Notre Dame Journal of Formal Logic
Volume 27, Number 1, January 1986

A Linearly Ordered Topological Space that is Not Normal

MELVEN KROM*

An old question of Birkhoff asks whether it can be shown that every linearly ordered topological space is normal, without using the axiom of choice ([1], p. 252, and [5]). An example shows that it is not possible.

Let *ZF* be Zermelo-Fraenkel set theory without the axiom of choice; we assume that *ZF* is consistent. Let $\mathfrak{O}$ be the *ordering principle*, the axiom which asserts that every set can be linearly ordered ([2], p. 19, and [4], p. 31). Let $\mathcal{S}$ be the *selection principle*, which says that for every family $\mathfrak{F}$ of sets, each containing at least two elements, there is a function f such that for $X \in \mathfrak{F}$, $f(X)$ is a nonempty proper subset of X ([2], p. 53, and [6], p. 207). It is known that there is a model of *ZF* and $\mathfrak{O}$ in which $\mathcal{S}$ is false ([2], p. 95, exercise 11). Assume that $\mathfrak{F} = \{X_\alpha\colon \alpha \in \Omega\}$ is a family of sets that is a counterexample for $\mathcal{S}$ in a model of *ZF* in which $\mathfrak{O}$ is true. We use $\mathfrak{F}$ to form a linearly ordered set such that with its order topology it is not normal. For simplicity we assume that the members of $\mathfrak{F}$ are pairwise disjoint, otherwise we could use a family obtained from $\mathfrak{F}$ by replacing each X_α with the Cartesian product $X_\alpha \times \{\alpha\}$. Let ω be the set of nonnegative integers, let ω^* be the set of negative integers, and let $L = \bigcup_{\alpha \in \Omega} (\omega^* \times \{\alpha\} \cup X_\alpha \cup \omega \times \{\alpha\})$. The construction of L does not require the axiom of choice (see [3], Chap. one, and particularly the bottom of p. 15). Using the principle $\mathfrak{O}$ we assume Ω is linearly ordered and for each $\alpha \in \Omega$, X_α is linearly ordered. For $\alpha \in \Omega$ let $\omega^* \times \{\alpha\}$ and $\omega \times \{\alpha\}$ be ordered according to the usual orders of ω^* and ω. Finally we extend the orders of these subsets of L to a linear ordering of L so that for each $\alpha \in \Omega$ the subset $\omega^* \times \{\alpha\} \cup X_\alpha \cup \omega \times \{\alpha\}$ is ordered with the elements of $\omega^* \times \{\alpha\}$ coming just before those in X_α and the elements of $\omega \times \{\alpha\}$ coming just after those in X_α. Between elements of such subsets corresponding to different members of Ω we order

*Thanks are due to the editor and the referee for noting that the result holds in *ZF* and not only in set theory with atoms.

Received October 10, 1984; revised January 7, 1985

according to the given linear ordering of Ω. For a definition of the order topology for a linearly ordered set and a proof, using the axiom of choice, that all such spaces are normal, see [7] pp. 66, 67. Let $A = \{(-1, \alpha): \alpha \in \Omega\}$ and $B = \{(0, \alpha): \alpha \in \Omega\}$. Then A and B are two disjoint closed subsets of the topological space consisting of L with the order topology. It is important to note that no elements from the sets X_α need to be specified to show that A and B are closed. For example, open intervals of the form $((-1, \alpha), (n, \alpha))$ and $((-n, \alpha), (-1, \alpha))$ show that A is closed. This space is not normal, otherwise there would be two disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Then for any $\alpha \in \Omega$, $U \cap X_\alpha \neq \phi \neq V \cap X_\alpha$. Thus the function f that assigns the nonempty proper subset $U \cap X_\alpha$ to the set X_α for any X_α in $\mathfrak{F}$ would contradict the assumption that $\mathfrak{F}$ is a counterexample to $\mathcal{S}$. This construction establishes the following theorem.

Theorem *In Zermelo-Fraenkel set theory it is not possible to prove that every linearly ordered topological space is normal without using the axiom of choice (in fact a version of the axiom of choice that is stronger than the ordering principle must be used).*

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Department of Mathematics
University of California, Davis
Davis, California 95616