Notre Dame Journal of Formal Logic
Volume 27, Number 1, January 1986

The Hanf Number of Stationary Logic

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The Hanf number of a logic is the least cardinal κ such that every sentence with a model of power $\geq\kappa$ in fact has arbitrarily large models. For $L(Q_1)$ where Q_1 = "there exist uncountably many", this number is known to be $\beth_\omega$. For the logic $L(aa)$ or "stationary logic" studied in [1], we show that the Hanf number κ is much larger than $\beth_\omega$, which answers a question from [1].[1] In fact, $\omega < \kappa = \beth_\kappa$ (Theorem 2.5) and κ is at least as large as the Hanf number for logic with quantification over countable sets (Theorem 4.1). In the universe L we show that the Hanf numbers of $L(aa)$ and of second-order logic are the same. The result also holds in $L(D)$ where D is a normal ultrafilter on some cardinal (see Theorems 4.3 and 4.6). We also relate the Hanf number of $L(aa)$ to large cardinals, by giving the consistency (relative to the existence of certain large cardinals) of the assertion that it exceeds many measurable cardinals (Theorem 3.5). The main tool for most of these results is Theorem 3.3, which says that $\aleph_1$ is an anomaly in the following sense: while $L(aa)$ behaves nicely on models of power $\aleph_1$ (as shown in [1], [8]), i.e., there is compactness and completeness, there is nevertheless a sentence ψ of $L(aa)$ with arbitrarily large models, such that ψ describes models of set theory in which every countable subset of the model is an element of the model, as long as the model has power greater than $\aleph_1$.

The groundwork is laid in Sections 1 and 2 by proving that one can in a sense force certain κ-like orderings to appear in all sufficiently large models of a fixed sentence of $L(aa)$. Section 3 contains the main result showing that a sentence ψ exists as described above. We conclude in Section 4 with a number of results on the relationships among the Hanf numbers of $L(aa)$, of logic with a well-ordering quantifier, of logic with a quantifier over countable sets, and of second-order logic. Further such results will appear in [9].

Received December 5, 1981; revised April 16, 1984

1 Basic lemmas The various results hinge upon the fact that there is a sentence ψ_1 of $L(aa)$ which is satisfied by well-orderings, such that every ordering satisfying ψ_1 is reasonably close to being a well-ordering.

Definition 1.1 ψ_1 is the sentence $aa\, s((\alpha) \wedge (\beta) \wedge$ "$<$ is a linear order"), where
(α) is $\forall x(\{y \in s: x < y\}$ has a minimal element), and
(β) is $\forall x(\{y \in s: y < x\}$ has a least upper bound).

Remark 1.1a: Clearly ψ_1 is true in every limit ordinal.

Then we may prove

Lemma 1.2 *If $(M, <) \models \psi_1$ and $\langle a_i: i < \alpha\rangle$ is increasing and bounded in $(M, <)$, where α has uncountable cofinality, then $\{a_i: i < \alpha\}$ has a least upper bound.*

Proof: First suppose that there exists $\langle b_n: n < \omega\rangle$, a decreasing sequence such that $b_n > a_i$ for all $n < \omega$, $i < \alpha$, and $\forall x\left[\bigwedge_{i<\alpha} x > a_i \rightarrow \bigvee_{n<\omega} x > b_n\right]$. Now for almost all $s \in P_{\omega_1}(M)$, $\{b_n: n < \omega\} \subseteq s$. And for any such s, if we choose $i < \alpha$ so that $(\forall x \in s)\left(\bigvee_{j<\alpha} x < a_j \rightarrow x < a_i\right)$ (using $cof(\alpha) > \omega$), then $\{d \in s: d > a_i\}$ has no least element. This contradicts clause (α) of ψ_1.

The proof now follows immediately from the following:

Claim *If $(M, <) \models \psi_1$, then there is no uncountable decreasing sequence in $(M, <)$.*

Suppose to the contrary that $\langle d_i: i < \omega_1\rangle$ is decreasing. It is easy to check that for almost all s:

$$(\forall x \in s)[\exists i < \omega_1(d_i < x) \rightarrow \exists i < \omega_1(d_i \in s \wedge d_i < x)] \ .$$

For any such s, if $j = sup\{i: d_i \in s\}$, then $\{x \in s: x > d_j\}$ has no least element, contradicting clause (α) of ψ_1.

Lemma 1.3 *Suppose $\langle a_i: i < \alpha\rangle$ is an increasing sequence in some linear order $(M, <)$, where $cof(\alpha) > \omega$. Suppose $X \subseteq \alpha$, where every member of X has cofinality ω. Then X is stationary in α iff $\{s \in P_{\omega_1}(M): sup\{i < \alpha: a_i \in s\} \in X\}$ is stationary in $P_{\omega_1}(M)$.*

Proof: It suffices to prove that if $Y \subseteq \alpha$ contains all members of α which have uncountable cofinality, then Y contains a *cub* in α iff $aa\, s[sup\{i < \alpha: a_i \in s\} \in Y]$. The direction $(\Rightarrow)$ is clear.

For $(\Leftarrow)$, first let $D = \{s \in P_{\omega_1}(M): sup\{i < \alpha: a_i \in s\} \in Y\}$, and suppose $aa\, s(s \in D)$. From [3] we know that there exist functions $f_n(n < \omega)$, where f_n is n-place, such that for all s closed under each $f_n(n < \omega)$, $s \in D$. (Think of the functions f_n as "strategies".) Now choose $C \subseteq \alpha$ such that C is *cub* in α, of order type $cof(\alpha)$. Let $Z = \{$limit β of C: for all x in the closure of $\{a_i: i \in \beta \cap C\}$ under $\{f_n: n < \omega\}$, $x < a_j$ for some $j \in \beta \cap C\}$. Then Z is *cub* in α, and it suffices to show $Z \subseteq Y$. Suppose $\beta \in Z$. If $cof(\beta) > \omega$, then $\beta \in Y$ by hypothesis. Otherwise we may choose $s_0 \subseteq \beta \cap C$ such that s_0 is countable and is cofinal in β. Let s be the closure of $\{a_i: i \in s_0\}$ under $\{f_n: n < \omega\}$. Then since $\beta \in Z$, $sup\{i < \alpha: a_i \in s\} = \beta$. Finally, by choice of $\{f_n: n < \omega\}$, $sup\{i < \alpha: a_i \in s\} \in Y$; i.e., $\beta \in Y$.

Remark 1.4: A similar proof shows the following. Suppose $\mathfrak{M} \models \psi_1$, and let $\langle a_i: i < \alpha\rangle$ be an increasing sequence in $\mathfrak{M}$ where $cof(\alpha) > \aleph_0$. Then the set $S = \{i < \alpha: \{a_j: j < i\}$ has a least upper bound$\}$ contains a closed unbounded set. Hence there is an increasing continuous sequence $\langle b_i: i < cof(\alpha)\rangle$ which has the same supremum as $\langle a_i: i < \alpha\rangle$.

2 Strongly κ-like initial segments We show that the sentence in Section 1 can be modified so that every model of power at least $\aleph_2$ has a continuous sequence $\langle a_\alpha: \alpha \leq \omega_2\rangle$, such that a_α has fewer than $\aleph_2$ predecessors for each $\alpha < \omega_2$. (This is what we need for Section 3.) In fact we prove a bit more in Lemma 2.4, and as a result we conclude without further ado that if κ is the Hanf number of $L(aa)$ then $\kappa = \beth_\kappa$. In particular, κ exceeds the Hanf number of $L(Q)$ (which is known to be $\beth_\omega$), which answers a question from [1].

Definition 2.1 Consider a structure $(\lambda, <, cf, F, g, \omega)$, where λ is a cardinal, cf is the one-place function giving the cofinality, $g(\alpha, \cdot)$ maps $\{i: i < cf(\alpha)\}$ cofinally into α, and whenever $\mu < \lambda$ is regular then $\{j < \mu: cof(j) = \aleph_0$ and $F(\mu, j) = i\}$ is stationary in μ, for each $i < \mu$. Now the following formula ψ_2 holds in this structure. (Recall that ψ_1 was defined in 1.1.)

$$\begin{aligned}\psi_2 \equiv\ & \psi_1 \wedge \forall x \exists y(x < y \wedge cf(y) = y)\\ & \wedge \forall x(cf(cf(x)) = cf(x))\\ & \wedge \forall x[cf(x) < x \rightarrow \text{“}g(x, \cdot) \text{ maps the predecessors of } cf(x)\\ & \quad \text{cofinally into the predecessors of } x\text{”}]\\ & \wedge (\gamma)\ ,\end{aligned}$$

where

$$\begin{aligned}(\gamma) \equiv\ & \forall x[cf(x) > \underline{\omega} \leftrightarrow (aa\, s)(\exists y < x)(\forall z < x)(s(z) \rightarrow z < y)]\\ & \wedge \forall x[cf(x) = x > \underline{\omega} \rightarrow (\forall y < x)(stat\ s)(\exists z < x)\\ & \quad (z = sup(\{v \in s: v < x\}) \wedge F(x, z) = y)]\ .\end{aligned}$$

For $\mathfrak{M} \models \psi_2$ and $a \in M$, we write $pred^{\mathfrak{M}}(a)$ for $\{x \in M: x <^{\mathfrak{M}} a\}$, and $cof(<a)$ for the cofinality of the order $(pred^{\mathfrak{M}}(a), <^{\mathfrak{M}} \restriction pred^{\mathfrak{M}}(a))$, and similarly for $card(<a)$.

Lemma 2.2 *If $\mathfrak{M} \models \psi_2 \wedge cf(a) = a$, then $cof(<a) = card(<a)$.*

Proof: Assume $\mathfrak{M} \models \psi_2 \wedge cf(a) = a > \underline{\omega}$ (as the case $a = \omega$ is trivial). The idea is that (γ) gives us $card(<a)$ disjoint stationary subsets of $pred^{\mathfrak{M}}(a)$, which must have disjoint intersections with any closed unbounded subset of $pred^{\mathfrak{M}}(a)$. Let $\langle a_i: i < \lambda\rangle$ be any increasing sequence with supremum a, where $\lambda = cof(<a)$. Now it's clear that $aa\, s(sup\{v \in s: v < a\} = sup\{a_i: a_i \in s\})$. Together with (γ), this shows that

(1) $(stat\ s)(F(a, sup\{a_i: a_i \in s\}) = b)$,

for all $b < a$. Let $Y_b = \{j < \lambda: F(a, sup\{a_i: i < j\}) = b\}$, for all $b < a$. Then

$b \neq c$ implies $Y_b \cap Y_c = \emptyset$, so we're done if we can show $Y_b \neq \emptyset$ for all $b < a$. By Lemma 1.3, Y_b is stationary in λ if

(2) $\{s \in P_{\omega_1}(M)\colon sup\{i < \lambda\colon a_i \in s\} \in Y_b\}$ is stationary in $P_{\omega_1}(M)$.

But if $j = sup\{i < \lambda\colon a_i \in s\}$ for fixed s, then j is a limit for $aa\, s$, hence $sup\{a_i\colon i < j\} = sup\{a_i\colon a_i \in s\}$. So (2) is implied by (1), which was shown above.

Definition 2.3 The sentence ψ_3 is the conjunction of the following sentences:

[1] $\forall x(P(x) \vee Q(x)) \wedge \neg\exists x(P(x) \wedge Q(x))$.

[2] ψ_2^P, the relativization of ψ_2 to P.

[3] Almost every countable subset of P is represented:

$$aa\, s\ \exists y(Q(y) \wedge (\forall x \in P)(s(x) \leftrightarrow xRy))\ .$$

[4] P is as large as the universe: $\forall x \exists y(P(y) \wedge g_1(y) = x)$.

[5] Enough set theory, formulated using the symbol R. The particular finite amount of set theory needed will be evident in the succeeding proofs. For example, we have finitely many instances of the subset scheme: $\forall x \exists y \forall u(uRy \leftrightarrow uRx \wedge \phi(u))$.

[6] The function f witnesses that every element of Q is countable:

$$\forall y[Q(y) \to \forall x(xRy \to f(y, x) < \underline{\omega}) \wedge \forall x_1 \forall x_2(f(y, x_1) = f(y, x_2) \wedge x_1Ry \wedge x_2Ry) \to x_1 = x_2)]\ .$$

[7] P has cofinality ω: $(aa\, s)(\exists x \in P)(\exists y \in P)(x < y \wedge s(y))$.

Lemma 2.4

(i) *Assume there exists $a \in M$ such that $card(<a) \geq \lambda$, where $\mathfrak{M} \models \psi_2$ and $\lambda > \omega$ is a regular cardinal. Then $card(<b) = \lambda$ for some $b \in M$ such that $\mathfrak{M} \models cf(b) = b$.*

(ii) *Assume $|M| \geq \lambda$ where λ has uncountable cofinality and $\mathfrak{M} \models \psi_3$. Then for some $b \in P^{\mathfrak{M}}$, $pred^{\mathfrak{M}}(b)$ is λ-like.*

Proof: (i) ψ_2 guarantees that for some $a' > a$, $\mathfrak{M} \models cf(a') = a'$. By Lemma 2.2, $cof(<a') = card(<a') \geq \lambda$. Choose an increasing sequence $\langle a_i\colon i < \lambda\rangle$ of elements less than a'. By Lemma 1.2, there exists $a'' = sup\{a_i\colon i < \lambda\}$. Let $b = cf^{\mathfrak{M}}(a'')$. Then $card(<b) = cof(<b) = cof(<a'') = \lambda$.

(ii) If λ is singular then this follows from clauses 4 and 7 in the definition of ψ_3 together with Lemma 1.2 and part (i) of this lemma. So assume λ is regular. Consider the initial segment $A = \{x \in M\colon |pred^{\mathfrak{M}}(x)| < \lambda\}$. If A has uncountable cofinality, then we may choose $b = sup(A)$ by Lemma 1.2, and this is the desired element. So we may assume that A has countable cofinality. Next, suppose that there are not arbitrarily large $b \in A$ such that $\mathfrak{M} \models cf(b) = b$; say this set is bounded by a. By an instance of clause 5 of ψ_3, we may choose $b \in M$ so that $\mathfrak{M} \models$ "b is the least element greater than a such that $cf(b) = b$". Then $b = sup(A)$, since otherwise there is $c < b$ with $c \notin A$; but then by clause 5 again, there is some function definable in $\mathfrak{M}$ that maps $pred^{\mathfrak{M}}(c)$ one-one into $pred^{\mathfrak{M}}(a)$, which contradicts $c \notin A$.

So we may assume that there is an increasing sequence $\langle b_n\colon n < \omega\rangle$ cofinal in A, with $\mathfrak{M} \models cf(b_n) = b_n > \underline{\omega}$ for all n. Since almost every countable set is represented in $\mathfrak{M}$ (by clause 3), we may choose $s^* \in M$ such that $\mathfrak{M} \models cf(x) = x$ for all $x \in s^*$ and $\mathfrak{M} \models b_n R s^*$ for all $n < \omega$. We may also assume (using (i)) that $card(<b) \leq \lambda$ and $b \geq b_0$, whenever $bR^{\mathfrak{M}}s^*$ and b exceeds each b_n. Let $s_0 = \left\{x\colon xRs^* \text{ and } \bigvee_n x < b_n\right\}$ and let $s_1 = \{x\colon xRs^*\} - s_0$. Clearly $x \in s_0$ implies $card(<x) < \lambda$, and $x \in s_1$ implies $cof(<x) = \lambda$ by Lemma 2.2 (i.e., $cof(<x) = card(<x)$ if $\mathfrak{M} \models cf(x) = x$). For all $d \in s_1$ let $\langle x_i^d\colon i < \lambda\rangle$ be an increasing sequence cofinal in d. Let $f_n (n < \omega)$ be n-place functions so that whenever T is a countable subset of M which is closed under each f_n, then $T = \{x\colon xRT^*\}$ for some $T^* \in M$. These functions exist by a theorem of Kueker [3] and because $\mathfrak{M} \models \psi_3$.

Define by induction $\mathfrak{N}_n$ and $i(n)$ such that:

(α) $\mathfrak{N}_n \prec (\mathfrak{M}, f_n)_{n<\omega}$ and $|N_n| < \lambda$, and $\forall x\left(\bigvee_m x < b_m \rightarrow x \in N_n\right)$;

(β) $i(n) < \lambda$ is chosen such that

$$(\forall d \in s_1)(\forall x < d)[x \in N_n \rightarrow x < x_{i(n)}^d];$$

(λ) $x_{i(n)}^d \in N_{n+1}$ for all $d \in s_1$.

Let T_1 be the closure of $\{x_{i(n)}^d\colon d \in s_1,\ n < \omega\}$ under the functions $f_n (n < \omega)$. For all $d \in s_0$, choose $y_d < d$ such that $y_d > y$ for all $y \in T_1$ with $y < d$. (y_d exists as $cof(<d) = card(<d)$ by Lemma 2.2 since $\mathfrak{M} \models cf(x) = x$ for all xRs^*, and $card(<d) \geq card(<b_0) > \aleph_0$.) Let T_2 be the closure of $T_1 \cup \{y_d\colon d \in s_0\}$. Now

(*) for all $d \in s_0$, $T_1 \cap \{x\colon x < d\}$ is not cofinal in $T_2 \cap \{x\colon x < d\}$,

but

(**) for all $d \in s_1$, $T_1 \cap \{x\colon x < d\}$ is cofinal in $T_2 \cap \{x\colon x < d\}$,

this because $T_1 \subseteq T_2 \subseteq \bigcup_{n<\omega} N_n$, and $\{x_{i(n)}^d\colon n < \omega\} \subseteq T_1$ is cofinal in $\left(\bigcup_{n<\omega} N_n\right) \cap \{x\colon x < d\}$. Now T_1 and T_2 both are coded in $\mathfrak{M}$ by some $T_1^*, T_2^* \in M$ (i.e., $T_i = \{x\colon xRT_i^*\}$). So by clause 5 of ψ_3, we can define s_0 in $\mathfrak{M}$ by using (*) and (**), so s_0 should have a least upper bound; contradiction.

Theorem 2.5 *Let μ be the Hanf number of $L(aa)$. Then $\mu = \beth_\mu$. (Hence $\mu > \beth_\omega$.)*

Proof: Let κ and λ be cardinals and let $\mathfrak{A}$ be a structure of power $\geq\kappa$, such that for some $\phi \in L(aa)$, $\mathfrak{A} \models \phi$ but ϕ has no model of power $>\lambda$. (That is: λ, ϕ, and $\mathfrak{A}$ witness $\kappa < \mu$.) It suffices to show that $\beth_\kappa < \mu$, since clearly μ is a limit cardinal. In fact the following sentence will be shown to have a model of power $\geq \beth_\kappa$, but no model of power $\geq \beth_{\lambda^+}$:

$$\psi_4 \equiv \psi_3 \wedge \forall x(U(x) \leftrightarrow \text{“}x \text{ is a strong limit”}) \wedge \phi^U ,$$

where ϕ^U is the relativization of ϕ to U, and “x is a strong limit” is $\forall y < x \exists u < x \exists v < x \forall w < y[R(x, u, w) \leftrightarrow R(x, v, w)]$. Clearly if ϕ has a model of power κ', then ψ_4 has a model of power $\beth_{\kappa'\cdot\omega}$. However, if $\mathfrak{B} \models \psi_4$ and $|B| \geq \beth_{\lambda^+}$, then by Lemma 2.3(ii) there exist elements b_α for all $\alpha < \lambda^+$ such that $(pred^{\mathfrak{B}}(b_\alpha), <^{\mathfrak{B}} \restriction b_\alpha)$ is $\beth_{\alpha\cdot\omega_1}$-like; so there are at least λ^+ "strong limits" in $\mathfrak{B}$, i.e. $|U^{\mathfrak{B}}| \geq \lambda^+$, which contradicts $\mathfrak{B} \models \phi^U$ and the choice of ϕ and ψ.

In Sections 3 and 4 we will see that the Hanf number of $L(aa)$ can be very large indeed.

3 Representing all countable subsets

Definition 3.1 Let T_{ω_2} be the tree of all strictly increasing finite sequences of ordinals less than ω_2, ordered by inclusion. An *ω_2-tree* is a subtree $T \subseteq T_{\omega_2}$ such that $\varnothing \in T$ and every element of T has ω_2-many immediate successors in T.

The key lemma is the following theorem. It is a special case of Theorem 4.7 of [7].

Lemma 3.2 [7] *Let T be an ω_2-tree and suppose that $\eta \mapsto A_\eta$ is a map from T to the set of countable subsets of ω_2. Then there is an ω_2-tree T_1 which is a subtree of T, such that for some map $\nu \mapsto \gamma_\nu$ from T_1 into ω_2, $A_\eta \cap A_\nu \subseteq \gamma_{\eta\cap\nu}$ for all η, $\nu \in T$.* (In fact, we may replace each γ_ν by a countable set, but we won't need this fact.)

Remark: Lemma 3.2 has a straightforward extension obtained by replacing ω_2 by an arbitrary ω_2-like linear order $(L, <)$. One simply uses a 1-1 map from L onto ω_2 to transfer the L-tree to an ω_2-tree, and then one applies the lemma and transfers back. We omit the details.

We now present the main theorem.

Theorem 3.3 *There is a sentence ψ of $L(aa)$ such that for all κ, ψ has a model of power κ iff $\kappa = \omega_1$ or $\kappa = \kappa^\omega$. Moreover, every model $(P \cup Q, P, Q, R, <, \ldots)$ of ψ of power greater than $\aleph_1$ is isomorphic to a model in which $R \subseteq P \times Q$ is membership and $Q = \mathcal{P}_{\omega_1}(P)$, and $(P, <)$ is well-ordered.*

Proof: Let ψ be the sentence ψ_3 of Definition 2.3, and suppose that $\mathfrak{M}$ is a model of ψ of power at least $\aleph_2$. Let us see that it suffices to prove that for every subset X of ω there is $a \in Q^{\mathfrak{M}}$ such that $n \in X$ iff $nR^{\mathfrak{M}}a$ for all $n < \omega$, where we identify n with the nth element of $P^{\mathfrak{M}}$ under $<^{\mathfrak{M}}$ (and hence that $(P^{\mathfrak{M}}, <^{\mathfrak{M}})$ is ω-standard). Given any countable set $X \subseteq P^{\mathfrak{M}}$, we may first represent some superset of X (by clause 3 of ψ_3), say by $a \in Q^{\mathfrak{M}}$. Then if we can represent $\{f^{\mathfrak{M}}(a, b): b \in X\}$ by some $c \in Q^{\mathfrak{M}}$, then by clauses 5 and 6 of ψ_3, we may choose $d \in Q^{\mathfrak{M}}$ such that for all x, $xRd \leftrightarrow [xRa$ and $f(a, x)Rc]$, i.e., $xRd \leftrightarrow x \in X$. The set theory in ψ_3 then guarantees that $<$ is a well-ordering of P.

So let $\mathfrak{M} \models \psi$, $|M| \geq \aleph_2$, and $X \subseteq \omega$; we show that X is represented in $\mathfrak{M}$. By Lemma 2.4(ii), there exists $a \in P^{\mathfrak{M}}$ such that $pred^{\mathfrak{M}}(a)$ is ω_2-like. Choose a strictly increasing sequence $\langle b(\alpha): \alpha < \omega_2\rangle$ with supremum a, such that $b(0) = 0$.

Again using the characterization of the cub filter in [3], since $\psi_3 \models aa\, s$ [s is represented], we may find a structure $\mathfrak{A}$ (countable vocabulary) with domain $P^{\mathfrak{M}}$, such that for all countable submodels $\mathfrak{B}$ of $\mathfrak{A}$, B is represented in $\mathfrak{M}$. For $\eta \in T_{\omega_2}$ let B_η be the universe of the submodel of $\mathfrak{A}$ generated by $\{b(\alpha)\colon \alpha \in \text{range } \eta\}$, and let $A_\eta = B_\eta \cap pred^{\mathfrak{M}}(a)$. Applying Lemma 3.2 (or more accurately, the remark immediately following it), we find an ω_2-subtree T_1 of T_{ω_2} and a map $\eta \to \gamma_\eta$ from T_1 into ω_2, such that for all $\eta, \nu \in T_1$, $A_\eta \cap A_\nu \subseteq [0, b(\gamma_{\eta \cap \nu}))$.

Choose η_0, η_1, η_2 of level 1 in T_1, all distinct (hence incomparable). We will define sequences $\langle \eta_i^k\colon k < \omega\rangle$ for $i = 0, 1, 2$. Though the notation is a bit cumbersome, the idea is rather simple. Each sequence $\langle \eta_i^k\colon k < \omega\rangle$ gives rise to a countable set that is represented, namely $A_i = \bigcup \{A_{\eta_i^k}\colon k < \omega\}$. The sets A_i will intertwine as in Figure 1.

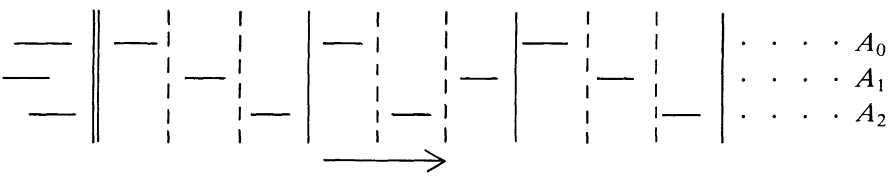

Figure 1.

That is, after a certain "bottom level" (indicated by the double vertical line), there are sets of three "porous blocks", one from each A_i. The first of the three is always from A_0. Whether the next block is from A_1 or A_2 depends on whether or not the level is an integer that belongs to the given subset X of ω. So, for example, according to the picture above we would have $0 \in X$, $1 \notin X$, $2 \in X, \ldots$. By a "porous block" we mean a subset of some interval $[\alpha_i^k, \beta_i^k)$, where these intervals are pairwise disjoint. Here then are the precise inductive hypotheses for the sequences $\langle \eta_i^k\colon k < \omega\rangle$ $(i = 0, 1, 2)$ and associated ordinals α_i^k and β_i^k $(i < 3,\ k < \omega)$. Fix $X \subseteq \omega$. We let A_i^k be an abbreviation for $A_{\eta_i^k}$.

(a) $\eta_0^0 = \eta_0$, $\eta_1^0 = \eta_1$, $\eta_2^0 = \eta_2$. $\alpha_0^0 = \alpha_1^0 = \alpha_2^0 = 0$.
(b) $\eta_i^k \subset \eta_i^{k+1}$; $|\eta_i^k| = k + 1$.
(c) If $k - 1 \in X$ then $\alpha_0^k < \beta_0^k < \alpha_1^k < \beta_1^k < \alpha_2^k < \beta_2^k$.
(d) If $k - 1 \notin X$ then $\alpha_0^k < \beta_0^k < \alpha_2^k < \beta_2^k < \alpha_1^k < \beta_1^k$.
(e) $\gamma_i^k < \beta_i^k < \alpha_0^{k+1}$, where we write γ_i^k to abbreviate $\gamma_{\eta_i^k}$.
(f) $A_i^k \subseteq \bigcup_{m \le k} [b(\alpha_i^m), b(\beta_i^m))$.
(g) For all $\nu \supseteq \eta_i^k$, $A_\nu \cap [0, b(\alpha_i^k)) \subseteq \bigcup_{m<k} [b(\alpha_i^m), b(\beta_i^m))$.

If we can indeed carry out this definition, then X is represented in $\mathfrak{M}$, as we now show. Let $A_i = \bigcup_{k<\omega} A_i^k$ for $i = 0, 1, 2$. Then A_i is represented in $\mathfrak{M}$, say by a_i. Now there is an equivalence relation on a subset of $\mathfrak{M}$, which is definable in $\mathfrak{M}$, given by: $x \sim y$ iff $x \ge \alpha_0^1$, $y \ge \alpha_0^1$, and $xRa_0 \wedge yRa_0 \wedge \forall z[x < z < y \vee y < z < x \to \neg zRa_1 \wedge \neg zRa_2]$. Moreover, each equivalence class has a least element, by clause 5 in ψ_3; in fact the set of these least elements is definable in $\mathfrak{M}$, hence represented in $\mathfrak{M}$ by some element b. Since $\{x: xRb\}$ has order type ω, clause 5 of ψ_3 guarantees that the order is ω-standard, and in fact there will be an internal bijection H from ω to $\{x: xRb\}$. So $k \in X$ iff $\mathfrak{M} \models \exists x[xRa_1 \wedge H(k+1) < x \wedge \forall y(H(k+1) < y < x \rightarrow \neg yRa_2)]$, by the construction of η_i^k, α_i^k, and β_i^k.

So, it suffices to carry out the construction. Assume (a) through (g) hold up to some $k < \omega$. (The case $k = 0$ is easy so we omit it.) Let us define η_0^{k+1}, α_0^{k+1}, and β_0^{k+1}. Choose $\alpha_0^{k+1} > max(\beta_1^k, \beta_2^k)$. First we guarantee that (g) remains true. For all $\delta < \omega_2$ let $\eta(\delta) = \eta_0^k \cup \{\langle k+1, \delta\rangle\}$, and let $S = \{\delta < \omega_2: \eta(\delta) \in T_1\}$. We claim that for some $\delta \in S$,

(*) for all $\nu \supseteq \eta(\delta)$, $A_\nu \cap [0, b(\alpha_0^{k+1})) \subseteq [0, b(\gamma_0^k))$.

For suppose (*) fails for all $\delta \in S$. Since T_1 is an ω_2-tree, $|S| = \omega_2$. For each $\delta \in S$ choose ν_δ witnessing the failure of (*), i.e., $\nu_\delta \supseteq \eta(\delta)$ and the set $B_\delta = A_{\nu_\delta} \cap [b(\gamma_0^k), b(\alpha_0^{k+1}))$ is nonempty. But notice that for $\delta, \rho \in S$, $B_\delta \cap B_\rho = A_{\nu_\delta} \cap A_{\nu_\rho} \cap [b(\gamma_0^k), b(\alpha_0^{k+1})) \subseteq [0, b(\gamma_0^k)) \cap [b(\gamma_0^k), b(\alpha_0^{k+1})) = \varnothing$. So the family of sets B_δ $(\delta \in S)$ is a family of ω_2-many nonempty pairwise disjoint subsets of $[b(\gamma_0^k), b(\alpha_0^{k+1}))$, which has power at most ω_1, a contradiction.

So, choose δ so that (*) holds, and let $\eta_0^{k+1} = \eta(\delta)$. Then we may verify (g): for $\nu \supseteq \eta_0^{k+1}$,

$$\begin{aligned} A_\nu \cap [0, b(\alpha_0^{k+1})) &\subseteq A_\nu \cap [0, b(\gamma_0^k)) && \text{(by (*))} \\ &\subseteq [A_\nu \cap [0, b(\alpha_0^k))] \cup [A_\nu \cap [b(\alpha_0^k), b(\beta_0^k))] && \text{(by (e))} \\ &\subseteq \bigcup_{m<k} [b(\alpha_0^m), b(\beta_0^m)) \cup [b(\alpha_0^k), b(\beta_0^k)) && \text{(by inductive hypothesis (g) for } k) \\ &= \bigcup_{m<k+1} [b(\alpha_0^m), b(\beta_0^m)). \end{aligned}$$

Finally, let β_0^{k+1} be any ordinal exceeding γ_0^{k+1} such that $A_{\eta_0^{k+1}} \subseteq [0, b(\beta_0^{k+1}))$. Then (f) for $k+1$ follows from (g) for $k+1$.

The constructions of α_i^{k+1}, β_i^{k+1}, and η_i^{k+1} for $i = 1, 2$ are similar to the construction of α_0^{k+1}, β_0^{k+1}, and η_0^{k+1}. The only added care is that if $k-1 \in X$ then we do the construction for $i = 1$ before the construction for $i = 2$, while if $k-1 \notin X$ then we do these in the opposite order. Then (c) and (d) continue to hold, and the proof is complete.

Remark 3.4: The following observations follow directly from Theorem 3.3, and give further examples of "bad behavior" for $L(aa)$ on cardinals greater than $\aleph_1$ (the opposites are true for $L(Q)$):

(i) Suppose $\omega_1 < \mu < \mu^\omega < \lambda = \lambda^\omega$. Then there is a model of power λ, for a countable language, with no $L(aa)$-elementary submodel of power μ.
(ii) The set of $L(aa)$ sentences which have arbitrarily large models is not arithmetic.

In the next section we will see that it is relatively consistent with the existence of a measurable cardinal that the Hanf number of $L(aa)$ exceeds the first measurable cardinal. The following theorem has a stronger hypothesis than this, but it also has a stronger conclusion and it follows quite easily from known results.

Theorem 3.5 *Suppose that $V \models ZFC + GCH +$ "κ is supercompact" + "for arbitrarily large $\mu < \kappa$, μ is λ-supercompact whenever $\mu < \lambda < \kappa$." Then there is a forcing extension $V[G]$ in which the Hanf number of $L(aa)$ exceeds κ, such that there are arbitrarily large $\mu < \kappa$ for which μ is λ-supercompact whenever $\mu < \lambda < \kappa$.*

Proof: First we use Laver's forcing [5] to get a model in which κ is still supercompact, $2^\lambda = \lambda^+$ for all $\lambda < \kappa$ such that λ is not strongly inaccessible, [μ is λ-supercompact whenever $\mu < \lambda < \kappa$] whenever this is true of μ in V ($\mu < \kappa$), and such that no κ-closed notion of forcing destroys the supercompactness of κ. Next use κ-closed forcing to add κ^{++} subsets of κ, and finally use Prikry forcing [6] to make $cof(\kappa) = \omega$; this is our model $V[G]$. Notice that $V[G]$ contains no bounded subsets of κ that are not already in the Laver model; hence it suffices to show that the Hanf number of $L(aa)$ exceeds κ in $V[G]$. Now work in $V[G]$. By Theorem 3.3 it is easy to produce a sentence of $L(aa)$ whose models of power greater than $\aleph_1$ are all expansions of a transitive model of set theory that contains all of its countable subsets as elements, and such that any such transitive set can be expanded to a model of this sentence. Let ϕ be the conjunction of this sentence with $\forall\lambda(cf(\lambda) = \omega \rightarrow \lambda^\omega = \lambda^+)$. Now ϕ has a model of power $\geq\kappa$, but since $\kappa^\omega = 2^\kappa = \kappa^{++}$, ϕ does not have arbitrarily large models.

4 Relations among Hanf numbers for certain logics In this section we relate the Hanf number of $L(aa)$ to the Hanf numbers of the following logics: logic with a well-ordering quantifier, denoted $L(wo)$; second-order logic with second-order quantification restricted to countable sets, denoted L^c; and full second-order logic, denoted L^{II}. We start with some relationships that can be proved in *ZFC*. For a logic $\underline{L}$, we write $h(\underline{L})$ for the Hanf number of $\underline{L}$.

Theorem 4.1 *$h(L(wo)) \leq h(L^c) \leq h(L(aa)) \leq h(L^{\mathrm{II}})$, and $h(L^c) < h(L^{\mathrm{II}})$.*

Proof: All of the inequalities in the first part are obvious except for $h(L^c) \leq h(L(aa))$. But this follows from Theorem 3.3. For let ψ be the sentence of Theorem 3.3; specifically, let ψ be the sentence ψ_3 from Definition 2.3. Suppose $\mathfrak{A} \models \phi$ where $\phi \in L^c$ and $\mathfrak{A}$ has power at least $h(L(aa))$. Expand $\mathfrak{A}$ to a model $(\mathfrak{A}, \mathfrak{B}, E, f)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are disjoint, $\mathfrak{B} \models \psi_3$, and $|B| = |A|$, with E and f as follows. E codes $\mathcal{P}_{\omega_1}(A)$ in the sense that for all countable $s \subseteq A$, there exists a unique $b \in B$ such that for all x, xEb iff $x \in s$. f is a one-one function from A into $P^{\mathfrak{B}}$. Now any sentence of $L(aa)$ that is true in this model has arbitrarily large models. By using $\mathfrak{B}$ and E to code $\mathcal{P}_{\omega_1}(A)$, so that ϕ becomes first-order in models of power at least $\aleph_2$, it follows easily that ϕ has arbitrarily large models.

The last inequality follows from Lemma 1 of [12]: $h(L_{\omega_1\omega_1}) < h(L^{\mathrm{II}})$.

It follows that the Hanf number of $L(aa)$ exceeds, for example, the least inaccessible cardinal (if one exists), since this is well-known for $h(L(wo))$ (and not hard to see). What happens under other set-theoretic hypotheses? First we consider a hypothesis that contradicts $V = L$.

Proposition 4.2 *Suppose that $\lambda \to (\omega_1)_2^{<\omega}$ for some λ (Erdös notation). Then $h(L(wo)) < h(L^c)$.*

Proof: In fact it's easier to prove a stronger result. Namely, let κ be least such that for every partition f of the finite subsets of κ into ω pieces, there is $g: \omega \to \omega$ such that for all $\alpha < \omega_1$, there exists $H \subseteq \kappa$ of order type α such that $f(X) = g(|X|)$ for all finite $X \subseteq H$. It's routine to construct a sentence ϕ of L^c such that ϕ has a model of power κ but has no model of power greater than κ: ϕ merely describes a well-ordering and a family of functions $f_\lambda(x) = f(\lambda, x)$ such that f_λ witnesses that λ does not have the property described above. However, it follows from Silver [10] that $h(L(wo)) \leq \kappa$.

Theorem 4.3 *Assume $V = L$. Then $h(L(wo)) = h(L^c) < h(L(aa)) = h(L^{\text{II}})$.*

Proof: The first equation follows from Lemma 2(ii) of [12]. Clearly $h(L(aa)) \leq h(L^{\text{II}}))$. To show that $h(L(aa)) \geq h(L^{\text{II}}))$, we recall the following theorem of [2].

($\Box$) *($V = L$) There is a Δ_1-definable sequence $\langle C_\delta$: δ a limit ordinal, $cof(\delta) < \delta\rangle$ such that for all δ, C_δ is a closed unbounded subset of δ of order type less than δ; moreover, for all limit points δ_1 of C_δ, $C_{\delta_1} = C_\delta \cap \delta_1$; and further, $|C_\delta| = \omega$ whenever $cf(\delta) = \omega$.*

The rough idea of the proof is to show that being a cardinal is sufficiently absolute for models of a certain sentence ϕ of $L(aa)$. More precisely, let ϕ be a sentence of $L(aa)$ that has the following properties. First, ϕ has arbitrarily large models, and every model of ϕ of power greater than ω_1 is isomorphic to an expansion of some (L_α, ϵ): this is easy by Theorem 3.3. ϕ should also assert that "aa" has the same meaning internally as externally: $\forall x \forall C$["C contains a cub in $\mathcal{P}_{\omega_1}(x)$" $\leftrightarrow aa\, s\ (s \cap x \in C)$]. Next, let ϕ also say that the Δ_1 definition of the $\Box$-sequence has the properties given in ($\Box$) above. Finally, ϕ should assert that λ is the largest cardinal and $\lambda = \kappa^+$, for constant symbols λ and κ. Notice that for all λ, $(L_{\lambda^+}, \lambda, \kappa, \epsilon) \models \phi$. If we can show that for every model $\mathfrak{A}$ of ϕ, $\lambda^{\mathfrak{A}}$ is really a cardinal, then the argument is completed as follows: Suppose that $\psi \in L^{\text{II}}$ has a model of power at least $h(L(aa))$. Let θ assert ϕ together with $(\exists X)((\kappa, X) \models \psi)$; so $\theta \in L(aa)$ and by choice of ψ, θ has arbitrarily large models. Choose $(L_\alpha, \kappa, \lambda) \models \theta$. If λ is a real cardinal, then the real power set of κ is a subset of L_λ, hence belongs to L_α. So if $L_\alpha \models$ "$(\kappa, X) \models \psi$", then indeed (κ, X) is really a model of ψ. By choosing α large enough we can make κ as large as we like.

Suppose $(L_\alpha, \epsilon, \kappa, \lambda, \ldots) \models \phi$; it remains to prove that λ is really a cardinal. Notice that for all δ, C_δ is the same internally as externally because Δ_1 definitions are absolute. Since $L_\alpha \models$ "λ is regular" $\wedge\, \lambda > \omega$, we may apply internally the countable completeness of the cub filter on λ (or, Fodor's lemma) to the function $f(\delta) =$ [order type of C_δ] on $\{\delta < \lambda: cof(\delta) = \omega\}$. This yields $S \subseteq \lambda$ which is stationary (according to L_α, hence also in V since $(L_\alpha, \ldots) \models \phi$) and an ordinal $\beta < \omega_1$ such that C_δ has order type β for all $\delta \in S$. Now if λ is not a cardinal, then C_λ exists. Let C'_λ be the set of all limit points of C_λ. Then clearly $|C'_\lambda \cap S| \leq 1$, since $C_\delta = C_\lambda \cap \delta$ for all $\delta \in C'_\lambda$. This is a contradiction, since $C'_\lambda \cap S$ is stationary.

Recall that it is relatively consistent with the existence of a measurable cardinal that $V = L[D]$ for some normal ultrafilter D (see [11] for basic information on $L[D]$, for example, $L[D]$ satisfies GCH). So the following theorem gives the relative consistency with the existence of a measurable cardinal that $h(L(aa))$ exceeds the least measurable cardinal. First we need some notions and lemmas about iterated ultrapowers.

Definition 4.4 Let δ be an ordinal of uncountable cofinality. D_δ is the filter generated by the cub subsets of δ together with $\{i < \delta: cof(i) = \omega\}$; i.e., $X \in D_\delta$ iff for some closed unbounded $Y \subseteq \delta$, $Y \cap \{i < \delta: cof(i) = \omega\} \subseteq X$. For any D, $L_\alpha[D]$ is the collection of sets constructible before stage α, using "$x \in D$" as a predicate; and $L[D] = \bigcup \{L_\alpha[D]: \alpha \text{ is an ordinal}\}$. Let $\alpha(\delta)$ be the least ordinal α such that for some $A \in L_\alpha[D_\delta]$, $A \subseteq \delta$ but $A \notin D_\delta$ and $(\delta - A) \notin D_\delta$. If $\alpha(\delta)$ does not exist then we write $\alpha(\delta) = \infty$.

Lemma 4.5 (Kunen [4]) *Let $V = L[D]$, where D is a normal ultrafilter on (a measurable cardinal) ρ.*

(i) *For all $\delta \leq \rho^+$, $\alpha(\delta) < \infty$. For all regular $\delta > \rho^+$, $\alpha(\delta) = \infty$.*

(ii) *Suppose E is a filter on δ such that $L[E] \models$ "E is a normal ultrafilter on a measurable cardinal δ". Then $E = D$, or else $\delta > \rho$ and $L[E]$ is an iterated ultrapower of $L[D]$.*

(iii) *If $L[E] \models$ "E is a normal ultrafilter on δ", then $\mathcal{P}(\rho) \subseteq L[E]$.*

(iv) *Suppose $L[E] \models$ "E is a normal ultrafilter on ν", where $\nu < \rho^{++}$. Then ρ^{+++} is not moved by the iterated ultrapower embedding from $L[D]$ to $L[E]$.*

(v) *Iterated ultrapower embeddings are elementary.*

Proof: These are consequences of the following theorems of [4]: Theorem 6.7 is (ii) above, if it is assumed that $\delta > \rho$. Then (iii) follows for the case $\delta > \rho$, from this and Theorem 3.1(iv). From what we have proved so far it is then not difficult to prove, using Theorem 5.11, that $\delta \geq \rho$ in (ii) and if $\delta = \rho$ then $E = D$ by Theorem 6.4; so (iii) is completed too. Part (v) is a consequence of Lemma 2.9. Next we check the first part of (i). Suppose $\alpha(\delta) = \infty$. Then clearly $L[D_\delta] \models$ "$(D_\delta \cap L[D_\delta])$ is a normal ultrafilter". It follows from (ii) that $D_\delta = D$ or else $\delta > \rho$. But D_δ is not an ultrafilter (as disjoint stationary subsets of δ, consisting of ordinals of cofinality ω, are known to exist). Hence $\delta > \rho$. Therefore by (iii) and (v), $\delta > \rho^+$. The second part of (i) is practically Theorem 5.8, and in fact it follows from the proof of 5.8. Part (iv) follows from Theorem 3.9(iii) and GCH in $L[D]$, using the observation that if (in the notation of [4]) $i_{0\theta}(\rho) = \nu$, then $\theta < \rho^{++}$ by Corollary 3.8 of [4].

Combining the proof of Theorem 8.8 of [4] with Theorem 3.3 above, one finds that $h(L(aa))$ exceeds the least measurable cardinal. However, since it is obvious that $h(L^{\mathrm{II}})$ exceeds the least measurable cardinal, this also follows from the following.

Theorem 4.6 *Suppose $V = L[D]$, D a normal ultrafilter on ρ. Then $h(L(aa)) = h(L^{\mathrm{II}})$.*

Proof: Choose a sentence $\phi \in L^{\mathrm{II}}$ that has a model of power $\lambda = \rho^{+++}$ but has no model of larger power. Let $\psi \in L(aa)$ say "enough" things about the structure $(H(\lambda^{++}), \epsilon, \lambda, \lambda^+, \rho, D)$ for the language $\{\epsilon, \underline{\lambda}, \underline{\Lambda}, \underline{\rho}, \underline{D}\}$. We show that ψ has no model of power greater than λ^{++}.

Suppose (for a contradiction) that $\mathfrak{M} \models \psi$ where $|M| > \lambda^{++}$. By (the proof of) Theorem 3.3, $\mathfrak{M}$ is well-founded, hence we may assume that $\mathfrak{M}$ is transitive. By Lemma 4.5(i) and the choice of ψ, $\mathfrak{M} \models$ "$\alpha(\delta) < \infty$ for all $\delta \leq \underline{\rho}$". Therefore $\alpha(\delta) < \infty$ in V for all $\delta \leq \underline{\rho}^{\mathfrak{M}}$, since we may require that ψ says that "aa" has the same meaning internally as externally. Then Lemma 4.5(i) implies that $\underline{\rho}^{\mathfrak{M}} < \rho^{++}$.

Now $M = L_\beta[\underline{D}^{\mathfrak{M}}]$ for some $\beta \geq \lambda^{+++} > \rho^{++}$. Since $\underline{\rho}^{\mathfrak{M}} < \rho^{++}$, $\mathcal{P}^{L[\underline{D}^{\mathfrak{M}}]}(\underline{\rho}^{\mathfrak{M}}) \subseteq L_{\rho^{++}}[\underline{D}^{\mathfrak{M}}]$ by the usual condensation argument. So

(*) $\quad L[\underline{D}^{\mathfrak{M}}] \models$ "$\underline{D}^{\mathfrak{M}}$ is a normal ultrafilter on $\underline{\rho}^{\mathfrak{M}}$",

since this holds in $L_\beta[\underline{D}^{\mathfrak{M}}]$. Suppose $D = \underline{D}^{\mathfrak{M}}$. Then the argument from the proof of $h(L(aa)) = h(L^{\mathrm{II}})$ assuming $V = L$ (cf. Theorem 4.3) shows that $\underline{\lambda}^{\mathfrak{M}}$ and $\underline{\Lambda}^{\mathfrak{M}}$ are cardinals in V. Therefore $\mathcal{P}(\underline{\lambda}^{\mathfrak{M}}) \subseteq L_{\underline{\Lambda}^{\mathfrak{M}}}[\underline{D}^{\mathfrak{M}}] \subseteq M$. Since $\mathfrak{M}$"thinks" that ϕ has a model of power $\underline{\lambda}$, this is really a model (in V). Therefore $\underline{\lambda}^{\mathfrak{M}} < \lambda^+$ and hence $|M| \leq (\lambda^{\mathfrak{M}})^{++} \leq \lambda^{++}$ (since $\mathfrak{M} \models V = L_{\lambda^{++}}[\underline{D}]$), a contradiction. So we may assume by Lemma 4.5(ii) and (*) that $L[\underline{D}^{\mathfrak{M}}]$ is an iterated ultrapower of $L[D]$. But as we have seen, $\underline{\lambda}^{\mathfrak{M}} \geq \lambda^+$. Also, $\underline{\Lambda}^{\mathfrak{M}}$ is a cardinal in $L[\underline{D}^{\mathfrak{M}}]$, by the proof of Theorem 4.3 again. Therefore $L[\underline{D}^{\mathfrak{M}}] \models$ "ϕ has a model of power $\geq \lambda^+$", hence this holds in $L[D]$ by Lemmas 4.5(iv) and (v); contradiction.

Shelah has recently shown that the relations given in Theorem 4.1 between the Hanf numbers of $L(wo)$, L^c, $L(aa)$, and L^{II} are the only ones provable in *ZFC* (subject to consistency of some large cardinal hypotheses). Proofs will appear in [9].

NOTE

1. The results in this paper are due to Shelah. The writeup was done by Kaufmann.

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