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Eventual Periodicity and "One-Dimensional" Queries

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Abstract We expand on the automata-like behavior of monadic second order relations investigated by Buchi and Ladner. We present a generalization of their representation theorem and use it to separate the intersection of the classes of monadic existential second order and monadic universal second order queries from the class of one-dimensional inductive queries.

0 Introduction In this article, we compare monadic second order logic to monadic least fixed point logic.

The first notion applies to second order sentences: the dimension of a second order sentence is the maximum arity of its quantified relation variables. Most of the work on this notion of dimension is restricted to one-dimensional ("monadic") second order relations, starting with the automata-like behavior of monadic second order queries described in Buchi [5] and Ladner [17]. These papers extended Ehrenfeucht's [9] pebble game-theoretic characterization of the Fraisse [12] equivalence relation for first order relations to monadic second order relations.

We will use this game in order to compare monadic second order logic with a slightly smaller logic whose relation with monadic second order logic is significant. Recall that on ordered structures, by Fagin [10], existential second order logic corresponds to NPTIME, whereas by Immerman [14] and Vardi [22] PTIME corresponds to a logic called Least Fixed Point (LFP) by computer scientists (see Aho & Ullman [3], Chandra & Harel [6], and Immerman [15]) and Positive Elementary Induction by logicians (see Moschovakis [20]).

One of the deepest problems in logic and theoretical computer science is the relationship between LFP and second order logic. First of all, LFP $\subseteq \Pi^1_1$. On $\mathfrak{N} = \langle \omega, +, \times, \uparrow, 0 \rangle$, where "$\uparrow$" refers to exponentiation, LFP $= \Pi^1_1$ (Kleene [16]). On the other hand, over any class of finite structures, LFP is closed under negation (see [15]) and hence over such a class LFP $\subseteq \Delta^1_1$. Since a number of que-

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ries (e.g., "the digraph has an even number of vertices") are known to be in Δ_1^1 but not LFP, over the class of all finite structures we have LFP $\nsubseteq \Delta_1^1$.

We will look at a simpler case of the problem: we will compare *monadic* second order logic (all second order quantifications range over unary relations) to *one*-dimensional LFPs, where "dimension" is the number of "recursion variables" needed for the induction. This notion of dimension in LFP is described in [6], de Rougemont [21], Dublish & Maheshwari [8], etc. Comparisons have already been made between one-dimensional LFPs and monadic second order formulas, mostly using the latter to investigate the former; much of the work seems concentrated on the expression of Transitive Closure–type queries. For example, Fagin [11] proved that connectivity (on finite graphs) is not monadic existential second order, and de Rougemont [21] proved that even on ordered graphs nonconnectivity is not monadic Π_1^1, from which it follows that nonconnectivity is not one-dimensional LFP on finite ordered graphs ([21] also has a small catalogue of basic results on Transitive Closure and Connectivity). Then Ajtai & Fagin [4] used probabilistic methods to prove that the negation of transitive closure (on finite directed graphs) is not monadic existential second order. On the other hand, Kanellakis (see [4]) observed that on finite nondirected graphs the negation of transitive closure is monadic universal second order, whereas McColm showed that the negation of transitive closure on finite nondirected graphs is not one-dimensional positive elementary inductive. This last result is unlike the others in that it distinguishes monadic Π_1^1 from one-dimensional LFP.

We will develop a representation theorem for monadic second order logic on certain "chain-like" classes of digraphs, and we will prove that on such classes of digraphs all monadic second order queries are both monadic existential and monadic universal second order definable; on the other hand, on such classes, there exist monadic second order queries that are not one-dimensional inductive. One consequence will be that there exist queries that are all at once two-dimensional inductive, two-dimensional coinductive, monadic second order, but not one-dimensional inductive—and these will not be too hard to find. In order to do all this, we will extend the results of [5] via [17] to these chain-like graphs. We will also use a surgical method introduced in McColm [18] for "constructing" least fixed points.

1 Second order definitions We will be working on relational structures. (A relational structure can be regarded as a sort of stripped-down relational database.) A *schema* is a tuple $\sigma = (R_1, d_1), (R_2, d_2), \ldots, (R_m, d_m); \{c_1, \ldots, c_n\}$, where each R_i is a relation symbol for d_i-ary relations, and each c_j is a constant symbol. A *relational structure* of schema σ is a tuple

$$\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, \ldots, R_m^{\mathfrak{A}}, c_1^{\mathfrak{A}}, \ldots, c_n^{\mathfrak{A}} \rangle$$

where A is some set, $R_i^{\mathfrak{A}} \subseteq A^{d_i}$ for each i, and $c_j^{\mathfrak{A}} \in A$ for each j. We call $|\mathfrak{A}| = A$ the *domain* of $\mathfrak{A}$, and $R_i^{\mathfrak{A}}$ is the *interpretation* of R_i in $\mathfrak{A}$, while $c_j^{\mathfrak{A}}$ is the *interpretation* of c_j in $\mathfrak{A}$. We will consider the constant symbols as *labels*, and refer to an interpretation of a constant as a *labelled vertex*. We will usually be interested in all structures of a particular schema σ, especially in all finite structures of a particular schema σ. For example, if $\sigma = (\mathrm{Arc}, 2); \varnothing$, then the set of all finite structures of the schema σ is precisely the set of all finite directed graphs with no labelled vertices (we call a directed graph a *digraph*; in a digraph, if there is an arc from vertex x to vertex y, we say that x is the *tail* of the arc, while y is the *head*).

Let $\mathcal{C}$ be a class of structures of a common schema, closed under isomorphism. A *query* on $\mathcal{C}$ is a set $\mathcal{R} \subseteq \mathcal{C}$. Given a logic $\mathcal{L}$, a query $\mathcal{R}$ is $\mathcal{L}$-definable if there exists an $\mathcal{L}$-sentence (viz., an $\mathcal{L}$-formula with no free variables) Θ such that for each $\mathfrak{A} \in \mathcal{C}$, $\mathfrak{A} \in \mathcal{R}$ iff $\mathfrak{A} \models \Theta$, where "$\mathfrak{A} \models \Theta$" means that "$\mathfrak{A}$ satisfies Θ". In the literature, such an $\mathcal{R}$ is often called a *Boolean query*; as is often the case, we will confuse $\mathcal{R}$ and Θ when no confusion would result.

One notational note: When we have a formula over structures of a particular schema σ, it is obvious that the constant symbols of σ may be used in the formula. Sometimes constant symbols may be added to a schema to prevent silly problems with the definitions. For example, in defining Transitive Closure (TC) on (di)graphs, we use the schema (Arc,2); $\{a, b\}$, and investigate the dimension of the (0-ary) query $\mathrm{TC}(a, b)$. That way no recursion variables need be used to remember a and b. (Often, when such constant symbols are added, they are called *parameters* to distinguish them psychologically from any "original" constants.)

We will define the First Order (FO) and Second Order (SO) formulas as usual; an SO formula is *monadic* if all of its second order variables range over unary relations. There is a hierarchy of monadic second order formulas. First, the *monadic existential SO* formulas are of the form

$$\exists S_1 \exists S_2 \cdots \exists S_k\, \theta(\ldots, S_1, S_2, \ldots, S_k),$$

where each variable S_i ranges over unary relations, and θ has no second order quantifications. Let ${}^1\Sigma_1^1$ denote the monadic existential formulas. The *monadic universal SO* formulas, which we denote ${}^1\Pi_1^1$, are of the form

$$\forall S_1 \forall S_2 \cdots \forall S_k\, \theta(\ldots, S_1, S_2, \ldots, S_k),$$

where each S_i is a unary relation variable and θ has no second order quantifications. Note that a relation query is ${}^1\Sigma_1^1$-definable iff its complement is ${}^1\Pi_1^1$-definable. Say that a query is ${}^1\Delta_1^1$-definable iff it is both ${}^1\Sigma_1^1$-definable and ${}^1\Pi_1^1$-definable.

This hierarchy can be built up further: a formula is ${}^1\Sigma_{n+1}^1$-definable iff it is of the form

$$\exists S_1 \exists S_2 \cdots \exists S_k\, \theta(\ldots, S_1, S_2, \ldots, S_k),$$

where $S_1, \ldots, S_k$ are unary relation variables and θ is ${}^1\Pi_n^1$-definable; ${}^1\Pi_{n+1}^1$ is defined similarly, and the intersection of these two is ${}^1\Delta_{n+1}^1$. Note that ${}^1\Pi_n^1, {}^1\Sigma_n^1 \subseteq {}^1\Delta_{n+1}^1$ for all n, and that a query is monadic SO iff it is ${}^1\Sigma_n^1$-definable for some n.

As an example, consider acyclicity on graphs. Acyclicity on finite graphs can be expressed as

$$\forall S\{\exists x S(x)$$
$$\quad \rightarrow \exists x\{S(x)\ \&\ [\forall u \forall v[(S(u)\ \&\ \mathrm{Edge}(u,x)\ \&\ S(v)\ \&\ \mathrm{Edge}(x,v)) \rightarrow u = v]\},$$

and hence acyclicity is universal monadic SO. Nevertheless, as an implicit consequence of [11], acyclicity is not existential monadic SO.

We now massage the results in Chandra et al. [7] to get them into the form that we want them. We will be playing pebble games. The first is a sort of pebble game that lives in the lore (see Moschovakis [19], Aczel [1], and [7]). Recall that a second order sentence is in *prenex form* if it is of the form

$$\Theta \equiv Q_1 S_1 Q_2 S_2 \cdots Q_t S_s Q_{s+1} x_{s+1} \cdots Q_t x_t\, \theta(S_1, \ldots, S_s, x_{s+1}, \ldots, x_t),$$

where each S_i, $i \le s$, is a second order variable ranging over the d_i-ary relations, and each x_j, $j > s$, is a first order variable, and θ is quantifier-free. Suppose that all of the $\neg$ symbols in θ have been pushed down to the atomic level. The game $G(\Theta)$, played on a structure $\mathfrak{A}$, works as follows.

There are two players, whom we shall call Eloise and Abelard following a recent text on such things. On the i-th move, $i \le s$, if Q_i is existential, Eloise chooses a d_i-ary relation $S_i \subseteq |\mathfrak{A}|^{d_i}$; if Q_i is universal, Abelard chooses S_i. On the j-th move, $s < j \le t$, if Q_j is existential, Eloise chooses an element $x_i \in |\mathfrak{A}|$; if Q_j is universal, Abelard chooses x_i. Finally, they reach $\theta(\mathbf{x}, \mathbf{S})$, where $\mathbf{x}$ is the tuple of chosen vertices and $\mathbf{S}$ is the tuple of chosen unary relations. The game continues as follows. If $\theta \equiv \varphi \vee \psi$, Eloise chooses $\vartheta = \varphi$ or $\vartheta = \psi$, and the game continues as $\vartheta(\mathbf{x}', \mathbf{S})$ (this is a disjunctive move) where $\mathbf{x}'$ is a list of some of the variables of $\mathbf{x}$. If $\theta \equiv \varphi \,\&\, \psi$, Abelard chooses $\vartheta = \varphi$ or $\vartheta = \psi$, and the game continues as $\vartheta(\mathbf{x}', \mathbf{S})$ (this is a conjunctive move). The game continues in this way until it reaches an atomic (or negated atomic) subformula ϑ of θ, with tuples $\mathbf{y}, \mathbf{S}$. If $\vartheta(\mathbf{y}) \equiv R(\mathbf{y})$, where R is a relative symbol interpreted by the relation $R^{\mathfrak{A}}$ or a relation variable interpreted by the previously chosen monadic relation $R^{\mathfrak{A}}$, then Elosie wins iff $\mathfrak{A} \models R(\mathbf{y})$; if $\vartheta \equiv \neg R$, then Eloise wins iff $\mathfrak{A} \not\models R(\mathbf{y})$. The ending is similar if $\vartheta(\mathbf{y})$ is $S_i(\mathbf{y})$ or $\neg S_i(\mathbf{y})$. In $G(\Theta)$, we say that a player wins if the game admits a winning strategy for that player. Clearly, Eloise wins on $\mathfrak{A}$ iff $\mathfrak{A} \models \Theta$. Call this first kind of game the Θ-*definition* game.

This game can be used to prove results about *comparison games*, which are more widespread in the literature. Inspired by the back-and-forth partial isomorphism construction of [12], Ehrenfeucht [9] proposed the following *r-comparison* game. Take two structures $\mathfrak{A}$ and $\mathfrak{B}$ of a common schema $\sigma = ((R_1, d_1), \ldots, (R_n, d_n); c_1, \ldots, c_m)$ and two sets of r pebbles, $p_1, \ldots, p_r$ for $\mathfrak{A}$ and $q_1, \ldots, q_r$ for $\mathfrak{B}$. There are two players, whom we call the Spoiler and the Duplicator after a recent paper on this sort of thing. There will be r pairs of moves, the i-th pair of moves consisting of the Spoiler choosing a structure, and placing the i-th pebble of that structure on some element of the structure, and the Duplicator responding by placing the i-th pebble of the other structure on an element of the other structure. In the end, $a_1, \ldots, a_r \in |\mathfrak{A}|$ and $b_1, \ldots, b_r \in |\mathfrak{B}|$ are pebbled. The Duplicator wins iff the map $c_i^{\mathfrak{A}} \mapsto c_i^{\mathfrak{B}}$, $a_i \mapsto b_i$ defines an isomorphism between the restrictions $\mathfrak{A} \upharpoonright A' = \langle A', R_1^{\mathfrak{A}} \upharpoonright A', \ldots \rangle$ and $\mathfrak{B} \upharpoonright B' = \langle B', R_1^{\mathfrak{B}} \upharpoonright A', \ldots \rangle$, where $A' = \{a_1, \ldots, a_r, c_1^{\mathfrak{A}}, \ldots, c_m^{\mathfrak{A}}\}$ and $B' = \{b_1, \ldots, b_r, c_1^{\mathfrak{B}}, \ldots, c_m^{\mathfrak{B}}\}$. If the Duplicator has a winning strategy, write

$$\mathfrak{A} \equiv_r \mathfrak{B}.$$

This game is associated with the notion of the *quantifier depth* of a (FO) formula. First of all, if θ is an atomic formula, then $\operatorname{depth}(\theta) = 0$. By induction, define $\operatorname{depth}(\theta \,\&\, \psi) = \operatorname{depth}(\theta \vee \psi) = \max\{\operatorname{depth}(\theta), \operatorname{depth}(\psi)\}$, $\operatorname{depth}(\neg\theta) = \operatorname{depth}(\theta)$, and $\operatorname{depth}(\exists x\theta) = \operatorname{depth}(\forall x\theta) = \operatorname{depth}(\theta) + 1$. We get:

Theorem 1.1 (see [9])

(i) *For each* r, $\equiv_r$ *is an equivalence relation with finitely many FO-definable equivalence classes.* (*Easily,* $r < s \Rightarrow \equiv_s$ *is a refinement of* $\equiv_r$.)

(ii) *If* $\mathfrak{A} \equiv_{\text{depth}(\theta)} \mathfrak{B}$ *and* $\mathfrak{A} \models \theta$, *then* $\mathfrak{B} \models \theta$.

In essence, the Spoiler is trying to distinguish between the two structures whereas the Duplicator is trying to demonstrate their similarity.

Many variations of this comparison game have been developed, especially a game developed in [11] for monadic SO logic. Here the game starts with a pair of sets of s crayons and a pair of sets of r pebbles for two structures $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, \ldots, c_1^{\mathfrak{A}}, \ldots \rangle$ and $\mathfrak{B} = \langle B, R_1^{\mathfrak{B}}, \ldots, c_1^{\mathfrak{B}}, \ldots \rangle$. The i-th pair of moves, $i \le s$, consists of the Spoiler choosing a structure and using the i-th crayon of that structure to color some of the elements of that structure; the Duplicator responds by using the i-th crayon to color some of the elements of the other structure (we permit an element to have several colors). The j-th pair of moves, $j > s$, consists of the Spoiler pebbling an element and the Duplicator responding likewise as above. In the end, we get s pairs of unary relations, $S_1^{\mathfrak{A}}, \ldots, S_s^{\mathfrak{A}}$ on $\mathfrak{A}$ and $S_1^{\mathfrak{B}}, \ldots, S_r^{\mathfrak{B}}$ on $\mathfrak{B}$, and the usual r pairs of elements: $a_1, \ldots, a_r \in |\mathfrak{A}|$ and $b_1, \ldots, b_r \in |\mathfrak{B}|$. Once again, the Duplicator wins if the correspondence $a_i \mapsto b_i$, $c_j^{\mathfrak{A}} \to c_j^{\mathfrak{B}}$ is a partial isomorphism with respect to $R_1, \ldots$, and $S_1, \ldots$. If the Duplicator wins the (s,r)-comparison game of $\mathfrak{A}$ and $\mathfrak{B}$, write

$$\mathfrak{A} \equiv_{s,r} \mathfrak{B}.$$

We get:

Theorem 1.2 (see [11], [17])

(i) *For each* s, r, $\equiv_{s,r}$ *is an equivalence relation with finitely many SO-definable equivalence classes.*

(ii) *If* depth$(\theta) \le r$ *and* $\mathbf{S}$ *is a list of no more than* s *unary relation variables, and if* $\mathbf{Q}$ *is a list of* (SO) *quantifications, then: if* $\mathfrak{A} \equiv_{s,r} \mathfrak{B}$ *and* $\mathfrak{A} \models \mathbf{QS}\ \theta(\mathbf{S})$, *then* $\mathfrak{B} \models \mathbf{QS}\ \theta(\mathbf{S})$.

The above machinery is used to prove the following result lurking under the surface of [5] and is brought out explicitly in [17]. Fix a schema $\sigma = (\text{Arc},2), (R_1,1), \ldots, (R_m,1);\ \varnothing$. We will be using the following definition a lot.

Definition A *directed chain* is a digraph of the form $\langle \{1,2,\ldots,n\}, \{(i,i+1) : 1 \le i < n\} \rangle$, where 1 is the *tail* and n is the *head*. A *nondirected chain* is an acyclic connected graph with no vertices of degree greater than 2.

Let Σ^+ be the class of all σ-structures where *Arc* defines a finite (directed) chain. Note that Σ^+ corresponds with all the finite strings of 2^m symbols as follows. Each string is actually an m-tuple of 0s and 1s, and the string $\iota_1 \cdots \iota_m$ applies to element $x \in |\mathfrak{A}|$ iff for each i, $\iota_i = \textbf{if } R_i^{\mathfrak{A}}(x) \textbf{ then } 1 \textbf{ else } 0$. Viewing the elements of Σ^+ as strings, we can define the *regular* subsets of Σ^+ as those defined by regular sets of strings, viz., by deterministic finite automata. A version of Theorem 1.2 is used to prove the following:

Theorem 1.3 (cf. [5] via [17]) *A set* $\mathfrak{C} \subseteq \Sigma^+$ *is* Σ*-regular iff there exists a monadic second order* ϕ *such that for each* $\mathfrak{A} \in \Sigma^+$, $\mathfrak{A} \in \mathfrak{C}$ *iff* $\mathfrak{A} \models \phi$.

The proof of this theorem actually establishes that the (finitely many) equivalence classes of $\equiv_{s,r}$ themselves correspond to Σ-regular sets.

2 Configurations We now generalize Theorems 1.2 and 1.3 to classes of graphs that consist of many chains glued together. In this section, we prove a representation theorem about monadic SO queries over such classes of graphs. We first construct representations of these classes of graphs. From graph theory we use the notion of a "multidigraph": a *multidigraph* is a digraph-like object with vertices and arcs from vertices to vertices where we allow several arcs to run from the same vertex to the same vertex (a pair of arcs from the same vertex to the same vertex is often called a *lune*); we also allow an arc to run from a vertex back to the same vertex—this is called a *loop* (a vertex is permitted to sport any number of loops). To represent multidigraphs in finite model theory, we can use the following: recalling that a structure is *two-sorted* if it has two mutually disjoint universes, we say that a *multidigraph* is a two-sorted structure $\mathfrak{M} = \langle V, A, H, T \rangle$, where V is the set of *vertices* of $\mathfrak{M}$, A is the set of *arcs* of $\mathfrak{M}$, and $H, T \subseteq V \times A$ are the relations

$$H(v,a) \equiv \text{“}v \text{ is the head of the arc } a\text{”},$$

$$T(v,a) \equiv \text{“}v \text{ is the tail of the arc } a\text{”},$$

where $\mathfrak{M} \models (\forall a \in A)(\exists ! v \in V)H(v,a)$ & $(\forall a \in A)(\exists ! v \in V)T(v,a)$. Say that the *indegree* of a vertex is the number of arcs going in and that the *outdegree* is the number of arcs going out.

Take a digraph $\mathfrak{G}$. A *subdigraph* $\mathfrak{H}$ of $\mathfrak{G}$ is a digraph whose vertices are vertices of $\mathfrak{G}$, such that for every x, y in $\mathfrak{H}$, $\mathfrak{H} \models \mathrm{Arc}(x,y)$ iff $\mathfrak{G} \models \mathrm{Arc}(x,y)$. We defined *subgraphs* of *graphs* similarly.

We will take a single multidigraph and assign to it a class of digraphs as follows. On a digraph $\mathfrak{G}$, a *chain* $\mathfrak{H}$ is a connected subdigraph of $\mathfrak{G}$ consisting of unlabelled vertices of indegree and outdegree 1 in $\mathfrak{G}$. Given a digraph $\mathfrak{G}$, let $\rho_1, \ldots, \rho_n$ be its maximal chains; as before, a maximal chain has a distinguished tail and a distinguished head, being the two "endpoints" of the chain: a vertex of $\mathfrak{G}$ is *distinguished* if either its indegree or its outdegree is not 1. Many technicalities are avoided by not considering infinite digraphs, e.g., digraphs with infinite one-way chains (viz., subdigraphs like $\langle \{1,2,3,\ldots\}, \{(i, i+1) : 1 \leq i\} \rangle$), or digraphs with arcs from one distinguished vertex to another.

We have a digraph $\mathfrak{G}$ with maximal chains $\rho_1, \ldots, \rho_n$ and distinguished vertices $v_1, \ldots, v_m$. The *configuration* $\mathcal{C}$ for $\mathfrak{G}$ is a multidigraph constructed as follows. The distinguished vertices of $\mathfrak{G}$ are the vertices (V) of $\mathcal{C}$. As for the arcs (A) of $\mathcal{C}$, given a maximal chain ρ of $\mathfrak{G}$, let E_ρ be an arc in $\mathcal{C}$ from the distinguished tail of ρ to the distinguished head. Note that the configuration for $\mathfrak{G}$ will have no nonisolated unlabelled nodes of indegree and outdegree 1. Hence if $\mathcal{C}$ has a nonisolated undistinguished node, it is not a configuration of any digraph; we will ignore such multidigraphs. Note that this definition flops if $\mathfrak{G}$ has any cyclic components (a cyclic component has no distinguished vertices, and hence its image in the configuration is ill-defined); for most of this paper, we will stick to (di)graphs without cyclic components.

(A minor modification allows us to deal with cyclic components: we can allow our configurations to have isolated nodes with single loops representing cyclic components. Note that such an isolated node w in such a configuration $\mathcal{C}$ does not correspond to any particular vertex v of a digraph $\mathfrak{G}$ of configuration $\mathcal{C}$ because all the nodes of that component are automorphically equivalent. All that one need do is notice that a relation holding for one node in the cycle holds for them all, so, in a sense, w somehow represents all of the nodes in that cycle; the only issue is how large the cycle is.)

If $\mathcal{C} = \langle V, A, H, T\rangle$ is a multidigraph, let $[\mathcal{C}] = \{\mathfrak{G}: \mathfrak{G}$ has $\mathcal{C}$ as a configuration$\}$. Since each digraph $\mathfrak{G}$ has precisely one configuration, for each maximal chain ρ of $\mathfrak{G}$, we can set $f_{\mathfrak{G}}(E_\rho) = |\rho|$, where $|\rho|$ is the number of vertices of ρ, and $f_{\mathfrak{G}}$ defines $\mathfrak{G}$; conversely, for each function $f: A \to \{0,1,2,3,\ldots\}$, there is precisely one structure $\mathfrak{G} \in [\mathcal{C}]$ such that $f = f_{\mathfrak{G}}$.

Incidentally, if $\mathfrak{G} \in [\mathcal{C}]$, then there exists a map π_1 from the nodes of $\mathcal{C}$ onto the distinguished vertices of $\mathfrak{G}$ (and perhaps some undistinguished vertices on cyclic components of $\mathfrak{G}$). If π_0 is an automorphism on $\mathcal{C}$, then $\pi_1 \circ \pi_0$ is a different witness to the fact that $\mathfrak{G} \in [\mathcal{C}]$. In addition, if $\mathfrak{G}$ has cyclic components, there may be many witnesses of $\mathfrak{G} \in [\mathcal{C}]$. For the rest of this paper, when we say that $\mathfrak{G} \in [\mathcal{C}]$ for some $\mathfrak{G}$ and $\mathcal{C}$, we will presume that some witness $\pi: \mathcal{C} \to \mathfrak{G}$ has been fixed.

We will be investigating "$\mathcal{C}$-w.ev.p." queries:

Definition Let $\mathcal{C} = \langle V, A, H, T\rangle$ and $S \subseteq [\mathcal{C}]$. S is $\mathcal{C}$-*weakly eventually periodic* (or just $\mathcal{C}$-w.ev.p.) if, for each $\mathfrak{G} \in [\mathcal{C}]$ and each $a \in A$, there exist N and τ such that for any $\mathfrak{G}_1, \mathfrak{G}_2 \in [\mathcal{C}]$, if $f_{\mathfrak{G}_1}(a') = f_{\mathfrak{G}_2}(a') = f_{\mathfrak{G}}(a')$ for each $a' \in A - \{a\}$, and if $N < f_{\mathfrak{G}_1}(a) = f_{\mathfrak{G}_2}(a) - \tau$, then $\mathfrak{G}_1 \in S$ iff $\mathfrak{G}_2 \in S$. If $\mathcal{C} = \langle \{x,y\}, \{a\}, \{(x,a)\}, \{(y,a)\}\rangle$ or $\mathcal{C} = \langle \{x,y\}, \{a,b\}, \{(x,a),(y,b)\}, \{(x,b),(y,a)\}\rangle$, and S is $\mathcal{C}$-w.ev.p., we say that S is *weakly eventually periodic*.

Given $\mathfrak{G} \in S$, the minimal N that works for each arc a of $\mathcal{C}$ is the *outset* of $\mathfrak{G}$ in S, and the minimal τ that works for each arc a of $\mathcal{C}$ is the *period* of $\mathfrak{G}$ in S.

A $\mathcal{C}$-w.ev.p. query S thus "eventually" becomes periodic in the sense that for any $\mathfrak{G}$, when we extend $\mathfrak{G}$ by lengthening its chain by adding new vertices one at a time to get a sequence $\mathfrak{G}_1, \mathfrak{G}_2, \ldots$, eventually we will reach a period of length τ such that $\mathfrak{G}_t \in S$ iff $\mathfrak{G}_{t+\tau} \in S$.

We now whip out the Buchi–Ladner theorem (Theorem 1.3) to get the following lemma. Note that by the world-famous Pumping Lemma of automata theory (see, e.g., Hopcroft and Ullman [13]), a set S of strings of *one* symbol a is regular iff $\{|a^n| : a^n \in S\}$ is weakly eventually periodic.

Lemma 2.1 *Fix a configuration $\mathcal{C}$. Every monadic second order query on $[\mathcal{C}]$ is $\mathcal{C}$-w.ev.p.*

Proof: Let $\mathcal{C} = \langle V, A, H, T\rangle$, and fix $\mathfrak{G} \in [\mathcal{C}]$ and $a \in A$. Let $\mathfrak{G}_n$ be the graph associated with the function

$$f(a') = \text{if } a = a' \text{ then } n \text{ else } f_{\mathfrak{G}}(a')$$

for each $n > 0$. Let $C \subseteq |\mathfrak{G}|$ be the maximal chain associated with the arc a, and for any SO Θ, let $G_\Theta = \{n : \mathfrak{G}_n \models \Theta\}$. Let d_1 be the distinguished tail and d_2 the distinguished head of C, and let $D = |\mathfrak{G}| - C = \{d_1, \ldots, d_m\}$ list the vertices outside the chain.

Now let $\mathfrak{C}_n$ be the digraph consisting of a directed chain of n vertices; we ignore the silly case $n = 0$. By Theorem 1.3, if Ψ is monadic SO and $G^{\Psi} = \{n : \mathfrak{C}_n \models \Psi\}$, then G^{Ψ} is weakly eventually periodic. If we can prove that for each monadic SO Θ on $[C]$, there exists monadic SO Ψ such that $G_{\Theta} = G^{\Psi}$, we will be done. The idea is to have Ψ be the same as Θ, as far as Abelard–Eloise games go, except that the moves made on vertices outside of C are made on "virtual vertices", i.e., they are not in the structure but are simulated by logical connectives.

From Θ, we construct Ψ using Abelard–Eloise games. But first we set some groundwork. In the Abelard–Eloise game of Ψ on $\mathfrak{C}_n$, the game is the same as that of Θ on $\mathfrak{G}_n$ except that if Abelard or Eloise choose a vertex to pebble, then either the pebble will be in $C = C_n$ (and the pebbling will take place as before), or the pebbler will pebble some d_i, which will correspond to a conjunctive or disjunctive move setting "pebble := d_i". Similarly, when (unary) relations are to be played, the player will color the vertices of C and also choose, via a conjunctive or disjunctive move, which vertices d_i should be colored as well. Several new relations now appear:

$$\mathfrak{C}_n \models \mathrm{Arc}_j^+(x) \equiv \mathfrak{G}_n \models \mathrm{Arc}(d_j, x),$$

$$\mathfrak{C}_n \models \mathrm{Arc}_j^-(x) \equiv \mathfrak{G}_n \models \mathrm{Arc}(x, d_j),$$

$$\mathfrak{C}_n \models \mathrm{Arc}_{ij}(\ ) \equiv \mathfrak{G}_n \models \mathrm{Arc}(d_i, d_j),$$

and for the relation variable S_i,

$$\mathfrak{C}_n \models S_{ij}(\ ) \equiv \mathfrak{G}_n \models S_i(d_j).$$

The idea will be to "replace" Θ, piece by piece, with another formula Ψ. We will actually replace Θ with an expression containing the subformulas of Θ, replace each subformula with an expression containing *its* subformulas, and so on down. When the replacing is done, we will replace the "atomic" expressions Arc_j^+, Arc_j^-, Arc_{ij}, and S_{ij} with formulas, and the result will be Ψ. For simplicity, suppose that

$$\Theta \equiv Q_1 S_1 \cdots Q_s S_s\, Q_{s+1} x_1 \cdots Q_{s+r} x_r\, \theta(\mathbf{x}, \mathbf{S}),$$

where θ is quantifier-free. Replacing Θ with some Ψ works as follows.

First, if $\Theta \equiv \exists S_i\ \vartheta(S_i)$ for some SO formula ϑ, Eloise will choose some $S_1 \subseteq C$ and $K_1 \subseteq \{1, \dots, m\} = [m]$ such that $S_1^{\mathfrak{G}} = S_1^{\mathfrak{C}} \cup \{d_j : j \in K_1\}$. Hence this SO quantification move consists of a coloring of points in C and a choice of K_1. Replace $\exists S_1\ \vartheta(S_1)$ with

$$\exists S_1 \bigvee_{K_1 \subseteq [m]} \varphi(S_1; K_1),$$

where $\varphi(S_1; K_1) \equiv \vartheta$, where K_1 is noted for future reference. Similarly, if $\Theta \equiv \forall S_1\ \vartheta(S_1)$, replace it with

$$\forall S_1 \bigwedge_{K_1 \subseteq [m]} \varphi(S_1; K_1).$$

Continue in the same vein down the SO quantifications. For example, if $\Theta_i(\mathbf{S};\mathbf{K}) \equiv \exists S_i\ \vartheta_i(\mathbf{S}, S_i)$, replace it with

$$\exists S_i \bigvee_{K_i \subseteq [m]} \varphi_i(\mathbf{S}, S_i; \mathbf{K}, K_i),$$

where $\varphi_i(\mathbf{S}, S_i; \mathbf{K}, K_i) \equiv \vartheta_i$, where K_i is noted for future reference.

Now for the sequence of FO quantifications. Given θ, let $\theta_{k\to j}$ be θ with each occurrence of x_k replaced by d_j, viz., $\mathrm{Arc}(x_k, x_l)$ is replaced by $\mathrm{Arc}_j^+(x_l)$, each occurrence of $\mathrm{Arc}(x_l, x_k)$ is replaced by $\mathrm{Arc}_j^-(x_l)$, each occurrence of $\mathrm{Arc}_l^+(x_k)$ is replaced by Arc_{lj}, each occurrence of $\mathrm{Arc}_l^-(x_k)$ is replaced by Arc_{jl}, and each occurrence of $S_i(x_k)$ is replaced by S_{ij}. If $\Theta_j(\mathbf{x},\mathbf{S};\mathbf{K}) \equiv \forall x_j\ \vartheta(\mathbf{x}, x_j, \mathbf{S}; \mathbf{K})$, replace it with

$$\forall x_k\ \vartheta(\mathbf{x}, x_k, \mathbf{S}; \mathbf{K}) \wedge \bigwedge_{j=1}^{m} \vartheta_{k\to j}(\mathbf{x}, \mathbf{S}; \mathbf{K}).$$

Similarly, if $\Theta_i(\mathbf{x},\mathbf{S}) \equiv \exists x_j\ \vartheta(\mathbf{x}, x_j, \mathbf{S}; \mathbf{K})$, replace it with

$$\exists x_k\ \vartheta(\mathbf{x}, x_k, \mathbf{S}; \mathbf{K}) \vee \bigvee_{j=1}^{m} \vartheta_{k\to j}(\mathbf{x}, \mathbf{S}; \mathbf{K}).$$

Continue until all quantifications are gone and you have a massive expression ϑ, which is actually an SO sentence over the class of finite chains except that it also contains Arc_j^+, Arc_j^-, Arc_{ij}, and S_{ij} for each i, j, as unary or 0-ary predicates.

As d_1 is the distinguished tail of C, we write

$$\mathrm{Arc}_1^-(x) \equiv \forall y\ \neg\mathrm{Arc}(y, x) \quad \text{and} \quad \mathrm{Arc}_1^+(x) \equiv \text{FALSE}.$$

Similarly, as d_2 is the distinguished head of C, write

$$\mathrm{Arc}_2^-(x) \equiv \text{FALSE} \quad \text{and} \quad \mathrm{Arc}_2^+(x) \equiv \forall y\ \neg\mathrm{Arc}(x, y).$$

And if $i > 2$, then $\mathrm{Arc}_i^+(x) \leftrightarrow \mathrm{Arc}_i^-(x) \leftrightarrow \text{FALSE}$. As Arc_{ij} is fixed throughout $\{\mathfrak{G}_n : n \in \omega\}$ for each fixed i, j, let $\mathrm{Arc}_{ij} \equiv \mathrm{Arc}(d_i, d_j)$, which is TRUE or FALSE, depending on d_i, d_j. Finally, the symbol S_{ij} is replaced with TRUE if it lies within an expression $\vartheta(-;-, K_i)$, where $j \in K_i$, and FALSE if it lies within an expression $\vartheta(-;-, K_i)$, where $j \notin K_i$. The result of all this replacing is an SO formula Ψ designed for chain structures.

It is straightforward, if tedious, to verify that Eloise wins the game for Θ on $\mathfrak{G}_n$ iff she wins for Ψ on $\mathfrak{C}_n$. It follows that $\mathfrak{G}_n \models \Theta$ iff $\mathfrak{C}_n \models \Psi$ for each n, and the lemma follows.

We now strengthen the notion of weakly eventually periodic.

Definition Let $\mathcal{C} = \langle V, A, H, T\rangle$ and $S \subseteq [\mathcal{C}]$. S is *$\mathcal{C}$-eventually periodic* (or just $\mathcal{C}$-ev.p.) if there exist N, τ such that for each $\mathfrak{G} \in [\mathcal{C}]$ and each $a \in A$, for any $\mathfrak{G}_1, \mathfrak{G}_2, \in [\mathcal{C}]$, if $f_{\mathfrak{G}_1}(a') = f_{\mathfrak{G}_2}(a') = f_{\mathfrak{G}}(a')$ for each $a' \in A - \{a\}$, and if $N < f_{\mathfrak{G}_1}(a) = f_{\mathfrak{G}_2}(a) - \tau$, then $\mathfrak{G}_1 \in S$ iff $\mathfrak{G}_2 \in S$. If $\mathcal{C} = \langle\{x, y\}, \{a\}, \{(x, a)\}, \{(y, a)\}\rangle$ or $\mathcal{C} = \langle\{x, y\}, \{a, b\}, \{(x, a), (y, b)\}, \{(x, b), (y, a)\}\rangle$, and S is $\mathcal{C}$-ev.p., we say that S is *eventually periodic*.

Notice the distinction: if S is C-ev.p., then there exists a single pair N, τ witnessing S being C-ev.p., whereas if S is merely C-w.ev.p., then for each $\mathfrak{G}$, there is a pair N, τ witnessing S being C-w.ev.p. in that case.

Lemma 2.2 *Fix a configuration C. Every monadic second-order query on $[C]$ is C-ev.p.*

Proof: Towards contradiction, suppose that

$$\Theta \equiv Q_1 S_1 \cdots Q_s S_s Q_{s+1} x_1 \cdots Q_{s+r} x_r\, \theta(\mathbf{x}, \mathbf{S})$$

has θ being quantifier-free, and that there exist $\mathfrak{G}_1, \mathfrak{G}_2, \ldots \in [C]$ such that if N_t, τ_t are the outset and period of $\mathfrak{G}_t$ respectively (and each N_t, τ_t exist by Lemma 2.1), then $N_t \leq N_{t+1}$ and $\tau_t \leq \tau_{t+1}$ for each t, and either $\lim_{t\to\infty} N_t = \infty$ or $\lim_{t\to\infty} \tau_t = \infty$.

Let $\mathfrak{G} \leq \mathfrak{G}'$ mean that for each arc a, $f_{\mathfrak{G}}(a) \leq f_{\mathfrak{G}'}(a)$, and using a thinning argument we can find a subsequence $\{\mathfrak{G}_{t_i}\}_{i\in\omega}$ such that $f_{\mathfrak{G}_{t_i}}(a) \leq f_{\mathfrak{G}_{t_{i+1}}}(a)$ for all a, i. In fact, without loss of generality we can suppose that $\{\mathfrak{G}_t\}_{t\in\omega}$ satisfies $f_{\mathfrak{G}_t}(a) \leq f_{\mathfrak{G}_{t+1}}(a)$ for all a, t. Let $l(a) = \lim_{t\to\infty} f_{\mathfrak{G}_t}(a)$ for each arc a, and let $M > l(a)$ for each finite $l(a)$ and let T be such that if $t \geq T$, then for each arc a, $l(a) < \infty \Rightarrow f_{\mathfrak{G}_t}(a) = l(a)$ and $l(a) = \infty \Rightarrow f_{\mathfrak{G}_t}(a) > M$.

Recall from the comment after Theorem 1.3 that each of the finitely many equivalence classes of $\equiv_{s,r}$ on finite chains is eventually periodic, and let N' be the maximum outset of any of these classes, and τ' the least common multiple of all the periods. Choose any $\mathfrak{G} = \mathfrak{G}_t$, $t \geq T$, and let

$$\begin{aligned} f_{(\mathfrak{G},a)}(a') = {} & \textbf{if } a = a' \;\&\; f_{\mathfrak{G}}(a) > N_T + N' + \tau_T\tau' \\ & \textbf{then } f_{\mathfrak{G}}(a) - \tau_T\tau' \\ & \textbf{else } f_{\mathfrak{G}}(a'), \end{aligned}$$

and let $[\mathfrak{G}; a]$ be the resulting digraph. Now, a $\equiv_{s,r}$ game, with all moves restricted to any chain of $\mathfrak{G}$ and the corresponding chain of $[\mathfrak{G}; a]$, gives the Duplicator a winning strategy. Hence, as all the SO moves are for unary relations, $\mathfrak{G} \equiv_{s,r} [\mathfrak{G}; a]$, and repeating for each maximal chain of $\mathfrak{G}$, it follows that $\mathfrak{G}$ and $[\mathfrak{G}; a]$ have the same outset and period. Choose another configuration arc a', and the same is true of $[\mathfrak{G}; a]$ and $[[\mathfrak{G}; a]; a']$, and so on. Repeating as often as possible, we find that the outset and period of $\mathfrak{G}_t$ is the same as that of some $\mathfrak{G} \in [C]$ all of whose chains are no longer than $N' + N_T + \tau_T\tau'$. Since this is true of all $\mathfrak{G}_t$, $t \geq T$, $\{N_t : t \geq T\}$ and $\{\tau_t : t \geq T\}$ are both finite, giving us our contradiction.

We now need a characterization of the C-ev.p. functions $f_{\mathfrak{G}}$.

Definition Let $S \subseteq \omega$. Recall that S is an *arithmetic progression* if there exist $N, \tau \in \omega$ such that $S = \{N + k\tau : k \in \omega\}$. Call N the *outset* and τ the *period* of S. Say that $S \subseteq \omega$ is *rational* if it is a finite union of arithmetic progressions. If $\mathrm{Arith}(N, \tau) = \{N + k\tau : k \in \omega\}$, say that $S = \bigcup_{k\leq m} \mathrm{Arith}(N_k, \tau_k)$ is of *pattern*

$$\{(s_1, \ldots, s_m) \in S : (\forall k \leq m)(s_k \leq N_k + \tau_k)\},$$

and call $\max_x N_k$ the *outset* and the least common multiple $\mathrm{lcm}_k \tau_k$ the *period* of S.

We say that $S \subseteq \omega^d$ is *rational* if there exists M and rational sets $J(i,j) \subseteq \omega$, $i \leq d$ and $j \leq M$, such that S is the union of Cartesian products $S = \bigcup\{\Pi_{i\leq d} J(i,j) : j \leq M\}$.

On the other hand, say that $S \subseteq \omega^d$ is *eventually periodic* if there exist N, τ such that for each $k \leq d$, if $n_k \geq N$, then

$$(n_1, \ldots, n_{k-1}, n_k, n_{k+1}, \ldots, n_d) \in S$$
$$\text{iff } (n_1, \ldots, n_{k-1}, n_k + \tau, n_{k+1}, \ldots, n_d) \in S.$$

Lemma 2.3 *A subset of ω^d is eventually periodic iff it is rational.*

Proof: First, suppose that $S \subseteq \omega^d$ is rational:

$$S = \bigcup\{\Pi_{i\leq d} J(i,j) : j \leq M\}.$$

For each $i,j \leq M$, $J(i,j)$ has outset $N(i,j)$ and period $\tau(i,j)$, and if N is the maximum $N(i,j)$ and τ is the least common multiple of all the $\tau(i,j)$, then N and τ witness the eventual periodicity of S.

Conversely, suppose that S is eventually periodic, with outset N and period τ. For each tuple $\mathbf{k} = (k_1, \ldots, k_d) \in \{1,2,\ldots,N+\tau\}^d$, and each $i \leq d$, let $J(i,\mathbf{k}) = \textbf{if } k_i \leq N \textbf{ then } \{k_i\} \textbf{ else } \{k_i\} \cup \{k_i + t\tau : t \in \omega\}$, and $J(i,\mathbf{k})$ is rational. Easily,

$$S = \bigcup\{\Pi_{i\leq d} J(i,\mathbf{k}) : \mathbf{k} \in S \cap \{1, \ldots, N+\tau\}^d\}.$$

It follows that if $\mathcal{C}$ is a configuration, all monadic SO queries are "rational" in the following sense.

Definition Fix a configuration $\mathcal{C}$. For any $\mathfrak{G} \in [\mathcal{C}]$, let $f_{\mathfrak{G}}$ be a function from the arcs of $\mathcal{C}$ to ω such that for any arc a, $f_{\mathfrak{G}}(a)$ is the number of vertices in the chain in $\mathfrak{G}$ corresponding to the arc a. Let $a_1, \ldots, a_d$ be the arcs of $\mathcal{C}$, and let $F(\mathfrak{G})$ be the tuple $(f_{\mathfrak{G}}(a_1), \ldots, f_{\mathfrak{G}}(a_d))$. For any set $S \subseteq [\mathcal{C}]$, let

$$F(S) = \{F(\mathfrak{G}) : \mathfrak{G} \in S\}.$$

Call a set $S \subseteq [\mathcal{C}]$ *$\mathcal{C}$-rational* if $F(S)$ is rational (in ω^d).

Lemma 2.4 *Let $\mathcal{C}$ be a configuration. Then every $\mathcal{C}$-rational query is ${}^1\Delta^1_1$-definable.*

Proof: Let $\mathcal{C} = \langle C, A, H, T\rangle$. As the class of $\mathcal{C}$-rational queries is closed under negation, it suffices to prove that every $\mathcal{C}$-rational query is ${}^1\Sigma^1_1$-definable.

The first step is to verify that every rational class of finite chains is ${}^1\Sigma^1_1$-definable. An arithmetic progression of chains, with distinguished tail t_0 and distinguished head h_0, and with outset N and period τ is ${}^1\Sigma^1_1$-definable using

$$\exists S\{(\forall s \in S)(\text{dist}(t_0, s) \geq N)\ \&$$
$$(\exists s \in S)(\text{dist}(t_0, s) = N)\ \&$$
$$(\forall s \in S - \{h_0\})(\text{“next } s' \in S \text{ up from } s \text{ is distance } \tau\text{”})\ \&\ S(h_0)\}.$$

As an eventually periodic set is a finite union of arithmetic sets, eventually periodic sets are ${}^1\Sigma^1_1$-definable.

Now, if S is $\mathcal{C}$-rational, it corresponds with some $\omega^{|A|}$-rational S. In ${}^1\Sigma^1_1$, given distinguished x, y, we can assert the existence of chains from x to y of lengths decreed by S. Hence we write, if the i-th arc is the one from $x_{t(i)}$ to $x_{h(i)}$,

$$\exists x_1 \cdots x_{|C|} \exists S_1 \cdots S_{|A|} \{\text{"There is a pattern for a } C \text{ rational set s.t. } S_1, \ldots, S_{|A|} \text{ is of that pattern"}\}.$$

This is ${}^1\Sigma_1^1$-definable, and we are done.

Theorem 2.5 *On* $[C]$, *a query is* C*-rational iff it is* C-ev.p. *iff it is monadic SO iff it is* ${}^1\Delta_1^1$*-definable.*

Hence on $[C]$, all monadic second order queries are ${}^1\Delta_1^1$-definable. One cyclic comment: When Fagin [11] proved that Connectivity is not ${}^1\Sigma_1^1$-definable, his proof actually achieved the following. Let $\mathfrak{G}$ be a digraph and let $[\mathfrak{G}, n]$ consist of all digraphs consisting of $\mathfrak{G}$ with n additional cyclic components.

Theorem 2.6 (Essentially, [11]) *Let* C *be a configuration, and fix* $n \geq 1$. *If* R *is a* ${}^1\Sigma_1^1$*-definable class of structures, and if* $R \cap [C]$ *is infinite, then for some* $\mathfrak{G} \in [C]$, $R \cap [\mathfrak{G}, n]$ *is also infinite.*

Given a configuration C, let

$$(C) = \bigcup_{\mathfrak{G} \in [C]} \left(\bigcup_{n \in \omega} [\mathfrak{G}, n] \right).$$

As far as ${}^1\Delta_1^1$ is concerned, cyclic components are invisible within any (C), so we could ignore them and restrict our attention to $[C]$. Nevertheless, in ${}^1\Sigma_1^1$ the existence of cyclic components can be asserted, whereas in ${}^1\Pi_1^1$ the existence of cyclic components can be denied. It follows that on any (C), all monadic second order relations are ${}^1\Delta_2^1$-definable.

Clearly, all of the above can be applied to (nondirected) graphs.

3 On positive elementary induction For logistical reasons, we will be using the formulation of LFP that appears in [20].

We start with the *positive SO formulas*. Intuitively speaking, an SO formula φ is *positive* if, for each relation variable in φ, there is no $\neg$ sign anywhere in front of it. The positive SO formulas are constructed as follows. First, if there are no relation variables in φ, then φ is positive SO. If S is a relation variable, then for any list $\mathbf{x}$ of (FO) variables and constants, $S(\mathbf{x})$ is positive SO. If φ and ψ are positive SO, then so are $\varphi \,\&\, \psi$, $\varphi \vee \psi$, $\exists x\varphi$, and $\forall x\psi$. (As a technicality, we will presume that a positive SO formula has no SO quantifications.) It is easy to show that if φ is positive SO, then ν is *monotonic* in the following sense: for each $\mathbf{x}, X, Y, \varphi(\mathbf{x}, X) \,\&\, X \subseteq Y \Rightarrow \varphi(\mathbf{x}, Y)$.

A system of SO formulas

$$\boldsymbol{\varphi} = \varphi_0(\mathbf{u}_0, S_0, \ldots, S_\nu), \ldots, \varphi_\nu(\mathbf{u}_\nu, S_0, \ldots, S_\nu)$$

is *operative* if, for each i, $\mathbf{u}_i$ and S_i have the same number of arguments (we allow 0-ary $\mathbf{u}_i, S_i$). We will be dealing with *positive SO operative systems* of formulas, i.e., operative systems as above where each φ_i is positive SO with relation variables $S_0, \ldots, S_\nu$.

Now suppose we have our positive elementary operative system $\boldsymbol{\varphi} = \varphi_0, \ldots, \varphi_\nu$. Since we will not be dealing with transfinite inductions, we restrict ourselves to the following: for each i and each number n, we get a system of iterates

$$\varphi_i^{n+1}(\mathbf{x}_i) \equiv \varphi_i(\mathbf{x}_i, \varphi_0^n, \ldots, \varphi_\nu^n),$$

and it is easily verified that for each $i, n, \varphi_i^n \subseteq \varphi_i^{n+1}$, and hence, on a finite structure, there exists an n such that $\varphi_i^{n+1} = \varphi_i^n$; we denote these iterates by $\boldsymbol{\varphi}^\infty = \varphi_0^\infty, \ldots, \varphi_\nu^\infty$. A query is a *positive elementary inductive least fixed point* (LFP) if it is one of the relations listed in some least (simultaneous) fixed point of some operative system of positive elementary SO formulas.

As an example, consider connectivity on digraphs. This uses the system

$$\varphi_0(S_1) \equiv \forall x \forall y\, S_1(x, y),$$

$$\varphi_1(x, y, S_1) \equiv x = y \vee \exists z[\mathrm{Arc}(x, z) \;\&\; S_1(z, y)].$$

Easily, for any digraph $\mathfrak{A}$, $\mathfrak{A} \models \forall x \forall y\, [\mathrm{TC}(x, y) \rightarrow \varphi_1^\infty(x, y)]$, and hence $\mathfrak{A} \models \varphi_0^\infty$ iff $\mathfrak{A}$ is connected.

One bit of lore: $(\varphi_0^\infty, \ldots, \varphi_\nu^\infty)$ is the *least* (simultaneous) fixed point of the system $\varphi_0, \ldots, \varphi_\nu$ in the following sense: if $X_0, \ldots, X_\nu$ was a fixed point of $\varphi_0, \ldots, \varphi_\nu$, then $\varphi_i^\infty \subseteq X_i$ for each i. To see this, note that for each $i, \varnothing \subseteq X_i$, and by the monotonicity of $\varphi_i, \varphi_i^{<\xi} \subseteq X_i \Rightarrow \varphi_i^\xi \subseteq X_i$ for all ξ. And so every LFP query is Π_1^1-definable.

We will need two definitions. First, the notion of *dimension*. Let $\boldsymbol{\varphi}$ be a positive SO operative system of formulas $\varphi_i(\mathbf{u}_i, S_0, \ldots, S_\nu)$, and let d_i be the arity of the relation variable S_i and thus the length of the tuple of "recursion variables" $\mathbf{u}_i$. Let $d_{\boldsymbol{\varphi}} = \max(d_0, \ldots, d_\nu)$ be the dimension of the system $\boldsymbol{\varphi}$. If R is a positive elementary inductive relation, let

$$d_R = \min\{d_{\boldsymbol{\varphi}}\colon \text{all } \boldsymbol{\varphi} \text{ such that } R = \varphi_0^\infty\}$$

be the *dimension* of R. Let ${}^d\mathrm{LFP}$ be the class of all d-dimensional LFP queries. Note that ${}^d\mathrm{LFP} \subseteq {}^d\Pi_1^1$.

Second, we can construct a hierarchy from ${}^d\mathrm{LFP}$ as follows. Let ${}^d\mathrm{LFP}_1 = {}^d\mathrm{LFP}$. If $\mathcal{C}$ is a class of structures of a given schema and R is ${}^d\mathrm{LFP}_n$-definable, let $(\mathcal{C}, R) = \{(\mathfrak{A}, R^{\mathfrak{A}}) : \mathfrak{A} \in \mathcal{C}\}$, where $(\mathfrak{A}, R^{\mathfrak{A}})$ is the expansion of the structure $\mathfrak{A}$ by adding the relation $R^{\mathfrak{A}}$. Then a query S is ${}^d\mathrm{LFP}_{n+1}$-definable on $\mathcal{C}$ if there exists a ${}^d\mathrm{LFP}_n$-definable R such that S is ${}^d\mathrm{LFP}$-definable over $(\mathcal{C}, R)$. Let ${}^d\mathrm{LFP}_\omega = \bigcup_{n\in\omega} {}^d\mathrm{LFP}_n$, and ${}^d\mathrm{LFP}_\omega$ is the least class of queries containing all the FO queries and closed under Boolean operations and d-dimensional inductions. Easily ${}^d\mathrm{LFP}_n \subseteq {}^d\Pi_n^1$, but that is all that is immediate. It is known (see Addison & Kleene [2]) that on $\mathfrak{N}$, $\bigcup_d {}^d\mathrm{LFP}_\omega \not\subseteq \Delta_2^1$, but perhaps things are different elsewhere.

4 One-dimensional inductions During the last decade, LFP has been separated from Σ_1^1 and Π_1^1 on classes of finite structures. The standard technique has been to choose some query that looks like it is not in LFP either because it is NP-complete (like $\exists$large clique, as in [14]) or because it is obviously not in $L_{\infty\omega}^{<\omega} = \bigcup_{k\in\omega} L_{\infty\omega}^k$, where $L_{\infty\omega}^k$ is FO logic with only k variables allowed, expanded by arbitrary disjunctions (like the oft-cited example, "there are evenly many vertices", which is implicitly dealt with in [11]). Of course, a wide number of techniques have been used, but these are the two kinds of queries that have been exhibited.

A toy version of this problem is the separation of ${}^1\Pi^1_1$ from ^{1}LFP. This is not always possible. For example, it is easily seen that for any monadic SO Θ, if C_n is the chain of n vertices, then $\{n : C_n \models \Theta\}$ is eventually periodic, and thus all monadic SO sentences are ^{1}LFP-definable on $\{C_n : n \in \omega\}$. The purpose of this section is to prove that this separation holds over the class of all graphs, and in fact it holds for certain highly behaved classes of graphs. We will actually prove something stronger. Recall that ${}^1\mathrm{LFP}_\omega$ is the least class of queries containing ^{1}LFP and closed under Boolean operations, first order quantifications, and one-dimensional inductions. We will prove that for some $\mathcal{C}_1, \mathcal{C}_2$, ${}^1\mathrm{LFP}_\omega \subsetneq {}^1\Delta^1_1$ on $[\mathcal{C}_1] \cup [\mathcal{C}_2]$, from which it will follow that on some $[\mathcal{C}]$, ${}^1\mathrm{LFP}_\omega \subsetneq {}^1\Delta^1_1$. (Note that ${}^1\mathrm{LFP}_\omega \subseteq \bigcup_k {}^1\Pi^1_k$, so that on any $[\mathcal{C}]$, all the ${}^1\mathrm{LFP}_\omega$-queries are $\mathcal{C}$-rational, and thus ${}^1\mathrm{LFP}_\omega \subseteq {}^1\Delta^1_1$.) First, we exhibit a query that ${}^1\Delta^1_1$ can represent.

Proposition 4.1 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be distinct configurations, and let $\mathfrak{C} = [\mathcal{C}_1] \cup [\mathcal{C}_2]$. "$\mathfrak{A} \in [\mathcal{C}_1]$" is ${}^1\Delta^1_1$-definable on $\mathfrak{C}$.*

Outline of proof: The proof is similar to the proof of Theorem 2.5. Let $\mathcal{C}_1$ have nodes $c_1, \ldots, c_n$, and arcs $a_1, \ldots, a_m$, and again, let $H(c, a) \equiv$ "c is the head of a", and $T(c, a) \equiv$ "c is the tail of a". Again, we write a ${}^1\Sigma^1_1$ sentence Θ_1 saying that

$$\exists x_1 \cdots x_n \, \exists S_1 \cdots S_m \left\{ \bigwedge_{i,j,k} [(H(c_i, a_j) \;\&\; T(c_k, a_j)) \to \text{"}S_j \text{ is the chain from distinguished node } x_k \text{ to distinguished node } x_i\text{"}] \;\&\; \forall x \bigvee_j x \in S_j \right\},$$

and for all $\mathfrak{A} \in \mathfrak{C}$, $\mathfrak{A} \models \Theta_1$ iff $\mathfrak{A} \in [\mathcal{C}_1]$. Similarly, we construct a ${}^1\Sigma^1_1$-sentence Θ_2 such that for all $\mathfrak{A} \in \mathfrak{C}$, $\mathfrak{A} \models \Theta_2$ iff $\mathfrak{A} \in [\mathcal{C}_2]$. Since Θ_1 is equivalent to $\neg\Theta_2$ on $\mathfrak{C}$, "$\mathfrak{A} \in [\mathcal{C}_1]$" is ${}^1\Delta^1_1$-definable on $\mathfrak{C}$.

We might as well mention 2-dimensional inductions somewhere in this article.

Proposition 4.2 *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct configurations, then "$\mathfrak{G} \in [\mathcal{C}_1]$" is ^{2}LFP expressible in $[\mathcal{C}_1] \cup [\mathcal{C}_2]$.*

Proof: Without loss of generality, suppose that neither $\mathcal{C}_1$ nor $\mathcal{C}_2$ have constant symbols; we also assume that the arc relation on each configuration is symmetric so we can deal with graphs. If $\mathcal{C}_1$ and $\mathcal{C}_2$ have a different number of distinguished vertices, or if, for some number $d \neq 2$, one has more distinguished vertices of degree d than the other, then certainly "$\mathfrak{G} \in [\mathcal{C}_1]$" is FO-expressible. So imagine that they both have the same number of distinguished vertices, of degrees $d_1 \geq \cdots \geq d_k$. Ordering the distinguished vertices by degree, let $R_l(i, j) \equiv$ "in $\mathcal{C}_l$, there is an arc from vertex i to vertex j".

The following is an operative system generating an LFP query:

$$\varphi_0(S_0,S_1,S_2) \equiv \exists x_1 \cdots x_k \left\{ \bigwedge_i \deg(x_i) = d_i \ \& \right.$$
$$\bigwedge \{S_2(x_i,x_j) : R_1(x_i,x_j)\} \ \&$$
$$\left. \forall x \left[\deg(x) \neq 2 \to \bigvee_i x = x_i \right] \right\}$$
$$\varphi_1(x,y,S_0,S_1,S_2) \equiv \deg(y) = 2 = \deg(x) \ \&$$
$$\{\mathrm{Edge}(x,y) \vee \exists z[\mathrm{Edge}(x,z) \ \& \ S_1(z,y)\}$$
$$\varphi_2(x,y,S_0,S_1,S_2) \equiv \deg(x) \neq 2 \neq \deg(y) \ \&$$
$$\exists x',y'\{\mathrm{Edge}(x,x') \ \& \ S_1(x',y') \ \& \ \mathrm{Edge}(y',y)\}.$$

As φ_0 fixes the degrees of all the distinguished vertices, for all $\mathfrak{G} \in [\mathcal{C}_1] \cup [\mathcal{C}_2]$, $\mathfrak{G} \in [\mathcal{C}_1]$ iff $\mathfrak{G} \models \varphi_0^\infty$.

In fact, if $\neg{}^2\mathrm{LFP} = \{\neg R : R \in {}^2\mathrm{LFP}\}$, then "$\mathfrak{G} \in [\mathfrak{C}_1]$" is $({}^2\mathrm{LFP} \cap \neg{}^2\mathrm{LFP})$-expressible in $[\mathcal{C}_1] \cup [\mathcal{C}_2]$. Thus if $\mathcal{C}$ is the disjoint union of $\mathcal{C}_1$ and $\mathcal{C}_2$, "$\mathfrak{G} \models$ 'x is in configuration #1'" is $({}^2\mathrm{LFP} \cap \neg{}^2\mathrm{LFP})$-expressible in $[\mathcal{C}]$. Nevertheless, this query cannot be represented by ${}^1\mathrm{LFP}_\omega$; hence ${}^2\mathrm{LFP} - {}^1\mathrm{LFP}_\omega \neq \varnothing$:

Theorem 4.3 *There exists a $({}^2\mathrm{LFP} \cap \neg{}^2\mathrm{LFP} \cap {}^1\Delta^1_1)$-definable query that is not ${}^1\mathrm{LFP}_\omega$-definable.*

Proof: Let $\mathcal{C}_1 = \langle\{0,1,2,3\},\{01,10,12,21,23,32,30,03\},H_1,T_1\rangle$ and $\mathcal{C}_2 = \langle\{0,1,2,3\},\{00,01,11,12,22,23,33,30\},H_2,T_2\rangle$, where $H_i(h,jk) \equiv h = j$ and $T_i(h,jk) \equiv h = k$, for all h,i,j,k. By Propositions 4.1 and 4.2, on $[\mathcal{C}_1] \cup [\mathcal{C}_2]$, "$\mathfrak{A} \in [\mathcal{C}_1]$" is $({}^2\mathrm{LFP} \cap \neg{}^2\mathrm{LFP} \cap {}^1\Delta^1_1)$-definable. We claim that "$\mathfrak{A} \in [\mathcal{C}_1]$" is not ${}^1\mathrm{LFP}_\omega$-definable on $[\mathcal{C}_1] \cup [\mathcal{C}_2]$. Let R be any relation on $[\mathcal{C}_1] \cup [\mathcal{C}_2]$ structures that is preserved under automorphism.

Let $\mathfrak{A}_n$ be the structure in $[\mathcal{C}_1]$ whose maximal chains are all of length n, and let $\mathfrak{B}_n$ be the structure in $[\mathcal{C}_2]$ whose maximal chains are all of length n. For each $i,j,t, 0 < t \leq n+1$, let $x_{ij}(t)$ be the t-th node on the maximal chain of arc ij, such that the distance from the distinguished node i to $x_{ij}(t)$ is t. By automorphism, for any $\mathfrak{A}_n$, and any $i,j,i',j',t, \mathfrak{A}_n \models R(x_{ij}(t)) \leftrightarrow R(x_{i'j'}(t))$ for any ${}^1\mathrm{LFP}_\omega$-definable R. Similarly, by automorphism, for any $\mathfrak{B}_n$, and any $i,j,t, \mathfrak{B}_n \models R(x_{ii}(t)) \leftrightarrow R(x_{jj}(t))$, and, if $j \cong i+1 \bmod 4$, $k \cong j+1 \bmod 4$, $\mathfrak{B}_n \models R(x_{ij}(t)) \leftrightarrow R(x_{jk}(t))$.

We claim that if R is ${}^1\mathrm{LFP}_\omega$-definable, then for each h, i, j, k, t, and sufficiently large n, $\mathfrak{A}_n \models R(x_{hi}(t)) \Leftrightarrow \mathfrak{B}_n \models R(x_{jk}(t))$. This will do the trick, for then, if $\varphi_0,\ldots,\varphi_\nu$ is a system of positive formulas whose atomic subformulas list only ${}^1\mathrm{LFP}_k$-definable relations, $\varphi_0^\infty,\ldots,\varphi_\nu^\infty$ will all be in ${}^1\mathrm{LFP}_{k+1}$, and it follows that for sufficiently large n, $\mathfrak{A}_n \models \varphi_0^\infty$ iff $\mathfrak{B}_n \models \varphi_0^\infty$. Hence it suffices to prove that *if n is sufficiently large and R is one-dimensional inductive over some system of queries $\mathbf{P} = P_0,\ldots,P_\eta$, where, for each p, and each appropriate h, i, j, k, and each $t \leq n$, $\mathfrak{A}_n \models P_p(x_{hi}(t)) \Leftrightarrow \mathfrak{B}_n \models P_p(x_{jk}(t))$, then for each appropriate h, i, j, k, and $t \leq n$, $\mathfrak{A}_n \models R(x_{hi}(t)) \Leftrightarrow \mathfrak{B}_n \models R(x_{jk}(t))$.*

To prove this, let $\boldsymbol{\varphi} = \varphi_0,\ldots,\varphi_\nu$ be a one-dimensional operative system of positive elementary formulas (each of quantifier depth $< r$) in the schema $(\mathrm{Arc},2),(P_0,1),\ldots,(P_\eta,1)$ such that $R \equiv \varphi_0^\infty$. Fix $n > 2^{r+1}$, and $\mathfrak{A} = \mathfrak{A}_n$ and

$\mathfrak{B} = \mathfrak{B}_n$. For each μ, let $f_{1\mu}(t) =$ if $\mathfrak{A} \models \varphi_\mu^\infty(x_{01}(t))$ then 1 else 0, and let $f_{20\mu}(t) =$ if $\mathfrak{B} \models \varphi_\mu^\infty(x_{00}(t))$ then 1 else 0 and let $f_{21\mu}(t) =$ if $\mathfrak{B} \models \varphi_\mu^\infty(x_{01}(t))$ then 1 else 0. Now suppose that we get a system $\mathbf{S} = S_0, \ldots, S_\nu$ such that

$$\mathfrak{B} \models S_\mu(x_{ij}(t)) \equiv \mathfrak{A} \models f_{1\mu}(t) = 1,$$

for all arcs ij in $\mathfrak{B}$ and all μ. We claim that $(\mathfrak{A}, \mathbf{P}, \boldsymbol{\varphi}^\infty) \equiv_r (\mathfrak{B}, \mathbf{P}, \mathbf{S})$, and hence $\mathbf{S}$ is a (simultaneous) fixed point of $\boldsymbol{\varphi}$: here is the Duplicator's winning strategy in the r-Fraisse game.

Suppose that it is the k-th move, and the Spoiler plays a_k on $(\mathfrak{A}, \mathbf{P}, \boldsymbol{\varphi}^\infty)$. If a_k is on chain ij, where $i + 1 \cong j \bmod 4$, then the Duplicator responds with b_k on chain ij of $(\mathfrak{B}, \mathbf{P}, \mathbf{S})$ in the corresponding position. If a_k is on chain ji, $i + 1 \cong j \bmod 4$, the Duplicator responds as follows. If a_k is within distance 2^{n-k} of any pebbled or distinguished points, the Duplicator responds with b_k the corresponding distance to that pebbled or distinguished point: b_k is on chain jj if $t \leq 2^{n-k}$, chain ii if $n + 1 - t \leq 2^{n-k}$, and within distance 2^{n-k} of the previously pebbled b_l if a_k was within distance 2^{n-k} of the previously pebbled a_l. On the other hand, if $a_k = x_{ji}(t)$ is not within distance 2^{n-k} of any pebbled or distinguished vertices, then by induction, at least one of $x_{11}(t)^{\mathfrak{B}}$, $l = 1,2,3,4$, is not within 2^{n-k} of any pebbled vertices and thus may be pebbled.

On the other hand, if the Spoiler plays b_k on $(\mathfrak{B}, \mathbf{P}, \mathbf{S})$, the Duplicator responds essentially with the above strategy in reverse. Again, if b_k is on chain ij, $i + 1 \cong j \bmod 2$, then the Duplicator responds with the corresponding a_k on chain ij in $(\mathfrak{A}, \mathbf{P}, \boldsymbol{\varphi}^\infty)$. If $b_k = x_{ii}(t)$, then the Duplicator responds with $a_k = x_{ij}(t)$ if t is small, $a_k = x_{jl}(t)$, $j + 1 \cong l \bmod 2$, if t is large, appropriately near or not near previously pebbled vertices if t is in between.

By playing the above strategy, the Duplicator wins; and thus $(\mathfrak{A}, \mathbf{P}, \boldsymbol{\varphi}^\infty) \equiv_r (\mathfrak{B}, \mathbf{P}, \mathbf{S})$, and $\mathbf{S}$ is a (simultaneous) fixed point of $\boldsymbol{\varphi}$. As $\boldsymbol{\varphi}^\infty$ is that least simultaneous fixed point of $\boldsymbol{\varphi}$, $f_{20\mu}(t), f_{21\mu}(t) \leq f_{1\mu}(t)$ for all μ, t.

On the other hand, consider the system $\mathbf{T} = T_0, \ldots, T_\nu$, where

$$\mathfrak{A} \models T_\mu(x_{ij}(t)) \equiv \mathfrak{B} \models f_{20\mu}(t) = 1$$

for all arcs $ij \in \{01, 10, 23, 32\}$, and all μ, in $\mathfrak{A}$, and for all other arcs, and all μ, let

$$\mathfrak{A} \models T_\mu(x_{ij}(t)) \equiv \mathfrak{B} \models f_{21\mu}(t) = 1.$$

Again, if n is sufficiently large, $(\mathfrak{A}_n, \mathbf{P}, \mathbf{T}) \equiv_r (\mathfrak{B}_n, \mathbf{P}, \boldsymbol{\varphi}^\infty)$ by a strategy quite similar to that of the previous game, and $\mathbf{T}$ is a fixed point of $\boldsymbol{\varphi}$. Again, as $\boldsymbol{\varphi}^\infty$ is the least fixed point of $\boldsymbol{\varphi}$, by using automorphisms we see that $f_{1\mu}(t) \leq f_{20\mu}(t), f_{21\mu}(t)$.

Hence, for all $\mu, t, f_{1\mu}(t) = f_{20\mu}(t) = f_{21\mu}(t)$. Hence, for each appropriate h, i, j, k, and $t \leq n$, $\mathfrak{A} \models R(x_{h,i}(t)) \Leftrightarrow \mathfrak{B} \models R(x_{j,k}(t))$. Using any ${}^1\mathrm{LFP}_n$-expressible queries $\mathbf{P}$, we find that ${}^1\mathrm{LFP}_{n+1}$ cannot distinguish between $\mathcal{C}_1$ and $\mathcal{C}_2$, and the theorem is proven.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the configurations of the above proof, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Using the same construction as above, we can see that on $[\mathcal{C}]$, the query "x is in the component of $\mathcal{C}_1$" is $({}^2\mathrm{LFP} \cap \neg{}^2\mathrm{LFP} \cap {}^1\Delta_1^1)$-expressible but not ${}^1\mathrm{LFP}_\omega$-expressible.

5 *Some future directions* Although there is more to be done with monadic logics, perhaps we should start thinking more about, say, d-dimensional logic, And although some techniques (like the surgery on monotonic relations described above) already developed are likely to be helpful, others (like eventual periodicity) seem more dubious. Even 2-dimensional induction is different. The $\mathcal{C}$-rational queries are a proper subset of the ^{2}LFP-definable queries: the reader is invited to confirm that the set of all finite chains of lengths $\{1,2,4,8,16,\dots\}$ is ^{2}LFP-definable. Investigating ^{2}LFP and ${}^2\Sigma_1^1$ will involve an entirely different kettle of fish.

Some general results would be desirable. For example, by Lemma 4.6 of [15], for any d, ${}^d\mathrm{LFP}_1$, $\neg{}^d\mathrm{LFP}_1 \subseteq {}^{2d}$LFP. The version we use is this: let $\varphi_0,\dots,\varphi_\nu$ be any d-dimensional system, with second order variables $S_0,\dots,S_\nu$. Let $\psi_i(\mathbf{x}) \equiv S_i(\mathbf{x})$ for all $\mathbf{x}$, so that $|\mathbf{x}|_{\varphi_i} + 1 = |\mathbf{x}|_{\psi_i}$ for all $\mathbf{x}, i$. One can construct the Stage Comparison relations $<_{\theta,\pi}$ as usual. Then an element of maximal stage (for the i-th induction) satisfies the Immermanesque formula

$$\max(\mathbf{x}) \equiv \forall\mathbf{y}[\mathbf{y} <_{\varphi_i,\psi_i} \mathbf{x} \vee \mathbf{x} <_{\psi_i,\varphi_i} \mathbf{y}],$$

so that $\neg\varphi_i(\mathbf{x}) \leftrightarrow \exists\mathbf{y}[\max(\mathbf{y})\ \&\ \mathbf{y} <_{\varphi_i,\varphi_i} \mathbf{x}]$. By repeatedly iterating this construction, we get ${}^d\mathrm{LFP}_n$, $\neg{}^d\mathrm{LFP}_n \subseteq {}^{2d}$LFP for all n. Thus, ${}^d\mathrm{LFP}_\omega \subseteq {}^{2d}$LFP. Is this inclusion proper? Even the inclusion ${}^d\mathrm{LFP} \subseteq {}^{2d}$LFP is not known to be proper, viz., for a fixed arity k, it is not known if ^{d}LFP, $d = 1,2,3,\dots$, generates an infinite hierarchy of k-ary queries (but see [6], [15], and especially [8] for varying k). Also, is there a d such that $d_{\mathrm{LFP}} = {}^d\mathrm{LFP}_\omega$? There must be a million of these questions.

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