(A6) $(\sim(A \wedge C) \wedge \sim(B \wedge \sim C)) \rightarrow \sim(A \wedge B)$,

if to preserve soundness we add two conditions to those governing the set S of stories:

(S5) if $\sim B \wedge (A \rightarrow B) \in s$, $\sim A \in s$
(S6) if $\sim(A \wedge C) \wedge \sim(B \wedge \sim C) \in s$, $\sim(A \wedge B) \in s$.

(R4) can be replaced by

(A7) $((\wedge X) \rightarrow A) \rightarrow ((\wedge X)' \rightarrow A)$
(S7) if $(\wedge X) \rightarrow A \in s$, $(\wedge X)' \rightarrow A \in s$,

and (R5) by

(A8) $(A \rightarrow \sim\sim A) \wedge (\sim\sim A \rightarrow A)$
(S8) $A \in s$ iff $\sim\sim A \in s$.

Severally and jointly the resulting axiomatics are sound and complete on the amended semantics, provided, of course, that in the canonical model the appropriate additional conditions are placed on the members of the set of stories S.

The resulting systems can again be enlarged to include other axioms that may be judged appropriate, given the intuitions with which we began. In the following each axiom is accompanied by its appropriate semantic condition(s):

(A9) $\sim(A \vee B) \rightarrow \sim A$
(S9) if $\sim(A \vee B) \in s$, $\sim A \in s$
(A10) $(\sim(\vee X) \rightarrow A) \rightarrow (\sim(\vee X)' \rightarrow A)$
(S10.1) if $\sim(\vee X) \in s$, $\sim(\vee X)' \in s$
(S10.2) if $\sim(\vee X) \rightarrow A \in s$, $\sim(\vee X)' \rightarrow A \in s$
(A11) $(A \wedge B) \rightarrow (A \vee B)$
(S11) if $A \wedge B \in s$, $A \vee B \in s$
(A12) $(A \rightarrow B) \rightarrow (A \rightarrow (A \wedge B))$
(S12) if $A \rightarrow B \in s$, $A \rightarrow (A \wedge B) \in s$ (as well as (S2*))
(A13) $(A \rightarrow B) \rightarrow (A \rightarrow (A \vee B))$

(S13) if $A \to B \in s$, $A \to (A \vee B) \in s$ (as well as (S12) and (S2*))
(A14) $((\vee X) \to A) \to ((\vee X)' \to A)$
(S14) if $(\vee X) \to A \in s$, $(\vee X)' \to A \in s$
(A15) $(A \to \sim A) \to \sim A$
(S15) if $A \to \sim A \in s$, $\sim A \in s$
(A16) $(A \wedge (A \to B)) \to B$
(S16) if $A \wedge (A \to B) \in s$, $B \in s$.

The conditions required for membership in the set S of stories have thus far all been straightforward and simple. But for

(A17) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

the case becomes more complex. We shall see what is required by working through a soundness proof:

ad (A17). Suppose that $O \notin V$(A17). Then by (V7) for some $w \in W$, $w \in V((A \to C) \wedge (B \to C))$ and $w \notin V((A \vee B) \to C)$. If $w \in S^*$, then by (VS) $(A \to C) \wedge (B \to C) \in w$ and $(A \vee B) \to C \notin w$, which is impossible by

(S17.1) if $(A \to C) \wedge (B \to C) \in s$, $(A \vee B) \to C \in s$.

So, $w = O$. $O \in V((A \to C) \wedge (B \to C))$ and $O \notin V((A \vee B) \to C)$. By ($V\wedge$) $O \in V(A \to C)$, $V(B \to C)$. By ($V\!\to$) for some $w \in W$, $w \in V(A \vee B)$ and $w \notin V(C)$, and for all $u \in W$ if $u \in V(A)$, $u \in V(C)$, and for all $u \in W$ if $u \in V(B)$, $u \in V(C)$. If $u \in S^*$, by (VS) $A \vee B \in u$ and $C \notin u$, for all $r \in S^*$ if $A \in r$, $C \in r$, and for all $r \in S^*$ if $B \in r$, $C \in r$, which is impossible by

(S17.2) if for all $r \in S$, $A \in r$ only if $C \in r$, for all $r \in S$, $B \in r$ only if $C \in r$, and $A \vee B \in s$, $C \in s$.

So $u = O$. $O \in V(A \vee B)$ and $O \notin V(C)$. By ($V\vee$) $O \in V(A)$ or $O \in V(B)$. If $O \in V(A)$, by ($V\!\to$) $O \in V(C)$, which is impossible. If $O \in V(B)$, by ($V\!\to$) $O \in V(C)$, which is again impossible.

Thus soundness requires two conditions on the members of S, (S17.1) and (S17.2). (S17.2) differs from any of the conditions we have so far encountered. Up to now each member of S has been treated in grand isolation. With (17.2), however, what goes on in each member of S depends upon what goes on in all its fellows.

In addition, the nature of (S17.2) poses a problem when it comes to condition (3) of the definition of the set $s[\sim(C \to D)]$ given the wff $\sim(C \to D) \in$ *Max*.

If (A17) is added to, say, our original axiom set, we can define $s[\sim(C \to D)]$ as follows: for each wff $\sim(C \to D) \in$ *Max*, let $s[\sim(C \to D)]$ be the intersection of all sets s that: (1) contain C; (2) for each wff $A \in s$ such that $A \to B \in$ *Max*, $B \in s$; and (3) satisfy (S2), (S2*), (S3), ($S\wedge$), (S4), (S17.1), and

(S17.2′) if $A \to C \in$ *Max*, $B \to C \in$ *Max*, and $A \vee B \in s$, $C \in s$.

Here we have simply substituted "$A \to C \in$ *Max*" and "$B \to C \in$ *Max*" for "for all $r \in S$ if $A \in r$, $C \in r$" and "for all $r \in S$ if $B \in r$, $C \in r$". Clearly if $A \to B \in$ *Max*, condition (2) will provide that if $A \in s[\sim(C \to D)]$, $B \in s[\sim(C \to D)]$ for all $\sim(C \to D) \in$ *Max*.

It also must be shown that the induction for (T5.6) holds when (S17.1) and (S17.2′) are added to condition (3). This is easy. We shall do just (S17.2′).

ad (S17.2′). $E \to A \in Max$. $F \to A \in Max$. $C \to (E \vee F) \in Max$ by IH. $(E \to A) \wedge (F \to A) \in Max$ by (A1), (D1), and (T5.5). $(E \vee F) \to A \in Max$ by (A17), (D1), and (T5.5). And $C \to A \in Max$ by (A3), (D1), and (T5.5).

It remains to show that S of the canonical model satisfies (S17.2). This is immediate from condition (2) and (S17.2′) of condition (3) of the construction of each $s[\sim(C \to D)]$.

Any or all of the following axiom schemes may also be added to L if we make the appropriate semantic adjustments.[4] Of course, given the motivation with which we began, many of these will seem out of place, if not downright silly:

(A18) $(A \to (B \to C)) \to ((A \wedge B) \to C)$
(A19) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
(A20) $(A \to ((B \to C) \to D)) \to ((B \to C) \to (A \to D))$
(A21) $((A \to B) \wedge (A \to C)) \to ((A \wedge B) \to C)$
(A22) $A \to (A \vee B)$
(A23) $(A \to C) \to ((A \wedge B) \to C)$
(A24) $(A \to B) \to ((C \to A) \to (C \to B))$
(A25) $(A \to B) \to ((B \to C) \to (A \to C))$
(A26) $(A \to B) \to (\sim B \to \sim A)$
(A27) $\sim A \to \sim(A \wedge B)$.

As an illustration of the "appropriate" semantic adjustments, let us go through the proof of the soundness of, say, (A24):

Suppose that $O \notin V(\text{A24})$. Then by (V→) for some $w \in W$, $w \in V(A \to B)$ and $w \notin V((C \to A) \to (C \to D))$. If $w \in S^*$, by (VS) $A \to B \in w$ and $(C \to A) \to (C \to B) \notin w$, which is impossible by

(S24.1) if $A \to B \in s$, $(C \to A) \to (C \to B) \in s$.

So $w = O$. $O \in V(A \to B)$ and $O \notin V((C \to A) \to (C \to B))$. By (V→) for all $u \in W$, $u \in V(A)$ only if $u \in V(B)$; and by (V→) for some $w \in W$, $w \in V(C \to A)$ and $w \notin V(C \to B)$. If $w \in S^*$, then by (VS) $C \to A \in w$ and $C \to B \notin w$, and also by (VS) for all $u \in S^*$ if $A \in u$, $B \in u$, which is impossible by

(S24.2) if for all $r \in S$, $A \in r$ only if $B \in r$ and $C \to A \in s$, then $C \to B \in s$.

So $w = O$. Thus $O \in V(C \to A)$ and $O \notin V(C \to B)$. By (V→) for all $u \in W$, $u \in V(C)$ only if $u \in V(A)$; and by (V→) for some $w \to W$, $w \in V(C)$ and $w \notin V(B)$. If $w \in S^*$, then by (VS) $C \in w$ and $B \notin w$ and for all $r \in S^*$, $C \in r$ only if $A \in r$, which is impossible by

(S24.3) if for all $r \in S$, $A \in r$ only if $B \in r$, for all $r \in S$, $C \in r$ only if $A \in r$, and $C \in s$, then $B \in s$.

So again $w = O$. $O \in V(C)$ and $O \notin V(B)$. By (V→) $O \to V(A)$ and $O \in V(B)$, which is impossible.

For this proof we need the semantic conditions (S24.1)–(S24.3). (S24.2) and (S24.3) will be amended further to

(S24.2′) if $A \to B \in Max$ and $C \to A \in s$, $C \to B \in s$
(S24.3′) if $A \to B$, $C \to A \in Max$ and $C \in s$, $B \in s$,

in the construction of the members of the set S of the canonical model.

An interesting issue arises in connection with the formula:

(A28) $(A \to (B \to C)) \to (B \to (A \to C))$,

especially since an instance of it, viz. (A20), raises no problem for the semantics as described thus far.

Let us attempt to show soundness for (A28) to get an idea of where things go awry.

Suppose that $O \notin V$(A28). Then by (V→) for some $w \in W$, $w \in V(A \to (B \to C))$ and $w \notin V(B \to (A \to C))$. If $w \in S^*$, by (VS) $A \to (B \to C) \in w$ and $B \to (A \to C) \notin w$, which is impossible by

(S28.1) if $A \to (B \to C) \in s$, $B \to (A \to C) \in s$.

So $w = O$. $O \in V(A \to (B \to C))$ and $O \notin V(B \to (A \to C))$. By (V→) for all $u \in W$, $u \in V(A)$ only if $u \in V(B \to C)$; and for some $w \in W$, $w \in V(B)$ and $w \notin V(A \to C)$. If $w \in S^*$, then by (VS) $B \in w$ and $A \to C \notin w$ and by (VS) for all $u \in S^*$, $A \in u$ only if $B \to C \in u$, which is impossible by

(S28.2?) if for all $r \in S$, $A \in r$ only if $B \to C \in r$ and $B \in s$, $A \to C \in s$.

So again $w = O$. Thus, to summarize $O \in V(A \to (B \to C))$, $V(B)$, and $O \notin V(A \to C)$. By (V→) $w \in V(B \to C)$. And here we seem to be stuck, for the fact is that $O \in V(B)$ tells us nothing about what goes on in w. It would be nice to be able to require

(S28.3?) if for all $r \in S$, $A \in r$ only if $B \to C \in r$, $O \in V(B)$, and $A \in s$, then $C \in s$.

Notice here that if B were an implication, as it is in (A20), no problem need occur. But in (S28.3?) we have made what happens in stories in the model depend upon our evaluations, when things should be the other way around. This is unsatisfactory.

*7 **The story of the world*** The reader may well have noted an inconsistency in motivation in Section 1. An author who claims to be unhappy with featureless "possible worlds" and setups should not be allowed to rest comfortably with the notion of model just presented. O may have been called an "actual world", but in itself it is about as featureless and barren a node as one could conjure up.

By taking a complete history of a world, an mc set of wffs of our language, as our model, we not only ensure that the world it represents has clear internal structure—the roughness of mountains and valleys, so to speak—we are also enabled to make use of this structure in putting conditions on stories. That we could not do so was precisely the problem with the troublesome (A28) just discussed.

It might, of course, be pointed out that O as used earlier is not entirely featureless. The set At of atomic wffs gives it features. This is true. In the presence of classical evaluation rules, At determines all classical truths in O. This being so, an objection can hardly be raised to a proposal that, in the presence of all evaluation rules, O should determine the whole truth, classical and non-classical.

Let us then redefine the notion of model for our original language L, keeping the valuation rules (VS)–(V→) as is. A model for L is now an mc set *Max* of wffs of L. Valuation rules (VAt)–(V→) presuppose that we have an actual world O. Let $O = Max$. Valuation rule (VAt) presupposes that we have a set AT of atomic wffs of L. Let the set At for *Max* be $\{A\colon A$ is an atomic wff and $A \in Max\}$. Valuation rule (VS) presupposes that we have a set S of "stories". We now define the set S for *Max*. As in Section 5, for each wff $\sim(C \to D) \in Max$, let $s[\sim(C \to D)]$ be the intersection of all sets s that (1) contain C, (2) for each wff $A \in s$ such that $A \to B \in Max$, $B \in s$, and (3) satisfy (S2), (S2*), (S3), (S∧), and (S4) of Section 3. Finally, let the set S of stories for *Max* be $\{s[\sim(C \to D)]\colon \sim(C \to D) \in Max\}$.

Theorem (T7.1) *(T5.6)–(T5.12) hold for our new notion of Model.*

Proof: Immediate.

Theorem (T7.2) *There is a model for L.*

Proof: The question reduces to whether or not there is an mc set for L.

It need only be shown that the set Ax of axioms of L is consistent. A set *Max-Ax* can then be constructed as in Section 5. By (T5.2) *Max-Ax* will be mc.

Suppose that Ax is not consistent. Then $Ax \vdash B$ where $\vdash \sim B$. Let J be a truth functional valuation that assigns exactly one member of $\{T, F\}$ to each wff of L treating "$\sim$", "$\wedge$", and "$\vee$" classically and treating "$A \to B$" as "$\sim A \vee B$". Then $J(\text{A0}), J(\text{A0.1}), \ldots, J(\text{A4}) = T$, and (R1)–(R5) are such that for each theorem A of L $J(A) = T$. Thus we have that for $A \in \{B\colon \vdash B\}$, $J(A) = T$. But $\{B\colon Ax \vdash B\} \subseteq \{B\colon \vdash B\}$. So if $Ax \vdash B$, $J(B) = T$. By the valuation rule for "$\sim$", $J(\sim B) \neq T$. Yet since $\vdash \sim B$, $J(\sim B) = T$, which is impossible.

We are now in a position to deal with wffs like (A28). Let us reconstruct our attempt at a soundness proof for (A28):

Suppose that $O \notin V((A \to (B \to C)) \to (B \to (A \to C)))$. By (V→) for some $w \in W$, $w \in V(A \to (B \to C))$ and $w \notin V(B \to (A \to C))$. If $w \in S^*$, by (VS) $A \to (B \to C) \in w$ and $B \to (A \to C) \notin w$, which is impossible by

(S28.1) if $A \to (B \to C) \in s$, $B \to (A \to C) \in s$.

So $Max = w$. $Max \in V(A \to (B \to C))$ and $Max \notin V(B \to (A \to C))$. By (V→) for some $x \in W$, $x \in V(B)$ and $x \notin V(A \to C)$. If $x \in S^*$, then by (VS) $B \in x$ and $A \to C \notin x$, which is impossible by

(S28.2) if $A \to (B \to C) \in Max$ and $B \in s$, $A \to C \in s$

and $(T?)$. So $x = Max$. $Max \in V(B)$ and $Max \notin V(A \to C)$. By (V→) for some $y \in W$, $y \in V(A)$ and $y \notin V(C)$. If $y \in S^*$, $A \in y$ and $C \notin y$, which is impossible by

(S28.3) if $A \to (B \to C) \in Max$, $B \in Max$, and $A \in s$, then $C \in s$

and $(T?)$. So $y = Max$. We have then $Max \in V(A \to (B \to C))$, $V(B)$, $V(A)$, and $Max \notin V(C)$. By (V→) $Max \in V(B \to C)$, $V(C)$, which is impossible.

Let us call the axiomatization that results when (A28) is added to L, L^*. In defining the set S of "stories" for a model *Max* for L^*, we will need to amend condition (3) of the definition of $s[\sim(C \to D)]$ to have s satisfy (S28.1)–(S28.3) as well as (S2), (S2*), (S3), (S∧), and (S4).

The remaining gap in the soundness proof above is $(T?)$. *For* $(T?)$ what we really need to show is

Theorem (T7.3) *For all wffs* $A \in L^*$, $A \in Max$ *iff* $Max \in V(A)$.

Proof: The only thing that needs to be done is to ensure that (R5.6) still holds when (S28.1)–(S28.3) are added to condition (3) defining $s[\sim(C \to D)]$. This is easy. We continue the proof of (T5.6) under the IH.

ad (S28.1). A is $F \to (E \to G)$ where $C \to (E \to (F \to G)) \in Max$ by IH. By (D1), (T5.3), (T5.5), and (A28) $(E \to (F \to G)) \to (F \to (E \to G)) \in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

ad (S28.2). A is $E \to G$ where $C \to F \in Max$ by IH and $E \to (F \to G) \in Max$. By (D1), (T5.5), and (A28), $F \to (E \to G) \in Max$. By (D1), (T5.5), and (A3) $C \to A \in Max$.

ad (S28.3). $C \to E \in Max$ by IH, and $E \to (F \to G)$, $F \in Max$. By (D1), (T5.5), and (A28) $F \to (E \to G) \in Max$. Since *Max* is mc, $E \to G \in Max$ or $\sim(E \to G) \in Max$. If $\sim(E \to G) \in Max$, by (D1), (T5.5), and (A1) $F \wedge \sim(E \to G) \wedge (F \to (E \to G)) \in Max$. But this is impossible, since by (A0.1), $\vdash \sim(F \wedge \sim(E \to G) \wedge (F \to (E \to G)))$ and *Max* is consistent. So $E \to A \in Max$. By (D1), (T5.5), and (A3) $C \to A \in Max$.

This completes the proof of (T7.3).

There are thus three levels of complexity in the notions of story that have been presented. In the first, what must be in a story depends only upon the story itself and what is already in it. In the second, it depends upon what is in the story and what is in other stories in the set S as well. In the third, it depends upon all this and in addition upon the history of the actual world that the model represents.

8 L modalized Let us return to the first notion of model and to the problem of (A28) once again. If we were to introduce "$\Box$" into L, then while the fact that $O \in V(B)$ need not tell us anything about what goes on in stories, that $O \in V(\Box B)$ would.

Let us define "$A \supset B$" as "$\sim A \vee B$", add to L:

(A□1) $\Box A \to A$
(A□2) $(A \to B) \to (\Box A \to \Box B)$
(A□3) $\Box A \supset (B \to A)$;

add to the conditions on $s \in S \in \langle O, A, t, S\rangle$:

(S□1) $\Box A \in s$ only if $A \in s$
(S□2.1) $(A \to B) \in s$ only if $\Box A \to \Box B \in s$
(S□2.2) if, for all $r \in S$, $A \in r$ only if $B \in r$ and $\Box A \in s$, then $\Box B \in s$;

and add to our valuation rules:

(V$\Box$) if, for all $w \in W$, $w \in V(A)$, $O \in V(\Box A)$; otherwise $O \notin V(\Box A)$.

Call the resulting language $L\Box$.

(A$\Box$3) may seem odd. What (A$\Box$3) does is to reflect the fact that if $O \in V(\Box A)$, A is in *every* story. Perhaps there are things like this. That there is exactly one prime number between four and six may be true in every story.

Theorem *$L\Box$ as just described is sound and complete.*

Proof: Given conditions (S$\Box$1), (S$\Box$2.1), and (S$\Box$2.2), the soundness proofs for (A$\Box$1)-(A$\Box$3) are straightforward.

Completeness: In the construction of the model M $(= \langle Max, At, S\rangle)$, condition (3) for $s[\sim(C \to D)]$ for $\sim(C \to D) \in Max$ will be expanded to include (S$\Box$1), (S$\Box$2.1), and

(S$\Box$2.2′) if $A \to B \in Max$ and $\Box A \in s$, $\Box B \in s$,

and in addition $s[\sim C \to D)]$ will have to satisfy

(4) if $\Box A \in Max$, $A \in s$.

Conditions (1) and (2) on $s[\sim(C \to D)]$ remain as before.

Let the set $\Box$ be the intersection of all sets s that satisfy (2), (3), and (4) as just described

Lemma (T8.1) *For all A in $s[\sim(C \to D)]$, $C \to A \in Max$.*

Proof: This is (T5.6) with four additional cases to be checked under IH:

ad (S$\Box$1). $C \to \Box A \in Max$ by IH. By (D1), (T5.3), (T5.5), and (A$\Box$1), $\Box A \to A \in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

ad (S$\Box$2.1). A is $\Box E \to \Box F$, where $C \to (E \to F) \in Max$ by IH. By (D1), (T5.3), (T5.5), and (A$\Box$2), (A$\Box$2) $\in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

ad (S$\Box$2.2′). A is $\Box F$, where $C \to \Box E \in Max$ by IH, and $E \to F \in Max$. By (D1), (T5.5), and (A$\Box$2), $\Box E \to \Box F \in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

ad (4). $\Box A \in Max$. We must show that $C \to A \in Max$. By (D1), (T5.3), (T5.5), and (A$\Box$3), $\Box A \supset (C \to A) \in Max$. Since Max is mc, $C \to A \in Max$ or $\sim(C \to A) \in Max$. If $\sim(C \to A) \in Max$, by (D1) and (A1), $Max \vdash \Box A \wedge (A \supset (C \to A)) \wedge \sim(C \to A)$. But by (A0) $\vdash \sim(\Box A \wedge (\Box A \supset (C \to A)) \wedge \sim(C \to A))$, and so Max is inconsistent, which is impossible. So $C \to A \in Max$.

This completes the proof of (T8.1).

Lemma (T8.2) *For all $A \in \Box$, there is some $\Box C \in Max$ such that $C \to A \in Max$.*

Proof: Let $\Box^* = \{B\colon \Box B \in Max\}$. We note first that condition (4) in the construction of $\Box$ concerns precisely $\Box^*$. By (D1), (T5.3), (T5.5), and (A1), each wff $A \in \Box^*$ is such that $\Box A \in Max$ and $A \to A \in Max$. The remainder of the proof proceeds by induction on the minimum number of applications of (2) and (3) to $\Box^*$ it takes to get $A \in \Box$.

The induction is almost a mirror of the proof of (T8.1), so not all cases will be spelled out. Assuming the lemma to hold for applications less than n, we have:

ad (2). $\Box C$, $C \to E$, $E \to A \in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

ad (S2). $\Box C$, $C \to (A \wedge E) \in Max$ by IH. By (D1), (T5.3), (T5.5), and (A2), $(A \wedge E) \to A \in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

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ad (S$\Box$2.1). A is $\Box E \to \Box F$, where $\Box C$, $C \to (E \to F) \in Max$ by IH. By (D1), (T5.3), (T5.5), and (A$\Box$2), (A$\Box$2) $\in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

ad (S$\Box$2.2′) A is $\Box F$, where $\Box C$, $C \to \Box E \in Max$ by IH and $E \to F \in Max$. By (D1), (T5.5), and (A$\Box$2), $\Box E \to \Box F \in Max$. By (D1), (T5.5), and (A3), $C \to A \in Max$.

This completes the proof of (T8.2).

Lemma (T8.3) *If* $\sim\Box A \in Max$, $A \notin \Box$.

Proof: Suppose that $A \in \Box$. By (T8.2) there is some $\Box C \in Max$ such that $C \to A \in Max$. By (D1), (T5.5), and (A$\Box$2), $\Box C \to \Box A \in Max$. If $\sim\Box A \in Max$, by (D1) and (A1), $Max \vdash \Box C \wedge \sim\Box A \wedge (\Box C \to \Box A)$. By (A0.1) $\vdash \sim(\Box C \wedge \sim\Box A \wedge (\Box C \to \Box A))$. So *Max* is inconsistent, which is impossible. So $\sim\Box A \notin Max$.

Now let $M = \langle Max, At, S\rangle$, where At is as in Section 5, and $S = \{s[\sim(C \to D)]\colon \sim(C \to D) \in Max\} \cup \{\Box\}$.

That M is a model is immediate.

Lemma (T8.4) *If* $\Box A \in Max$, $A \in Max$.

Proof: By (D1), (T5.5), and (A$\Box$1).

Theorem (T8.5) $B \in Max$ *iff* $Max \in V(B)$.

Proof: We need to examine one case in addition to those already covered in the proof of (T5.12):

(5) B is $\Box C$. $\Box C \in Max$ iff, by the construction of S ($\Box$ in particular) and *Max*, (T8.4), and (T8.3), for all $w \in W$, $C \in w$ iff, by IH, $w \in V(C)$ iff, by (V$\Box$), $Max \in V(\Box C)$.

This concludes the proof of (T8.5).

It is plain that we can use the second notion of model set out in Section 7 appropriately amended for $L\Box$.

It turns out, too, that we can add

(A$\Box$28) $(A \to (B \to C)) \to (\Box B \to (A \to C))$ to $L\Box$

and produce the semantic conditions to make axiomatization sound and complete. Furthermore,

(A$\Box$5) $\Box A \to \Box\Box A$,

and

(A□6) $\sim\Box\sim A \rightarrow \Box\sim\Box\sim A$

can also be added in this way to $L\Box$.

The rule of necessitation is absent from $L\Box$. That it should be absent is straightforward upon reflection. We have been concerned with what is in a story, with what is implied by what. To have a rule of necessitation in $L\Box$ is tantamount to demanding that all theorems must appear in all stories; for by the rule if $\vdash A$, $\vdash \Box A$. yet the whole idea of a story is that of a fiction in which even the impossible may happen and in which not everything is settled—indeed, sometimes not even logical truths. This is what allows stories to do the job of illustrating what follows from what.

Clearly when

(R□1) if $\vdash A$, $\vdash \Box A$

is added to the rules of $L\Box$, the appropriate semantic condition to add for the construction of $s[\sim(C \rightarrow D)]$ and the set $\Box$ is:

(SR□1) if $\vdash A$, $A \in s$,

which just will not do.

Of course, one might introduce a new modal operator '$\underline{\underline{\Box}}$' with the valuation rule

(V$\underline{\underline{\Box}}$) if, for all $w \in \{O\} \cup S\underline{\underline{\Box}}$, $w \in V(A)$, $O \in V(\underline{\underline{\Box}}A)$; otherwise $O \notin V(\underline{\underline{\Box}}A)$,

where $S\underline{\underline{\Box}}$ is $\{s\colon s \in S$ and s is mc$\}$, and expect to validate a rule of necessitation. The details are left for the interested reader.

9 Identity We extend the language L to Li by including a set of names, for each $n \geq i$ a set of n-place predicates, and a two-place predicate "$=$". Where F is an n-place predicate and $\langle c_1, \ldots, c_n\rangle$ is an n-tuple of names, $Fc_1 \ldots c_n$ is a wff. "$= bc$" is also written "$b = c$". Where A is a wff $Ab//a$ is a result of replacing zero or more occurrences of the name b in A by the name a.

The axiom schemes of Li are those of L and

(Ai1) $a = a$
(Ai2) $a = b \rightarrow b = a$
(Ai3) $(a = b \wedge A) \rightarrow Ab//a$.

In order to see what kind of model we need, let us try to prove soundness for (Ai3).

Suppose that $O \notin V$(Ai3). Then by (V→) for some $w \in W$, $w \in V(a = b \wedge A)$ and $w \notin V(Ab//a)$. If $w \in S^*$, then by (VS) $a = b \wedge A \in w$ and $Ab//a \notin w$, which is impossible by

(Si3) if $a = b \wedge A \in s$, $Ab//a \in s$.

So $w = O$. $O \in V(a = b \wedge A)$ and $O \notin V(Ab//a)$. By (V∧) $O \in V(a = b)$, $V(A)$. Thus by a normal valuation rule for predicates $\langle V(a), V(b)\rangle \in V(=)$, i.e., $V(a) = V(b)$. Now all cases in which A is not of the form $B \rightarrow C$ are straightforwardly proved in an induction. But where A is, say $(Fb \wedge B) \rightarrow Fb$ and $Ab//a$ is $(Fb \wedge B) \rightarrow Fa$, we have $O \in V((Fb \wedge B) \rightarrow Fb)$ and $O \notin V((Fb \wedge B) \rightarrow Fa)$. What would help is some condition like

(S?) if $O \in V(a = b)$ $[V(a) = V(b)]$ and $B \in s$, $Bb//a \in s$,

just the kind of condition that led us in Section 7 to take an mc set as a model. We take the hint.

A model for *Li* is an mc set *Max*. Given *Max* we can construct the set *At* as before. For each wff $\sim(C \rightarrow D) \in Max$, let $s[\sim(C \rightarrow D)]$ be the intersection of all sets s such that: (1) $C \in s$; (2) if $A \in s$ and $A \rightarrow B \in Max$, $B \in s$; and (3) s satisfies (S2)–(S4) and

(Si2) if $a = b \in s$, $b = a \in s$
(Si3.1) if $a = b \wedge A \in s$, $Ab//a \in s$
(Si3.2) if $a = b \in Max$ and $A \in s$, $Ab//a \in s$.

Let $|a| = \{b\colon a = b \in Max\}$. For an n-place predicate F, let $|F| = \{\langle |a_1|, \ldots, |a_n|\rangle\colon Fa_1 \ldots a_n \in Max\}$.

We then have the following valuation rules in addition to (VS)–(V→):

(VN) If c is a name, $V(c) = |c|$.
(VP) If F is a predicate, $V(F) = |F|$.
(VB) If F is an n-place predicate and $\langle a_1, \ldots, a_n\rangle$ is an n-tuple of names, $1 \leq n$, $\langle V(a_1), \ldots, V(a_n)\rangle \in V(F)$ only if $O \in V(Fa_1 \ldots a_n)$; $O \notin V(Fa_1 \ldots a_n)$ otherwise.

Lemma (T9.1) *For all $A \in s[\sim(C \rightarrow D)]$, $C \rightarrow A \in Max$.*

Proof: The proof proceeds as in (T5.6) with three additional cases:

ad (Si2). A is $b = a$, where $C \rightarrow a = b \in Max$ by IH. By (D1), (T5.3), (T5.5), and (Ai2), $a = b \rightarrow b = a \in Max$. By (D1), (T5.3), and (A3), $C \rightarrow A \in Max$.

ad (Si3.1). A is $Bb//a$, where $C \rightarrow (a = b \wedge B) \in Max$ by IH. The proof proceeds by (D1), (T5.3), (T5.5), (Ai3), and (A3).

ad (Si3.2). A is $Bb//a$, where $C \rightarrow B \in Max$ by IH and $a = b \in Max$. By (D1) and (Ai3), $C \rightarrow Bb//a \in Max$.

This completes the proof of (T9.1).

Lemma (T9.2) *(T5.7)–(T5.11) all hold for Li.*

Proof: Immediate.

Theorem (T9.3) *$B \in Max$ iff $Max \in V(B)$.*

Proof: Proceeds as in (T5.12) with one addition to the Base Case of the induction:

B is $Fc_1 \ldots c_n$. $Fc_1 \ldots c_n \in Max$ iff, by construction, $\langle |c_1|, \ldots, |c_n|\rangle \in |F|$ iff, by (VN) and (VP), $\langle V(c_1), \ldots, V(c_n)\rangle \in V(F)$ iff, by (VB), $Max \in V(Fc_1 \ldots c_n)$.

This concludes the proof of (T9.3).

Theorem (T9.4) *There is a model for Li.*

Proof: That the set of axioms of Li is a consistent set can be shown easily in the style of the proof of (T7.2). This set then can be extended to an mc set.

10 Quantification We extend the language Li to Lq by including a denumerable set of names, a denumerable set of variables, and a constant $\forall$. Where A is a wff and A' is the result of replacing zero or more occurrences of names in A by a variable x, $(\forall x)A'$ is a wff. The definitions of $\exists$ and of a free occurrence of a variable x in a formula A are the usual. Ax/c is the result of replacing all free occurrences of a variable x in a formula A by a name c. We write $A(-c-)$ in case the name c does *not* occur in the formula A.

Intuition provides no help when it comes to deciding how to treat open formulas in stories. All the wffs of Lq are therefore closed.

The axioms and rules of Lq are those of Li plus:

(AQ1) $((\forall x)A \wedge B) \rightarrow Ax/c$, *where x does not occur free in A or the name c does occur in B*

(AQ2) $Ax/c \rightarrow (\exists x)A$

(AQ3) $(\forall x)(A \rightarrow B) \rightarrow ((\forall x)A \rightarrow (\forall x)B)$

(RQ1) if $\vdash A(-c-)x/c$, $\vdash (\forall x)A$

(RQ2) if $\vdash (A(-c-) \wedge (\forall x)C(-c-)) \rightarrow D$, $\vdash (B(-c-) \wedge {\sim}(\forall x)C(-c-) \wedge {\sim}C(-c-)x/c) \rightarrow E$, $\vdash {\sim}D$, and $\vdash {\sim}E$, then $\vdash {\sim}(A(-c-) \wedge B(-c-))$

(RQ3) if $\vdash (A(-c-) \wedge c = b) \rightarrow B$ and $\vdash {\sim}B$, $\vdash {\sim}A(-c-)$.

Let us begin by stating the normal valuation rules for a first-order language. These are (VS)-(V$\rightarrow$), (VN), (VP), (VB), and

(V$\forall'$) If $O \in Vc(A(-c-)x/c)$ for all Vc, $O \in V((\forall x)A)$; otherwise $O \notin V((\forall x)A)$.

Here Vc will turn out to be any valuation that differs at most from V in its assignment to the name c.

Prior to setting out the nature of a model for Lq, let us try our tactic of sketching out a soundness proof for (AQ1):

Suppose that $O \notin V$(AQ1). Then by (V$\forall$) for some $w \in W$, $w \in V((\forall x)A \wedge B)$ and $w \notin V(Ax/c)$. If $w \in S^*$, then by (VS) $(\forall x)A \wedge B \in w$ and $Ax/c \notin w$, which is impossible by

(SQ1) if $(\forall x)A \wedge B \in s$ and either c occurs in B or x does not occur free in A, $Ax/c \in s$.

So $w = O$. $O \in V((\forall x)A \wedge B)$ and $O \notin V(Ax/c)$. By (V$\wedge$) $O \in V((\forall x)A)$, $V(B)$. By (V$\forall$) for all Va, $O \in Va(A(-a-)x/a)$. At this point what is required is the following lemma:

Lemma ? *If $Va(a) = V(c)$, then $O \in Va(A(-a-)x/a)$* iff *$O \in V(Ax/c)$.*

In the inductive proof of this lemma we shall have one case in which A is $B \rightarrow C$. Here, as in Section 9,

(S?) if $a = b \in Max$ and $B(-a-)x/a \in s$, $Bx/b \in s$

seems to be just what we need, but cannot have. So we must search for a notion of model similar to that used in Section 7.

(D5) A set X of wffs is *instanced* iff for all wffs $\sim(\forall x)A \in X$, there is some name c such that $\sim Ax/c \in X$.

Let A be a wff of Lq such that $\{A\}$ is consistent. We shall construct an mc instanced (mci) set *Max-A* such that $A \in$ *Max-A* by extending Lq to $Lq+$ with the help of a denumerable set Na of new names. Let $\langle A_1, \ldots, A_n, \ldots\rangle$ be an ordering of the wffs of Lq and $\langle c_1, \ldots, c_n, \ldots\rangle$ be an ordering of the members of Na. Then where $S_i = \langle B_1, \ldots, B_n, \ldots\rangle$ is an ordering of the wffs of $Lq \cup \{c_1, \ldots, c_i\}$, let S_{i+1} be an ordering of the wffs of $Lq \cup \{c_1, \ldots, c_i, c_{i+1}\}$ that begins with S_i. In this way we order $Lq \cup Na$ ($Lq+$) and ensure that however far we go along the ordering there will be some name in Na that has not yet appeared in any wff.

Let $S_0 = \{A\}$.

(1) *If neither* A_n nor $\sim A_n$ is of the form $\sim(\forall x)B$, let $S_{n+1} = S_n \cup \{A_n\}$, where $S_n \cup \{A_n\}$ is consistent, and $S_n \cup \{\sim A_n\}$ otherwise.

(2) If A_n is $\sim(\forall x)B$, let $S_{n+1} = S_n \cup \{\sim(\forall x)B \wedge \sim Bx/c\}$, where c is the first name not to appear in A_n or in any wff in S_n and where $S_n \cup \{\sim(\forall x)B \wedge \sim Bx/c\}$ is consistent, otherwise $S_{n+1} = S_n \cup \{(\forall x)B\}$.

(3) If $\sim A_n$ is $\sim(\forall x)B$, let $S_{n+1} = S_n \cup \{(\forall x)B\}$, where $S_n \cup \{(\forall x)B\}$ is consistent, otherwise $S_{n+1} = S_n \cup \{\sim(\forall x)B \wedge \sim Bx/c\}$, where c is the first name not to appear in A_n or in any wff in S_n.

Finally, let *Max-A* be the union of the S_m as m ranges from 1 to ∞.

Lemma (T10.1) *Max-A is consistent.*

Proof: Suppose not. Then for some A_n: (1) $S_n \cup \{A_n\}$ and $S_n \cup \{\sim A_n\}$ are both inconsistent, (2) A_n is $\sim(\forall x)B$ and both $S_n \cup \{\sim(\forall x)B \wedge \sim Bx/c\}$ and $S_n \cup \{(\forall x)B\}$ are inconsistent, or (3) $\sim A_n$ is $\sim(\forall x)B$ and both $S_n \cup \{(\forall x)B\}$ and $S_n \cup \{\sim(\forall x)B \wedge \sim Bx/c\}$ are inconsistent. Let A_i be the first such wff. Since S_0 is consistent, S_i is consistent.

(1) As in the proof of (T5.2).

(2) By (D1) and (D2), we have $B_1, \ldots, B_h$, $C_1, \ldots, C_k \in S_i$ such that $\vdash(B_1 \wedge \ldots \wedge B_h \wedge \sim(\forall x)B \wedge \sim Bx/c) \rightarrow D$, where $\vdash \sim D$ and $\vdash(C_1 \wedge \ldots \wedge C_k \wedge (\forall x)B) \rightarrow E$, where $\vdash \sim E$. Since c occurs in none of $B_1, \ldots, B_h$, $C_1, \ldots, C_k$, $(\forall x)B$, by (RQ2), $\vdash \sim(B_1 \wedge \ldots \wedge B_h \wedge C_1 \wedge \ldots \wedge C_k)$. By (A1), (D1), and (D2) S_i is inconsistent, which is impossible.

(3) As in the proof of (2).

This completes the proof of (T10.1).

Clearly *Max-A* is mci.

Remark: Since Lq contains denumerably many names, any consistent set of wffs that contains a finite number of names can be extended to an mci set *in* Lq by an appropriate ordering of the remaining wffs of Lq.

Let Lq^* be Lq or any $Lq \cup X$, where X is a finite subset of Na. Let *Max* be any mci set in Lq^*.

Let a be a member of Na not in Lq^* and b be a name in Lq^*. Let *Max-a/b* be the set of wffs $Max \cup \{a = b\}$ in $Lq^* \cup \{a\}$.

Lemma (T10.2) $Max \cup \{a = b\}$ *is consistent.*

Proof: Suppose not. Then there are $A_1, \ldots, A_n \in Max$ such that $\vdash (A_1 \wedge \ldots \wedge A_n \wedge a = b) \rightarrow B$, where $\vdash \sim B$. By (R4) and (RQ3), $\vdash \sim (A_1 \wedge \ldots \wedge A_n)$. By (A1), (D1), and (D3) *Max* is inconsistent. But this is impossible.

Lemma (T10.3) *$\{B$: Max-$a/b \vdash B\}$ $(= Y)$ is an mci set in $Lq^* \cup \{a\}$.*

Proof: Where A is a wff containing a, let A' be the result of replacing all occurrences of a in A by b. Since *Max* is mci in Lq^*, either $A' \in Max$ or $\sim A' \in Max$, but not both. Since $a = b \in Y$, $a = b \wedge A' \in Y$ or $a = b \wedge \sim A' \in Y$. By (Ai3) $A \in Y$ or $\sim A \in Y$. By similar reasoning it can be shown that if $A \in Y$ and $\sim A \in Y$, $A' \in Max$ and $\sim A' \in Max$, which is impossible.

For convenience let "*Max-a/b*" now designate $\{B$: Max-$a/b \vdash B\}$.

Let *Max-a* stand for any *Max-a/b*.

We are ready to define a model for Lq^*. A model for Lq^* is an mci set *Max*. The set *At* is defined as before. $s[\sim(C \rightarrow D)]$ is the intersection of all sets s that meet conditions (1) and (2) as in Section 5 and (3) satisfy (S2)–(S4), (Si2)–(Si3.2), (SQ1), and

(SQ2) if $Ax/c \in s$, $(\exists x)A \in s$,
(SQ3.1) if $(\forall x)(A \rightarrow B) \in s$, $(\forall x)A \rightarrow (\forall x)B \in s$,
(SQ3.2) if $(\forall x)(A \rightarrow B) \in Max$ and $(\forall x)A \in s$, $(\forall x)B \in s$.

$|a|$ and $|F|$ are as in Section 9. The set S is defined as before in Section 5.

The evaluation rules apply to all languages Lq^* starting simultaneously from the least complex wffs and proceeding to those of greater and greater complexity. The rules for all connectives but $\forall$ have already been given. There remains:

(V$\forall$) If for every model *Max-a*, Max-$a \in V(A(\text{-}a\text{-})x/a)$, $Max \in V((\forall x)A(\text{-}a\text{-}))$; otherwise $Max \notin V((\forall x)A(\text{-}a\text{-}))$.

A few words are in order concerning (V$\forall$). A *Max-a* is an mci set that contains *Max*, is in a language that adds one name to the language of *Max*, and adds or subtracts no entities from the ones *Max* says there are. What may change from one *Max-a* to another is the identity of a. So by taking *all Max-a*'s, we succeed in evaluating Ax/a at all the entities *Max* says there are.

A model $Max \Vdash A$ if $Max \in V(A)$. $\Vdash A$ if $M \Vdash A$ for all models M in Lq^*.

Lemma (T10.4) *For all $A \in s[\sim(C \rightarrow D)]$, $C \rightarrow A \in$* Max.

Proof: The proof proceeds as in (T9.1) with four additional cases:

ad (SQ1). A is Gx/c, where $C \rightarrow ((\forall x)G \wedge B) \in Max$ by IH. By (D1), (T5.3), (T5.5), and (AQ1), $((\forall x)G \wedge B) \rightarrow Gx/c \in Max$. By (D1), (T5.5), and (A3), $C \rightarrow A \in Max$.

ad (SQ2). A is $(\exists x)B$, where $C \rightarrow Bx/c \in Max$ by IH. By (D1), (T5.3), (T5.5), (AQ2), and (A3).

ad (SQ3.1). A is $(\forall x)F \rightarrow (\forall x)G$, where $C \rightarrow ((\forall x)(F \rightarrow G)) \in Max$ by IH. By (D1), (T5.3), (T5.5), (AQ3), and (A3).

ad (SQ3.2). A is $(\forall x)G$, where $C \rightarrow (\forall x)F \in Max$ by IH and $(\forall x)(F \rightarrow G) \in Max$. By (D1), (T5.5), (AQ3), and (A3).

This completes the proof of (T10.4).

Lemma (T10.5) *(T5.7)–(T5.11) all hold for Lq.*

Proof: Immediate.

Lemma (T10.6) *Where* $(\forall x)A$ *is a wff in* Lq^*, *for all Max-a* $Ax/a \in$ *Max-a iff for all names b in* Lq^* $Ax/b \in$ *Max.*

Proof: (1) Assume the hypothesis for the left-right direction and that for some name b in Lq^* $Ax/b \notin Max$. $\sim Ax/b \in Max$. *Max-a/b* is a *Max-a*. Since $Max \subseteq$ *Max-a* for all *Max-a*, $\sim Ax/b \in$ *Max-a* for all *Max-a*. Thus $\sim Ax/b \in$ *Max-a/b*. $a = b \in$ *Max-a/b* by construction. $a = b \wedge \sim Ax/b \in$ *Max-a/b*. By (Ai3) $\sim Ax/a \in$ *Max-a/b*. By (T10.3) *Max-a/b* is mci. So $Ax/a \notin$ *Max-a/b*, which is impossible.

(2) Assume the hypothesis in the right-left direction and that for some *Max-a* $Ax/a \notin$ *Max-a*. Since all *Max-a*'s are mci, $\sim Ax/a \in$ *Max-a*. By construction for some b in Lq^* *Max-a* is *Max-a/b*. $a = b \in$ *Max-a/b*. By (Ai2) $b = a \in$ *Max-a/b*. Thus $b = a \wedge \sim Ax/a \in$ *Max-a/b*. By (Ai3) $\sim Ax/b \in$ *Max-a/b*. Since a does not occur in $\sim Ax/b$, $\sim Ax/b \in Max$. So $Ax/b \notin Max$, which is impossible.

Theorem (T10.7) *For all languages* Lq^* *and all wffs B and mci sets Max such that* $\{B\}$, $Max \subseteq Lq^*$, $B \in Max$ *iff* $Max \in V(B)$.

Proof: By induction on the complexity of B.
Base Cases. (1) B is atomic. By construction and (VAt).

(2) B is $Fa_1 \ldots a_n$. $\langle |a_1|, \ldots, |a_n| \rangle \in |F|$ by construction. $\langle V(a_1), \ldots, V(a_n) \rangle \in V(F)$ by (VN) and (VP), and finally $Max \in V(B)$ by (VB). The reasoning is clearly reversible.

Induction Hypothesis: The theorem holds for all wffs of complexity less than n in all Lq^*.

(3) B is $\sim C$. $\sim C \in Max$ iff, by mci, $C \notin Max$ iff, by IH, $Max \notin V(C)$ iff, by ($V\sim$), $Max \in V(\sim C)$.

(4) B is $C \wedge D$. $C \wedge D \in Max$ iff, by (T5.9), $C, D \in Max$ iff, by IH, $Max \in V(C), V(D)$ iff, by ($V\wedge$) $Max \in V(C \wedge D)$.

(5) B is $C \vee D$. The proof uses (T5.10).

(6) B is $C \rightarrow D$. $C \rightarrow D \in Max$ iff, by (T5.8), (T5.11), and construction, for all $w \in W$, $C \in w$ only if $D \in w$ iff, by IH and (VS), for all $w \in W$, $w \in V(C)$ only if $w \in V(D)$ iff, by ($V\rightarrow$), $Max \in V(C \rightarrow D)$.

(7) B is $(\forall x)C$. $(\forall x)C \in Max$ iff, by mci, for all names $d \in Lq^*$ $Cx/d \in Max$ iff, by (T10.6), for all *Max-c* $C(\text{-}c\text{-})x/c \in$ *Max-c* iff, by IH, for all *Max-c*, *Max-c* $\in V(C(\text{-}c\text{-})x/c)$ iff, by ($V\forall$), $Max \in V((\forall x)C)$.

This completes the proof of (T10.7).

Theorem (T10.8) (Completeness) $\Vdash A$ *iff* $\vdash A$.

Proof: $\nVdash A$ iff for some model M, $M \nVdash A$ iff for some model M, $M \notin V(A)$ iff, by (T10.7), for some model M, $A \notin M$ iff, by mci, for some model M, $\sim A \in M$ iff, by mci, $\nvdash \sim\sim A$ iff, by (R5), $\nvdash A$.

11 Existence In quantification, some philosophers distinguish between being and existence. Among these, some provide for two quantifiers, one to range over the members of the wider category of being, the other to range over the narrower category of existence.[5] These quantifiers can be treated separately, which is what is done in this and the preceding section. Occasionally philosophers refuse to distinguish between the categories and insist that the one quantifier they happen to use ranges over whichever category they take to have ontological status. To them the statement "There are things that do not exist" is false, or at least somehow wicked.[6]

In this section an attempt will be made to limit the quantifier to range over just existents, so that the purely fictional entities of stories will not be captured by it. Of course, not all entities in fiction need be *purely* fictional, so to speak. Flashman is a creature purely of fiction, an individual who never did exist. In his travels, however, he had dealings with people, like Abraham Lincoln and Victoria, who did in fact exist.

The language of *Le* is that of *Lq* with the addition of a unary predicate "*E*" to represent the predicate "exists". The axioms and rules of *Le* are those of *Li* plus:

(AQE1) $((\forall x)A \wedge Ec) \rightarrow Ax/c$
(AQE2) $(Ax/c \wedge Ec) \rightarrow (\exists x)A$,

(AQ3), (RQ1),

(RQE2) if $\vdash(A(\text{-}c\text{-}) \wedge (\forall x)C(\text{-}c\text{-})) \rightarrow D$, $\vdash(B(\text{-}c\text{-}) \wedge \sim(\forall x)C(\text{-}c\text{-}) \wedge \sim C(\text{-}c\text{-})x/c \wedge Ec) \rightarrow E$, $\vdash\sim D$, and $\vdash\sim E$, $\vdash\sim(A(\text{-}c\text{-}) \wedge B(\text{-}c\text{-}))$
(RQE3) if $\vdash(A(\text{-}c\text{-}) \wedge Ec) \rightarrow B$ and $\vdash\sim B$, $\vdash\sim A(\text{-}c\text{-})$.

The valuation rules for *Le* are those for *Li* and

(VE∀′) if for all Vc such that $Vc(c) \in Vc(E)$, $O \in Vc(A(\text{-}c\text{-})x/c)$, $O \in V((\forall x)A)$; otherwise $O \notin V((\forall x)A)$.
(D6) A set X of wffs is *e-instanced* iff for all wffs $\sim(\forall x)A \in X$, there is some name c such that $\sim Ax/c \wedge Ec \in X$.

Let A be a wff of *Le* such that $\{A\}$ is consistent. An mc *e*-instanced (mce) set *Max-A* can be constructed such that $A \in$ *Max-A* by extending *Le* to *Le+* in the usual way with the help of a denumerable set *Na* of new names. The proof of the consistency of *Max-A* uses (RQE2).

It is clear, given this construction of an mce set *Max*, that if for all names c such that $Ec \in$ *Max*, $Ax/c \in$ *Max*, $(\forall x)A \in$ *Max*; and, conversely, if $\sim(\forall x)A \in$ *Max*, there is some name c such that $Ec \in$ *Max* and $\sim Ax/c \in$ *Max*.

*Le** is *Le* or any of $Le \cup X$, where X is a finite subset of *Na*. *Max-a/b* and *Max-a* are defined in the spirit of the preceding section. Clearly where *Max* is mce, *Max-a/b* is mce.

A model for *Le** is an mce set *Max*. The set *At* is defined as previously. $s[\sim(C \rightarrow D)]$ is the intersection of all sets s that meet conditions (1) and (2) as in Section 5 and (3) satisfy (S2)–(S4), (Si2)–(Si3.2), (SQ3.1), (SQ3.2), and

(SQE1) if $(\forall x)A \wedge Ec \in s$, $Ax/c \in s$
(SQE2) if $Ax/c \wedge Ec \in s$, $(\exists x)A \in s$.

$|a|$ and $|F|$ are as in Section 9. The set S is defined as before. As in the preceding section, the evaluation rules apply to all languages *Le** starting simultaneously from the least complex wffs and proceeding to those of greater complexity. Instead of (V∀), the Le^*'s use:

(VE∀) If, for every model *Max-a* such that $Ea \in Max\text{-}a$, $Max\text{-}a \in V(A(\text{-}a\text{-})x/a)$, $Max \in V((\forall x)A(\text{-}a\text{-}))$; otherwise $Max \notin V((\forall x)A(\text{-}a\text{-}))$.

That for all $A \in s[\sim(C \to D)]$, $C \to A \in Max$ is shown in the usual way. (T5.7)-(T5.11) hold.

Lemma (T11.1) *Where $(\forall x)A$ is a wff in Le^*, for all Max-a such that $Ea \in Max\text{-}a$, $Ax/a \in Max\text{-}a$ iff for all names b in L^* such that $Eb \in Max$ $Ax/b \in Max$.*

Proof: As in Section 10 by (Ai2) and (Ai3).

The proof of completeness is essentially that of the preceding section.

NOTES

1. It might be remarked that stories are not sets, at least the texts and tellings that sometimes give rise to stories are not. The order in which the sentences occur in texts and tellings is an important feature they have.

   This is, indeed, a fact and something that someday must be reckoned with. But *all* use of sentences by human beings has an order which is important to the sense of what they say and write. Yet philosophers have offered analyses of necessity, counterfactuals, and many other notions, and *none* of these analyses, to my knowledge, takes account of the order of sentences in human interchange. So my failure to do so does not disturb me much.

2. Jeff Rueger has pointed out that the motivation here is much akin to that of Parry's Analytical Implication. His *Proscriptive Principle*, given in [6], says "no formula with analytic implication as the main relation holds universally if it has a free variable occurring in the consequent but not in the antecedent".

3. The view that quantifiers ought to range over just existing things is discussed at length in [5], [7], [8], and [9].

4. Up to this point all the axioms and rules proposed in this section and in the original system L can be shown to obey Parry's *Proscriptive Principle* by means of matrices due to Parry (see [1]). These are:

~	
0	2
1	3
*2	0
*3	1

∧	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

→	0	1	2	3
0	2	2	2	2
1	0	3	0	3
2	0	0	2	2
3	0	1	0	3

∨	0	1	2	3
0	0	0	2	2
1	0	1	2	3
2	2	2	2	2
3	2	3	2	3

⊃	0	1	2	3
0	2	2	2	2
1	2	3	2	3
2	0	0	2	2
3	0	1	2	3

↔	0	1	2	3
0	2	0	0	0
1	0	3	0	1
2	0	0	2	0
3	0	1	0	3

5. Of these analyses one of the most interesting is that of Cocchiarella in [2] and [3]. For Cocchiarella there are attributes or properties that entail existence. Existence for him is defined in a second-order way as the possession of some existence-entailing attribute.

6. It may sound here as if I am taking a stand on whether there are possibilia or even impossibilia. But I think talk about "possibilia" misleads in that it brings to mind Meinong and "impossibilia". There really are a number of issues here, as Cocchiarella nicely points out in [4]. One is whether the quantifier "there is" differs in its range from "there exists". This is really the only point on which I take a firm stand. Another separate problem is that of analyzing definite descriptions, like "the round square", "the golden mountain", and "the existent". Yet another is whether real individuals, like Lincoln, *can* figure in stories.

   If the ranges of "there is" and "there exists" somehow *have to* coincide in their range, then to say "There are things that do not exist" is tantamount to saying "There exist things that do not exist", which I for one wish to avoid. Detailed discussions of this issue may be found in [7], [8], [9], and in a sequel in which I extend the language *Lq* of Section 10 by adding an operator "the story... says that...", as in "The story *Moby Dick* says that Ahab is a ship's captain". This enhanced language permits us to say, for instance, "There are at least two purely fictional individuals who are said by more than one story to live at 221b Baker Street" (actually there are at least three such individuals). Now unless one wishes to argue that Holmes, Watson, and Mrs. Hudson are (were?) really Conan Doyle's brain states, or that this statement is itself fictional, one cannot make sense of it and yet hold that to be and to exist are the same thing.

   How then does one treat definite descriptions like "the round square", etc.? I do not know. But interesting views and discussions of this and other issues concerning nonexistent individuals may be found in [4], [7], and [9].

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