Notre Dame Journal of Formal Logic
Volume 27, Number 2, April 1986

Analytic Implication

KIT FINE*

Parry presented a system of analytic implication in [7] and [8]. Dunn [2] gave an algebraic completeness proof for an extension of this system and Urquhart [10] later gave a semantic completeness proof for Dunn's system with necessity. This paper establishes completeness for Parry's original system,[1] thereby answering a question of Gödel's [6], and then, on the basis of the completeness result, derives decidability; it also deals with quantificational versions and other modifications of his system.

Section 1 contains some informal remarks on the notion of analytic implication. They are not strictly relevant to the later analysis, although they may help to place it in perspective. Section 2 presents the semantics and Section 3 exhibits a system of analytic implication. Section 4 helps to demonstrate that the system is equivalent to Parry's, and Section 5 establishes completeness. Finally, Section 6 outlines the theory for some related systems.

1 Informal remarks One may distinguish between the propositional content (*p*-content) and constitutive content (*c*-content) of a sentence. The *p*-content is the proposition or thought conveyed by the sentence. The *c*-content consists of elements from which the proposition is formed. Thus the *c*-content of "Socrates is a philosopher" consists of the concept philosopher and, perhaps, an element corresponding to the term "Socrates".

This is a rough and intuitive distinction; no particular interpretation for "proposition", "concept", "element", etc. is presupposed.

*This paper originally appeared in the Proceedings of the Conference on the Philosophy of Language and Logic held at the University of Keele in April 1979. These Proceedings were put out by the Keele University Library and received a very limited circulation. An account of the present semantics has been given in Part 2, Section 11 of [5], and similar ideas have been pursued in [3]. I should like to thank Alasdair Urquhart for helpful conversations on the subject of my paper.

Received May 15, 1984

A strictly implies B if the p-content of A contains that of B. For *analytic* implication it is also required that the c-content of A contains that of B. There is thus a certain symmetry in the conditions. It is possible, of course, that the propositional content is so colored by the c-content that an independent requirement for analytic implication cannot be formulated. However, in all cases that come to mind, it is possible to separate out the p- and c-content in a significant manner.

Strict implication can be taken in various ways. It can be structural or intensional, i.e., a question of logical form alone or of nonlogical material as well. It can be syntactic or semantic, i.e., characterized "concretely" in terms of rules or "abstractly" in terms of models.

It is natural to take the pairs structural/syntactic and intensional/semantic together, but this is not necessary. For example, if the structural, syntactic notion is a matter of derivation within a logic, then the intensional, syntactic notion is a matter of derivation within an applied theory, say a logic with meaning postulates.

Similar possibilities exist for containment of c-content: it can be structural or intensional, syntactic or semantic. Suppose we define "the c-content of A contains that of B" as follows: any concept of B is *definable* in terms of the concepts of A. Then the concepts can be identified by logical form alone or by nonlogical material as well, and definition can be characterized in terms of *derivable* equivalence, or *valid* equivalence. Thus there is a structural relationship between "x is a bachelor" and "x is not a bachelor" (for not-P is definable in terms of P), but only an intensional relationship between "x is a bachelor" and "x is not an unmarried man".

In defining analytic implication, one could cross the boundaries of this mutual classification. For example, one could take containment of p-content as structural and containment of c-content as intensional. However, this would be highly unnatural, for the categories of the classification represent various points of view. Thus it is natural to regard analytic implication itself as structural or intensional, syntactic or semantic.

For containment of c-content, there is another important ambiguity. Are the concepts of a sentence the ones from which the proposition is *ultimately* or *immediately* formed? For example, if P is a complex predicate, say "grandfather", then is Father a concept of "x is a grandfather" or only Grandfather? In the first case, "x is a grandfather" analytically implies "x is a grandfather or x is a father", in the second case it does not.

There are advantages to both positions. The first position ties in well with the view that the elements of a sentence are the primitive terms (of some fixed vocabulary) from which the sentence is formed. However, there are no interesting structural relationships between primitive terms, and so there is no theory of structural definability in analogy to the theory of deducibility or logical consequence. The second position, on the other hand, allows one to develop such a theory within the object-language; for "$Pa \rightarrow (Qa \rightarrow Qa)$" will express the fact that the (possibly complex) predicate Q is definable in terms of the (possibly complex) predicate P.

Where does Parry stand on these issues? [7], pp. 152–153, suggests that analytic implication is syntactic and either structural or intensional, and that the concepts of a sentence are the ultimate or primitive ones. Thus, in accordance with this conception, one might define "A analytically implies B" as: whenever A is a formula of a system, B is a formula of the system too and deducible from A. In other words, whenever A is a formula of a system, B must be obtainable from A by means of both the formation and transformation rules.

2 Semantics Formulas are constructed from an infinite set of sentence-letters SL, the truth-functional connectives & and $-$, and the further connectives $\rightarrow$, $\Box$, and $\preccurlyeq$. $A \rightarrow B$, $\Box A$, and $A \preccurlyeq B$ are read as "A analytically implies B", "necessarily A", and "the *c*-content of A is included in that of B", respectively. In Parry's system, the connectives $\Box$ and $\preccurlyeq$ are not used. A *model* $\mathfrak{A}$ is a sextuple $(W, R, I, \cup, \phi, \gamma)$, where:

W (worlds) is a nonempty set
R (accessibility) is a reflexive and transitive relation on W
I_w (contents) is, for each $w \in W$, a nonempty set
$\cup_w$ (compound) is, for each $w \in W$, an associative, commutative, and idempotent operation on I_w
ϕ (valuation) $\subseteq W \times SL$
γ_w (content-assignment) is, for each $w \in W$, a function from SL into I_w

$(I_w, \cup_w)$ is, of course, a semilattice: for $a, b \in I_w$, $a \cup_w b$ represents the compound or union of the contents a and b; $\leq_w$ is the corresponding semilattice ordering, i.e., $a \leq_w b$ iff $a \cup_w b = b$.

For A a formula, $\mathfrak{L}(A)$, the *language* of A, is the set of sentence-letters occurring in A. γ_w is extended to all formulas through the condition:

$$\gamma_w(A) = \gamma_w(p_1) \cup_w \ldots \cup_w \gamma_w(p_n), \text{ for } \mathfrak{L}(A) = \{p_1, \ldots, p_n\} \ .$$

Relative to a model $\mathfrak{A}$, the truth-predicate $\models$ is then defined by the following clauses:

(i) $w \models p$ iff ϕwp
(ii) $w \models -B$ iff not $w \models B$
(iii) $w \models B \,\&\, C$ iff $w \models B$ and $w \models C$
(iv) $w \models B \rightarrow C$ iff $(\forall v)\ (wRv \Rightarrow \gamma_v\ (C) \leq_v \gamma_v\ (B))$ and $(\forall v)\ (wRv \,\&\, v \models B \Rightarrow v \models C)$
(v) $w \models \Box B$ iff $(\forall v)\ (wRv \Rightarrow v \models B)$
(vi) $w \models B \preccurlyeq C$ iff $(\forall v)\ (wRv \Rightarrow \gamma_v\ (B) \leq_v \gamma_v\ (C))$.

The set of formulas Δ *has a model* $\mathfrak{A}$ if, for some world w in $\mathfrak{A}$, $w \models A$ (relative to $\mathfrak{A}$) for each A in Δ. The formula A is *valid if* $\{-A\}$ does not have a model.

Two variants on the modelling are worth noting. The first is that, by the representation theorem for semilattices, each content can be regarded as a set of concepts and $\cup_w$ can then be treated as set-theoretical union. This is the approach in [10]. The second is that each model can be replaced by an equivalent model in which $(I_w, \cup_w, \gamma_w)$ extends $(I_v, \cup_v, \gamma_v)$ whenever wRv. In terms of the concept modelling, this means that any model is equivalent to one in which: (a) each concept of v is one of w if wRv; and (b) a concept in A at w is in A at v if it is a concept of v and wRv. These equivalences are established by "injecting" the contents (or concepts) of accessible worlds into any given world.

How does the modelling above accord with the informal discussion in Section 1? We can regard the structure of possible worlds as approximating, in some ill-understood way, to syntactic notions of deducibility and definability. Alternatively, we can regard the possible worlds as being of independent interest in a semantic account of these notions. It is then natural to give a conceivability interpretation of R, along the lines of [4]. wRv if v is conceivable relative to w, i.e., if v can be completely described by using the concepts that are required to describe w. Condition (a) above is now mandatory, for new concepts cannot be used in the description of an accessible world, although some old concepts may not be used at all.

3 The systems The postulates of my system are:

I 1. Taut = the set of tautologous formulas
2. Modus Ponens (for $\supset$)
II 3. $\Box(A \supset B) \supset (\Box A \supset \Box B)$
4. $\Box A \supset A$
5. $\Box A \supset \Box\Box A$
6. $A/\Box A$
III 7. $(A \preccurlyeq B \,\&\, B \preccurlyeq C) \supset (A \preccurlyeq C)$
8. $(A \preccurlyeq C \,\&\, B \preccurlyeq C) \supset (A \,\&\, B \preccurlyeq C)$
9. $A \preccurlyeq B$ if $\wp(A) \subseteq \wp(B)$
IV 10. $(A \preccurlyeq B) \supset \Box(A \preccurlyeq B)$
V 11. $(A \rightarrow B) \equiv \Box(A \supset B) \,\&\, (B \preccurlyeq A)$.

The postulates for this system are particularly perspicuous: System I gives the propositional calculus; II the system S4 for $\Box$; III states, in effect, that $\preccurlyeq$ forms a semilattice, IV says that content-inclusions hold necessarily; and V gives an equivalent for $\rightarrow$ in terms of $\Box$ and $\preccurlyeq$ and could, of course, be replaced by a definition.

For reference, we shall list the postulates for Parry's system (A14 has been simplified slightly):

A1 $A \,\&\, B \rightarrow B \,\&\, A$
A2 $A \rightarrow A \,\&\, A$
A3 $A \rightarrow --A$
A4 $--A \rightarrow A$
A5 $A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$
A6 $A \vee (B \,\&\, {\sim}B) \rightarrow A$
A7 $(A \rightarrow B) \,\&\, (B \rightarrow C) \rightarrow (A \rightarrow C)$
A8 $(A \rightarrow B \,\&\, C) \rightarrow (A \rightarrow C)$
A9 $(A \rightarrow B) \,\&\, (C \rightarrow D) \rightarrow (A \,\&\, C \rightarrow B \,\&\, D)$
A10 $(A \rightarrow B) \,\&\, (C \rightarrow D) \rightarrow (A \vee C \rightarrow B \vee D)$
A11 $A \rightarrow B \rightarrow (A \supset B)$
A12 $A \leftrightarrow B \,\&\, f(A) \rightarrow f(B)$
A13 $f(A) \rightarrow (A \rightarrow A)$
A14 $((-A \rightarrow A) \,\&\, (A \rightarrow B)) \rightarrow (-B \rightarrow B)$

A15 $-(A \supset B) \to -(A \to B)$
MP $A, A \to B/B$
Adj $A, B/A \;\&\; B$.

In A12 and A13, $f(A)$ is a formula that contains an occurrence of A and $f(B)$ is the result of substituting B for that occurrence of A.

4 Equivalence We wish to establish that the two systems are equivalent. In fact, we shall only establish that my system can be derived from Parry's. Equivalence will then follow from the completeness of the former system and the soundness of the latter.

For the connectives $\Box$ and $\preccurlyeq$, we use the definitions:

$$\Box A = (A \to A) \to A$$
$$A \preccurlyeq B = B \to (A \to A).$$

Parry's definition of $\Box A$ is, in effect, $\sim A \to A$; in his system, the two definitions are equivalent.

The derivation proper, under Theorem 1, is preceded by some lemmas:

Lemma 1 *$\vdash A$ for A tautologous.*

Proof: As in [1].

Lemma 2 $\vdash A \to A$.

Proof: By $A3$ and $A4$.

Lemma 3 $\vdash A \to B \Rightarrow \vdash A \supset B$.

Proof: By A11 and modus ponens.

Lemma 4 $\vdash A \leftrightarrow B$ *and* $\vdash f(A) \Rightarrow \vdash f(B)$.

Proof: Assume $\vdash A \leftrightarrow B$ and $\vdash f(A)$. Then $\vdash (A \leftrightarrow B) \;\&\; f(A)$ by Adj. and so $f(B)$ by modus ponens.

Lemma 5 $\vdash (A \to B) \;\&\; (A \to C) \to (A \to (B \;\&\; C))$.

Proof: $\vdash (A \to B) \;\&\; (A \to C) \to (A \;\&\; A \to B \;\&\; C)$ by A9. By A2 and A8, $\vdash A \leftrightarrow A \;\&\; A$. So the result follows from Lemma 4.

Lemma 6 $\vdash \bigwedge_{i=1}^{n} (p_i \to p_i) \to (B \to B)$ *if* $\mathfrak{V}(B) \subseteq \{p_1, \ldots, p_n\}$.

Proof: By induction on the construction of B. Details are in [2].

Lemma 7 $\vdash A \to (B \to B)$ *if* $\mathfrak{V}(B) \subseteq \mathfrak{V}(A)$.

Proof: Mainly from Lemma 6 and A13. Again, details are in [2].

Lemma 8 $\vdash ((A \to A) \;\&\; (B \to B) \to B) \supset ((B \to B) \to B)$.

Proof: Mainly from A14. See [2].

Lemma 9 $\vdash [A \wedge (A \supset B)] \to B$.

Proof: We wish to show $\vdash [A \;\&\; (\sim A \vee B)] \to B$. $\vdash [A \;\&\; (\sim A \vee B)] \to (A \;\&\; \sim A) \vee (A \;\&\; B)$ by A5. $\vdash [(A \;\&\; \sim A) \vee (A \;\&\; B)] \to (A \;\&\; B) \vee (A \;\&\; \sim A)$ by

A1, Lemma 4 and the definition of $\vee$. $\vdash[(A \,\&\, B) \vee (A \,\&\, {\sim}A)] \to A \,\&\, B$ by A6. Finally, $\vdash A \,\&\, B \to B$ by A8 and Lemma 2.

Lemma 10 $\vdash A$ *and* $\vdash A \supset B \Rightarrow \vdash B$.

Proof: From Adj, Lemma 9, and modus ponens for $\to$. We refer to Lemmas 1 and 10 together, as PC.

Lemma 11 $\vdash (B \to C) \supset ((A \to B) \to (A \to C))$.

Proof: $\vdash (B \to C) \supset (B \leftrightarrow B \,\&\, C)$ by PC, and Lemmas 2 and 5. $\vdash[(A \to B) \to (A \to B)] \,\&\, (B \leftrightarrow B \,\&\, C) \supset [(A \to B) \to (A \to B \,\&\, C)]$ by PC, A12, and Lemma 3. $\vdash[(A \to B) \to (A \to B \,\&\, C)] \supset [(A \to B) \to (A \to C)]$ by A8, A7, and Lemma 3. The result now follows by PC.

Lemma 12 $\vdash A \Rightarrow \vdash B \to A$ *if* $\mathfrak{L}(A) \subseteq \mathfrak{L}(B)$.

Proof: By induction on the proof of A. It suffices to show, in any particular case, that there is a formula C such that $\vdash (C \to C) \to A$ and $\mathfrak{L}(C) \subseteq \mathfrak{L}(A)$. For then $\vdash B \to (C \to C)$ by Lemma 7 and $\vdash B \to A$ by A7.

(a) A an axiom. Then A is of the form $C \to D$. By Lemma 11 and PC, $\vdash (C \to C) \to (C \to C)$ and we are done.

(b) A from C and D by adjunction. Suppose $\mathfrak{L}(A) \subseteq \mathfrak{L}(B)$. Then $\vdash B \to C$ and $\vdash B \to D$ by IH. Hence $\vdash B \to C \,\&\, D$ by Lemma 5.

(c) A from C and $C \to A$ by modus ponens. By IH, $\vdash[(C \to C \,\&\, (A \to A)] \to C$ and $\vdash[(C \to C) \,\&\, (A \to A)] \to (C \to A)$. So by Lemma 5, $\vdash[(C \to C) \,\&\, (A \to A)] \to C \,\&\, (C \to A)$. $\vdash (C \to A) \to (C \supset A)$ by A11, and so $\vdash C \,\&\, (C \to A) \to C \,\&\, (C \supset A)$ by A9 and Lemma 2. But $\vdash C \,\&\, (C \supset A) \to A$ by Lemma 9; and so $\vdash (C \,\&\, (C \to A)) \to A$ by A7. Therefore $\vdash[(C \to C) \,\&\, (A \to A)] \to A$ by A7 again. Hence $\vdash (A \to A) \to A$ by Lemma 8.

Lemma 13 $\vdash A \Rightarrow \vdash (B \to A) \supset ((B \to B) \to A)$.

Proof: Suppose $\vdash A$. $\vdash (B \to A) \supset [(B \to B) \to (B \to A)]$ by Lemma 11. Since $\vdash A$, $\vdash (B \to A) \to A$ by Lemma 12. So, by A7 and PC, $\vdash (B \to A) \supset ((B \to B) \to A)$.

Lemma 14 $\vdash (A \to B) \supset (C \to (A \to B))$ *if* $\mathfrak{L}(C) \supseteq \mathfrak{L}(A)$.

Proof: $\vdash (A \to B) \supset ((A \to A) \to (A \to B))$ by Lemma 11. $\vdash C \to (A \to A)$ by Lemma 12. Hence $\vdash (A \to B) \supset (C \to (A \to B))$ by A7 and PC.

Theorem 1 *A a theorem of Parry's system* $\Rightarrow$ *A a theorem of our first system.*

Proof: We go through the postulates of our system one by one. $\vdash$ refers to provability within Parry's system.

I: 1. By Lemma 1.
2. By Lemma 10.

II: The definition of $\Box A$ is $(A \to A) \to A$. By Lemmas 2, 4, and 12, we can replace $\Box A$ by $T_A \to A$, where T_A is any theorem with the same language as A. Similarly, $A \preccurlyeq B = B \to (A \to A)$ can be replaced by $B \to T_A$.

3. The postulate is equivalent to $[(T_{A\&B} \to (A \supset B))\ \&\ (T_A \to A)] \supset (T_B \to B)$. Letting the antecedent be Ant, $\vdash Ant \supset [T_A\ \&\ T_{A\&B} \to A\ \&\ (A \supset B)]$ by A9. But $\vdash (A \supset B)\ \&\ A \to B$ by Lemma 9. So $\vdash Ant \supset (T_A\ \&\ T_B \to B)$ by A7 and PC. But then $\vdash Ant \supset (T_B \to B)$ by Lemma 8.

4. $\vdash ((A \to A) \to A) \supset ((A \to A) \supset A)$ by Lemma 3 and A11. $\vdash A \to A$ by Lemma 2. Therefore $\vdash ((A \to A) \to A) \supset A$ by PC.

5. $\vdash (T_A \to A) \supset [T_A \to (T_A \to A)]$ by Lemma 14.

6. From Lemma 12.

III: 7. $\vdash Ant(=(B \to T_A)\ \&\ (C \to T_B)) \supset (B \to B) \to T_A$ by Lemma 13. So $\vdash Ant \supset C \to T_A$ by A7 and PC.

8. $\vdash Ant(=(C \to T_A)\ \&\ (C \to T_B)) \supset C \to T_A\ \&\ T_B$ by Lemma 5. But $T_A\ \&\ T_B$ can now be replaced by $T_{A\&B}$.

9. For $\mathfrak{R}(A) \subseteq \mathfrak{R}(B)$, $\vdash B \to (A \to A)$ by Lemma 12.

IV. 10. $\vdash (B \to T_A) \supset (T_{A\&B} \to (B \to T_A))$ by Lemma 14.

V. 11. There are three things to prove:

(a) $(A \to B) \supset \Box(A \supset B)$. $\vdash (A \to B) \supset T_{A\&B} \to (A \to B)$ by Lemma 14. $\vdash (A \to B) \to (A \to B)$ by A11. Therefore $\vdash (A \to B) \supset T_{A\&B} \to (A \supset B)$ by A7 and PC.

(b) $(A \to B) \supset (B \preccurlyeq A)$. $\vdash B \to (B \to B)$ by Lemma 12. Therefore $(A \to B) \supset A \to (B \to B)$ by Lemma 11.

(c) $(\Box(A \supset B)\ \&\ B \preccurlyeq A) \supset (A \to B)$. $\vdash Ant(=(T_{A\&B} \to (A \supset B))\ \&\ (A \to T_B)) \supset (A \to T_B)$. $\vdash A \to T_A$ by Lemma 12. Therefore $\vdash Ant \supset (A \to T_{A\&B})$ by Lemma 5. $\vdash Ant \supset (T_{A\&B} \to (A \supset B))$ and $\vdash A \to A$ by Lemma 2. So by A7, PC and Lemma 5, $\vdash Ant \supset (A \to (A\ \&\ (A \supset B)))$. But $\vdash ((A \supset B)\ \&\ A) \to B$ by Lemma 9. Hence $\vdash Ant \supset (A \to B)$ by A7 and PC.

It is worth noting that the theorem $\Box(A \supset B)\ \&\ B \preccurlyeq A \supset (A \to B)$ gives a uniform way of establishing $A \to B$ in Parry's system: first establish $\Box(A \supset B) = T_{A\&B} \to (A \supset B)$, and then establish $B \preccurlyeq A = A \to T_B$. For example, it immediately follows from Lemmas 1 and 12 that $A \to B$ is a theorem if $A \supset B$ is a tautology and $\mathfrak{R}(B) \subseteq \mathfrak{R}(A)$. All of the schemes A1–A6 are instances of this principle.

5 Completeness We use the Henkin method. Define *deduction* ($\vdash$), *theory*, *consistent*, *m(aximally) c(onsistent)* in the usual, i.e., classical, way.

Lemma 15 $\Delta, A \vdash B \Rightarrow \Delta \vdash A \supset B$.

Lemma 16 *Each consistent set of formulas is contained in an m.c. theory.*

Given a theory Δ, *define* $\sim_\Delta$ *by:*

$$A \sim_\Delta B \textit{ iff } A \preccurlyeq B,\ B \preccurlyeq A \in \Delta\ .$$

$\sim_\Delta$ is reflexive by postulate 9, symmetric by definition, and transitive by postulate 7. Now let $I_\Delta = \{|A|\sim_\Delta$: A a formula$\}$ and define $\cup_\Delta$ by:

$$|A|\sim_\Delta \cup_\Delta |B|\sim_\Delta = |A\ \&\ B|\sim_\Delta\ .$$

That this definition is good follows from:

Lemma 17
(a) $A \preccurlyeq A' \vdash A \,\&\, B \preccurlyeq A' \,\&\, B$, *and*
(b) $B \preccurlyeq B' \vdash A \,\&\, B \preccurlyeq A \,\&\, B'$.

Proof: (a) $A' \preccurlyeq A' \,\&\, B$ by postulate 9; and so $A \preccurlyeq A' \vdash A \preccurlyeq A' \,\&\, B$ by postulate 7. Also, $\vdash B \preccurlyeq A' \,\&\, B$ by postulate 7 again. Hence $A \preccurlyeq A' \vdash A \,\&\, B \preccurlyeq A' \,\&\, B$ by postulate 8. The reasoning is similar for (b).

We define the canonical model $\mathfrak{A} = (W, R, I, \cup, \phi, \gamma)$ by:

$$W = \{\Delta\colon \Delta \text{ is an m.c. theory}\}$$
$$R = \{\langle w, v\rangle \in W^2\colon A \in v \text{ whenever } \Box A \in w\}$$
$$I = \{\langle w, I_w\rangle\colon w \in W\}$$
$$\cup = \{\langle w, \cup_w\rangle\colon w \in W\}$$
$$\phi = \{\langle w, p\rangle \in W \times SL\colon p \in w\}$$
$$\gamma = \{\langle w, \{\langle p, |p|{\sim_w}\rangle : p \in SL\}\colon w \in W\}.$$

Lemma 18 $\mathfrak{A}$ *is a model.*

Proof: W is nonempty by Lemma 16 and consistency of the system. R is reflexive and transitive by the postulates for $\Box$ in group II. $\langle I_w, \cup_w\rangle$ is a semilattice for each $w \in W$ by postulate 9.

Lemma 19 $w \models A$ *(relative to* $\mathfrak{A}$*) iff* $A \in w$.

Proof: It suffices to establish that the relation $A \in w$ satisfies the truth-conditions (i)–(vi). (i)–(iii) follow from the group I postulates. (iv) follows from postulate II,(v) and II(vi). (v) follows from the postulates for $\Box$. That leaves (vi). First, we show that $\gamma_w(A) = |A|{\sim_w}$. (For ease of reading, we may drop the subscript w.) If $\mathfrak{P}(A) = \{p_1, \ldots, p_n\}$ then $\gamma(A) = \gamma(p_1) \cup \ldots \cup \gamma(p_n) = |p_1| \cup \ldots \cup |p_n| = |p_1 \,\&\ldots\&\, p_n| = |A|$ by postulate 9. Second, we show that $\gamma_w(B) \leq_w \gamma_w(C)$ iff $B \preccurlyeq C \in w$. $\gamma_w(B) \leq_w \gamma_w(C)$ is equivalent to the following statements: $|B| \leq |C|$ (by above); $|B| \cup |C| = |C|$ (by definition of $\cup$); $|B \,\&\, C| = |C|$; $B \,\&\, C \sim C$; $B \,\&\, C \preccurlyeq C \in w$ (by postulate 9); $B \preccurlyeq C \in w$ (by postulates 7, 8, and 9). Finally, we show that condition (vi) is satisfied. By the above, it suffices to show that $B \preccurlyeq C \in w$ iff $(\forall v)(wRv \Rightarrow B \preccurlyeq C \in w)$. But by (v), the R.H.S. is equivalent to $\Box(B \preccurlyeq C) \in w$; and $B \preccurlyeq C \in w$ iff $\Box(B \preccurlyeq C) \in w$ by postulates 4 and 10.

It is now standard work to establish:

Theorem 2 *Every consistent set of formulas has a model.*

Corollary $\vdash A$ *iff* A *is valid.*

Our models yield to the method of filtrations. Hence:

Theorem 3 *Any consistent formula has a model.*

Corollary *The two systems of analytic implication are decidable.*

6 Further systems

1. We can impose different conditions on the accessibility relation R. In the system, this merely corresponds to adopting different postulates for $\Box$. For example, suppose that R is to be an equivalence relation. Then the postulate

$-\Box A \supset \Box-\Box A$ should be added to the system in $\Box$, $\preccurlyeq$, and $\rightarrow$. Alternatively, postulate $-A \rightarrow (A \rightarrow -A)$ or $-A \supset (A \rightarrow -A)$, A implicational, should be added to a system in $\rightarrow$ alone.

The structure of conceptual relations can also be varied. The most natural condition is that this structure be the same from world to world, i.e., that $(I_w, \cup_w) = (I_v, \cup_v)$ and $\gamma_w = \gamma_v$ for any worlds w and v. This corresponds to adding $-(B \preccurlyeq C) \supset \Box-(B \preccurlyeq C)$ or $-A \rightarrow (A \rightarrow -A)$, A of the form $C \rightarrow (B \rightarrow B)$, to the respective systems.

It should be noted that the two additions above are not independent of one another. The S5 postulates imply the postulates for world-invariance of conceptual relations. Indeed, this fact could be established directly by the model-theoretic construction mentioned at the end of Section 2.

2. Another change, suggested by Parry [6], arises from treating analytic implication as a concept. No proposition not containing this concept could then analytically imply a proposition containing that concept. Semantically, γ_w will now assign a content to both $\Box$ and $\preccurlyeq$, and $\gamma_w(A) = \gamma_w(c_1) \cup_w \ldots \cup_w \gamma_w(c_n)$, where $\{c_1, \ldots, c_n\}$ is $\mathfrak{L}'(A)$, the set of sentence-letters and non-truth-functional connectives occurring in A. In the system, postulate 9 should now be subject to the condition that $\mathfrak{L}'(A) \subseteq \mathfrak{L}'(B)$.

No system in $\rightarrow$ alone will then be equivalent to the system above. For $\Box$ and $\preccurlyeq$ will have to be defined in terms of $\rightarrow$, so that the same content will attach to both of them.

Going one step further in this direction, one can allow that the truth-functional connectives contribute to the content of a sentence. This raises the question of whether there is any connection between the contents of different truth-functional connectives. At the one extreme, we can give the same content to all of the connectives. At the other extreme, we can require that the content of c be included in that of d iff d is definable in terms of c. Thus on the first proposal, p will not (analytically) imply $--p$ although $p \vee p$ will imply $-p$; while on the second proposal, the previous implication will not hold, although $p \supset p$ will imply $(p \supset p) \vee p$ since $\vee$ is definable in terms of $\supset$.

Let us say that a definition of a connective is *analytic* if each connective in the definiens is definable in terms of the definiendum. For example, any definition of Sheffer stroke is analytic, whereas the definition of $\supset$ in terms of $\vee$ and $-$ is not. Now on the second proposal, only analytic definitions are acceptable for otherwise the definiens will contain more content than the definiendum. Thus this proposal raises anew the problem of functional completeness.

The smallest complete basis will consist of each connective that cannot be analytically defined in terms of connectives that individually do not suffice to define the given connective. In terms of the lattice of all systems of truth-functions that are closed under definition, the connectives of the smallest complete basis will correspond to those systems that are not the least upper bound of two other systems. By consulting the diagram on page 101 of [9], we see that these are the systems D_2, L_4, R_1, O_1, O_4, O_2, O_3, R_2, R_3, S_1, P_1, $F_1^{\mu+2}$, $F_5^{\mu+2}$, F_2, F_6, $F_3^{\mu+2}$, $F_7^{\mu+2}$, $F_4^{\mu+2}$, $F_8^{\mu+2}$, $\mu > 2$, F_1^∞, F_4^∞, F_2^∞, F_6^∞. Some of the entries are rather strange: for example, $(p \equiv q \equiv r)$ is in the basis but $(p \equiv q)$ is not, since it has the analytic definition $(p \equiv q \equiv t)$.

It is not clear that systems which are functionally complete in this strong sense have much interest outside the context of analytic definition. Perhaps the best that can be said for them is that a stratified system of this sort would allow one to obtain every (functional) subsystem of propositional logic in a straightforward manner.

3. Systems for analytical implication can be extended to the predicate calculus. A simple system is obtained from our sentential one by adding the standard quantificational rules and by requiring of postulate 9 that every predicate of B be a predicate of A. The resulting system is complete for the semantics in which the domains of possible worlds are nondecreasing and in which contents are assigned to predicates.

So far, we have thought of *c*-content primarily in terms of concepts. However, one could also include within the content of a sentence any object which it was about. Thus "Socrates is a philosopher" and "Plato is a philosopher" would differ in content because they are about different individuals.

In terms of the semantics, each object would be assigned a content (perhaps the object itself). Content could then be extended to all sentences in one of two ways. Either the content of a sentence is the union of the contents of its predicates and of the objects assigned to its free variables, or the content of A is the union of the above content and the content of all objects in the domain in case A contains a bound variable. Thus on the first account, the content of $\forall x A(x)$ is the intersection of the contents of $A(a)$ for a any name of an object in the domain; while on the second account, the content of $\forall x A(x)$ is the union of the contents of all such $A(a)$.

The first account seems to be more natural, for in order to understand $\forall x A(x)$ I need not know (or, at least, possess names for) the objects in the domain of the quantifier.

In terms of the system, the first account requires of postulate 9 that any variable in A also occur in B and the second account requires, in addition, that A contains a bound variable only if B does. Other changes may also be required.

4. It should be possible to construct the analytical analogue of other implication notions, e.g., those of intuitionism or of E. The semantics could be straightforwardly modified, although more work would be required for proofs of completeness.

NOTE

1. I mean the full system of [7] with adjunction, A14 and A15.

REFERENCES

[1] Anderson A. R. and N. D. Belnap, Jr., "A simple treatment of truth-functions," *The Journal of Symbolic Logic*, vol. 25 (1959), pp. 301-302.

[2] Dunn, J. M., "A modification of Parry's analytic implication," *Notre Dame Journal of Formal Logic*, vol. 13, no. 2 (1972), pp. 195-205.

[3] Epstein, D., "The semantic foundations of logic," to appear.

[4] Hughs, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 1968.

[5] Kielkopf, C. F., *Formal Sentential Entailment*, University Press of America, Washington, D.C., 1977.

[6] Parry, W. T., "Ein Axiomsystem für eine neue Art von Implikation (analytische Implikation)," *Ergebrisse eines Mathematischen Colloquiums*, vol. 4 (1933), pp. 5-6.

[7] Parry, W. T., "The logic of C. I. Lewis," pp. 115-154 in *The Philosophy of C. I. Lewis*, ed., P. A. Schilpp, Cambridge University Press, 1968.

[8] Parry, W. T., "Comparison of entailment theories," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 441f.

[9] Post, E. L., *The Two-Valued Iterative Systems of Mathematical Logic*, Princeton, University Press, Princeton, New Jersey, 1941.

[10] Urquhart, A., "A semantical theory of analytical implication," *Journal of Philosophical Logic*, vol. 2 (1973), pp. 212-219.

Centre for Cognitive Science
University of Edinburgh
2 Buccleuch Place
Edinburgh EH8 9LW
Scotland

and

Department of Philosophy
University of Michigan
Ann Arbor, Michigan 48109