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Iterated Images on Manifolds

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Many results of classical recursion theory carry over nicely when generalized to recursive manifolds. However, this paper shows that a standard classical property of the basic operation of iteration does not hold in the generalized setting. Specifically, the result

(*) If f is p.r., then $\{f^n(x_0)\}_{n=0}^{\infty}$ is r.e. for any x_0

does not carry over to recursive manifolds.

By way of comparison and contrast, consider the state of affairs for uniformly reflexive structures (URS). The result of this paper has a parallel in Friedman's example [1] of a URS with a nonsemicomputable splinter.[1] However, if a URS has one nonsemicomputable splinter, then it cannot, in fact, have any infinite semicomputable splinter; this implication does not transfer to manifolds, as can easily be seen.

To keep the paper self-contained, this paragraph briefly reviews the relevant terminology from [2]. A simple example of a manifold is N^2, written as the disjoint union $\bigcup_{i=0}^{\infty} A_i$, where $A_i = \{i\} \times N$ is enumerated by α_i, with $\alpha_i(j) = (i, j)$. B, a subset of N^2, is *𝔄-r.e.* iff $\alpha_i^{-1}(B)$ is r.e. for every i. A function f from N^2 to N^2 is *𝔄-𝔄-rec* iff for each m and n there exists partial recursive $f_{m,n} : \alpha_m^{-1}(f^{-1}(A_n)) \to N$ such that $f \circ \alpha_m = \alpha_n \circ f_{m,n}$. A *compact* f from N^2 to N^2 is one such that each $f(A_i)$ is contained in a finite union of A_k's. If f is 1-1 and 𝔄-𝔄-rec such that both f and f^{-1} are compact, f is an *embedding*. And, as usual, for $S \subseteq N$, χ_S denotes the characteristic function of S.

We will show that (*) does not generalize to N^2 (much less to other, more complicated manifolds). In fact, there is an embedding f from N^2 onto N^2 and $x_0 \in N^2$ such that not only is $\{f^n(x_0)\}_{n=0}^{\infty}$ not 𝔄-r.e., but $\alpha_k^{-1}(\{f^n(x_0)\}_{n=0}^{\infty})$ is not r.e. for *any* k.

Proof: Define $h : N^2 \to N^2$ by

$$h(\alpha_k(2n+1)) = \alpha_{k+1}(2n+1)$$

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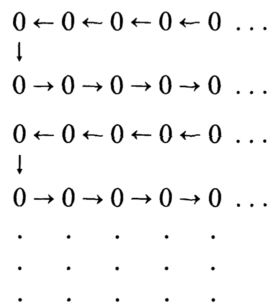
$h(\alpha_0(2n)) = \alpha_0(2n+1)$
$h(\alpha_k(2n)) = \alpha_{k-1}(2n),\ k \geq 1.$

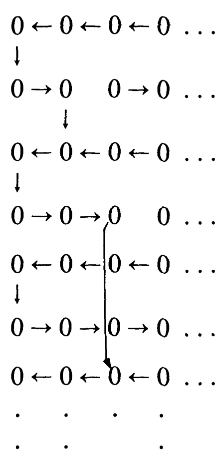
Clearly, h is an onto embedding. Using an infinite lattice in which the circle in the i'th row, j'th column represents $\alpha_j(i)$, $i, j = 0, 1, 2, \ldots$, the function h may be graphically represented as follows:



Let $A \subseteq \{2n \mid n \in N\}$ be a non-r.e. set containing 0. Let $a_0 = 0$ and $a_k =$ the k'th nonzero element of A for $k = 1, 2, 3, \ldots$. Set

$$g(\alpha_1(1)) = \alpha_1(a_1),$$
$$g(\alpha_2(a_1+1)) = \alpha_2(a_2),$$
$$g(\alpha_3(a_2+1)) = \alpha_3(a_3),$$
$$\vdots$$
$$g(\alpha_k(a_{k-1}+1)) = \alpha_k(a_k),$$
$$\vdots$$
$$g(x) = h(x), \text{ otherwise.}$$

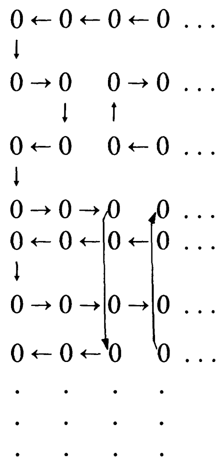
The picture for g looks like this:



Finally, to produce a bijection, set

$$
\begin{aligned}
f(\alpha_2(a_1)) &= \alpha_2(1), \\
f(\alpha_3(a_2)) &= \alpha_3(a_1 + 1), \\
f(\alpha_4(a_3)) &= \alpha_4(a_2 + 1), \\
&\ \ \vdots \\
f(\alpha_k(a_{k-1})) &= \alpha_k(a_{k-2} + 1), \\
&\ \ \vdots \\
f(x) &= g(x), \text{ otherwise.}
\end{aligned}
$$

The diagram for f is:



Both f and f^{-1} are compact, since $f(A_k)$ and $f^{-1}(A_k)$ are contained in $A_{k-1} \cup A_k \cup A_{k+1}$. Further, because on each A_k f and h agree at all but at most two points, f is $\mathfrak{A}$-$\mathfrak{A}$-rec. Thus f is an embedding. Let $x_0 = (0, 0)$, $S = \{f^n(x_0)\}_{n=0}^{\infty}$. Then, defining B as $\alpha_0^{-1}(S)$, $B = A \cup \{a + 1 | a \in A\}$ and so is not r.e. In addition, for each k, $\alpha_k^{-1}(S)$ almost equals B. More precisely, $\chi_{\alpha_k^{-1}(S)}(x) = \chi_B(x)$ for all $x \geq a_{k-1}$. Thus $\alpha_k^{-1}(S)$ is non-r.e. for each k.

NOTE

1. The author wishes to thank the referee for calling this parallel to his attention.

REFERENCES

[1] Friedman, H., cited (with construction outlined) in [3].

[2] Harkleroad, L., "Recursive equivalence types on recursive manifolds," *Notre Dame Journal of Formal Logic*, vol. 20, no. 1 (1979), pp. 1-31.

[3] Strong, H. R., "Construction of models for algebraically generalized recursive function theory," *The Journal of Symbolic Logic*, vol. 35, no. 3 (1970), pp. 401-409.

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