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# Classical Logic, Intuitionistic Logic, and the Peirce Rule

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**Abstract** A simple method is provided for translating proofs in Gentzen's LK into proofs in Gentzen's LJ with the Peirce rule adjoined. A consequence is a simpler cut elimination operator for LJ + Peirce that is primitive recursive.

***Introduction*** In Gentzen's formalization of first-order logic, the essential difference between the classical system LK and the intuitionistic system LJ is that sequents $A_1, \ldots, A_m \rightarrow B_1, \ldots, B_n$ are restricted to the case $n \leq 1$ for the intuitionistic system LJ (see, for example Kleene [5], p. 444). The restriction $n \leq 1$ for the intuitionistic system applies both to the rules of inference, which are otherwise the same for LJ and LK, and to the form of the provable sequents. Thus it is possible for a sequent like $\rightarrow A \vee \neg A$ with $n \leq 1$ to be provable in LK without being provable in LJ.

Occasionally one might wish to work within the formalism of LJ and still be able to derive all sequents with $n \leq 1$ that are provable in the classical system LK. This can be done by adjoining to LJ the *Peirce rule*

$$\frac{A \supset B, \Gamma \rightarrow A}{\Gamma \rightarrow A}$$

(see Curry [1], p. 193). Yet there does not appear to be any simple method in the literature of translating proofs in LK into proofs in LJ + Peirce (Gordeev [4], p. 148). It is the purpose of this paper to provide such a method.

Felscher ([2], p. 150) has observed that the proof in Curry [1] (pp. 208–215, 329–331) of the cut elimination theorem for LJ + Peirce cannot be formalized in primitive recursive arithmetic. Thus Gordeev in [4] proves the cut elimination theorem for LJ + Peirce anew, using a complex series of transformations to obtain such a primitive recursive operator. A consequence of our work will be a much simpler operator which makes use of the standard cut elimination theorem for LK.

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Finally, we will argue that the simpler form of the Peirce rule (see [1], p. 260)

$$\frac{\neg A, \Gamma \to A}{\Gamma \to A}$$

is more appropriate for these purposes, and we will restate our results for this modified version of the Peirce rule.

***0 Preliminaries*** We will follow the notation and formulation of LK and LJ as given in [5], pp. 442–443. Uppercase letters $A$, $B$, $C$, and $D$ stand for formulas, Greek uppercase letters $\Gamma$, $\Delta$, $\Theta$, and $\Lambda$ stand for finite sequences of zero or more formulas, and the restrictions on variables are all as stated in [5]. We restate the axiom schema and rules of inference:

**Axiom schema** $A \to A$

**Rules of inference**

$\to\supset$: $\dfrac{A, \Gamma \to \Theta, B}{\Gamma \to \Theta, A \supset B}$

$\supset\to$: $\dfrac{\Delta \to \Lambda, A \qquad B, \Gamma \to \Theta}{A \supset B, \Delta, \Gamma \to \Lambda, \Theta}$

$\to\&$: $\dfrac{\Gamma \to \Theta, A \qquad \Gamma \to \Theta, B}{\Gamma \to \Theta, A \,\&\, B}$

$\&\to$: $\dfrac{A, \Gamma \to \Theta}{A \,\&\, B, \Gamma \to \Theta} \quad \dfrac{B, \Gamma \to \Theta}{A \,\&\, B, \Gamma \to \Theta}$

$\to\vee$: $\dfrac{\Gamma \to \Theta, A}{\Gamma \to \Theta, A \vee B} \quad \dfrac{\Gamma \to \Theta, B}{\Gamma \to \Theta, A \vee B}$

$\vee\to$: $\dfrac{A, \Gamma \to \Theta \qquad B, \Gamma \to \Theta}{A \vee B, \Gamma \to \Theta}$

$\to\neg$: $\dfrac{A, \Gamma \to \Theta}{\Gamma \to \Theta, \neg A}$

$\neg\to$: $\dfrac{\Gamma \to \Theta, A}{\neg A, \Gamma \to \Theta}$

$\to\forall$: $\dfrac{\Gamma \to \Theta, A(b)}{\Gamma \to \Theta, \forall x A(x)}$
subject to restriction on variables

$\forall\to$: $\dfrac{A(t), \Gamma \to \Theta}{\forall x A(x), \Gamma \to \Theta}$

$\to\exists$: $\dfrac{\Gamma \to \Theta, A(t)}{\Gamma \to \Theta, \exists x A(x)}$

$\exists\to$: $\dfrac{A(b), \Gamma \to \Theta}{\exists x A(x), \Gamma \to \Theta}$
subject to restriction on variables

$\to$Thinning: $\dfrac{\Gamma \to \Theta}{\Gamma \to \Theta, C}$

Thinning$\to$: $\dfrac{\Gamma \to \Theta}{C, \Gamma \to \Theta}$

$\to$Contraction: $\dfrac{\Gamma \to \Theta, C, C}{\Gamma \to \Theta, C}$

Contraction$\to$: $\dfrac{C, C, \Gamma \to \Theta}{C, \Gamma \to \Theta}$

$\to$Interchange: $\dfrac{\Gamma \to \Lambda, C, D, \Theta}{\Gamma \to \Lambda, D, C, \Theta}$

Interchange$\to$: $\dfrac{\Delta, C, D, \Gamma \to \Theta}{\Delta, D, C, \Gamma \to \Theta}$

Cut: $\dfrac{\Delta \to \Lambda, C \qquad C, \Gamma \to \Theta}{\Delta, \Gamma \to \Lambda, \Theta}$

For LJ, $\Lambda$ and $\Theta$ are presumed empty in all inference rules except Thinning$\rightarrow$, Contraction$\rightarrow$, and Interchange$\rightarrow$, where $\Theta$ may be empty or contain a single formula. $\rightarrow$Contraction and $\rightarrow$Interchange do not apply to LJ. To save space, we will omit steps involving just an Interchange in our proofs.

We denote by LJP the system LJ together with the Peirce rule:

$$\frac{A \supset B, \Gamma \rightarrow A}{\Gamma \rightarrow A}$$

Note that the Peirce rule is readily derivable in LK using the cut rule:

**Lemma 1** *If $A \supset B, \Gamma \rightarrow A$ is provable in LK then so is $\Gamma \rightarrow A$.*

*Proof:*

$$\dfrac{\dfrac{\dfrac{\dfrac{A \rightarrow A}{A \rightarrow A, B}}{\rightarrow A, A \supset B} \quad A \supset B, \Gamma \rightarrow A}{\Gamma \rightarrow A, A}}{\Gamma \rightarrow A}$$

*1* Let $F$ be an abbreviation for the formula $\forall x(P(x) \,\&\, \neg P(x))$ where $P$ is any available unary predicate in the language.

**Lemma 2** *$F \rightarrow$ is provable in LJ.*

*Proof:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(a) \rightarrow P(a)}{\neg P(a), P(a) \rightarrow}}{P(a) \,\&\, \neg P(a), P(a) \rightarrow}}{P(a) \,\&\, \neg P(a), P(a) \,\&\, \neg P(a) \rightarrow}}{P(a) \,\&\, \neg P(a) \rightarrow}}{\forall x(P(x) \,\&\, \neg P(x)) \rightarrow}$$

If $\Theta$ is the sequence of formulas $B_1, \ldots, B_n$, let $\Theta \supset F$ be the sequence $B_1 \supset F, \ldots, B_n \supset F$. The following lemma leads immediately to our main result:

**Lemma 3** *If $\Gamma \rightarrow \Theta$ is provable in LK then $\Theta \supset F, \Gamma \rightarrow$ is provable in LJP. If $\Gamma \rightarrow \Theta$ is provable in LK without use of the cut inference then $\Theta \supset F, \Gamma \rightarrow$ is provable in LJP without use of the cut inference. The transformation of the proof of $\Gamma \rightarrow \Theta$ to the proof of $\Theta \supset F, \Gamma \rightarrow$ is linear.*

*Proof:* By induction on the number $k$ of inferences in the proof figure of $\Gamma \rightarrow \Theta$.

*Case* $k = 0$. $\Gamma \rightarrow \Theta$ is $A \rightarrow A$.

$$\frac{A \rightarrow A \qquad F \rightarrow}{A \supset F, A \rightarrow}$$

*Case* $k > 0$. We consider the last inference in the proof figure of $\Gamma \rightarrow \Theta$. There is a subcase for each rule of inference of LK. The subcases $\rightarrow\supset$, $\supset\rightarrow$, $\rightarrow\&$, $\rightarrow\vee$, $\rightarrow\neg$, $\rightarrow\forall$, $\rightarrow\exists$, and cut require the Peirce rule; the others do not. We consider here just the subcases $\rightarrow\supset$ and cut. The other cases are all essentially similar.

*Subcase* $\to\supset$. The last inference is of the form

$$\frac{A,\Gamma \to \Theta', B}{\Gamma \to \Theta', A \supset B}$$

where $\Gamma \to \Theta$ is $\Gamma \to \Theta'$, $A \supset B$. By the inductive hypothesis $\Theta' \supset F$, $B \supset F$, $A,\Gamma \to$ is provable in LJP. We have:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Theta' \supset F,\ B \supset F,\ A,\Gamma \to}{B \supset F,\ \Theta' \supset F,\ A,\Gamma \to B}}{\Theta' \supset F,\ A,\Gamma \to B}\ \text{Peirce rule}}{\Theta' \supset F,\ \Gamma \to A \supset B} \qquad F \to}{\Theta' \supset F,\ (A \supset B) \supset F,\ \Gamma \to}$$

*Subcase, cut.* The last inference is of the form

$$\frac{\Delta \to \Lambda, C \qquad C,\Gamma' \to \Theta'}{\Delta,\Gamma' \to \Lambda, \Theta'}$$

where $\Gamma \to \Theta$ is $\Delta,\Gamma' \to \Lambda, \Theta'$. By the inductive hypothesis $\Lambda \supset F$, $C \supset F$, $\Delta \to$, and $\Theta' \supset F$, $C,\Gamma' \to$ are provable in LJP. We have

$$\cfrac{\cfrac{\cfrac{\Lambda \supset F,\ C \supset F,\ \Delta \to}{\Lambda \supset F,\ C \supset F,\ \Delta \to C}}{\Lambda \supset F,\ \Delta \to C}\ \text{Peirce} \qquad \cfrac{\Theta' \supset F,\ C,\Gamma' \to}{C, \Theta' \supset F,\ \Gamma' \to}}{\Lambda \supset F,\ \Theta' \supset F,\ \Delta,\Gamma' \to}\ \text{cut}$$

Note that the cut rule is requried to prove $\Theta \supset F$, $\Gamma \to$ in LJP only when the cut rule is used in the proof of $\Gamma \to \Theta$ in LK. Also note that the only additional inferences required in transforming the proof of $\Gamma \to \Theta$ to the proof of $\Theta \supset F$, $\Gamma \to$ are those needed to transfer the principal formula $A$, $B$, or $C$ from the antecedent to the succedent and from the succedent to the antecedent and for the proof of $F \to$. Thus the transformation is linear.

**Theorem 1** *If* $\Gamma \to A$ *is provable in LK then* $\Gamma \to A$ *is provable in LJP. If* $\Gamma \to A$ *is provable without the cut inference in LK then* $\Gamma \to A$ *is provable without the cut inference in LJP. The transformation of the proof of* $\Gamma \to A$ *in LK to the one in LJP is linear.*

*Proof:* By Lemma 3, $A \supset F$, $\Gamma \to$ is provable in LJP. Hence

$$\frac{A \supset F, \Gamma \to A}{\Gamma \to A}\ \text{Peirce rule}$$

and $\Gamma \to A$ is provable in LJP.

**Corollary 1** (Gordeev [4]) *There is a primitive recursive operator which transforms a derivation of* $\Gamma \to A$ *in LJP into a cut-free derivation of* $\Gamma \to A$ *in LJP.*

*Proof:* Suppose $\Gamma \to A$ is provable in LJP. Since the Peirce rule is derivable in LK with cut (Lemma 1) and LJ is a subsystem of LK, $\Gamma \to A$ is derivable in LK. By the standard cut elimination theorem for LK, $\Gamma \to A$ is derivable in LK without cut. By Theorem 1, $\Gamma \to A$ is derivable in LJP without cut. This entire process is primitive recursive since cut elimination for LK is primitive recursive (see, for example, Gallier [3], pp. 280 and 374) and the transformation of Theorem 1 is linear.

*Note.* Our cut elimination theorem for LJP is nominally inferior to Gordeev's in that the subformula property is not satisfied. This is because the formula $F$ may not be a subformula of $\Gamma$ or $A$. This deficiency is overcome in the modified formulation of LJP given in the next section.

*2* The reader has no doubt noticed that the introduction of the formula $F$ is merely a device to allow $\neg A$ to be expressed in the form $A \supset F$. The use of $F$ can be entirely eliminated, thereby simplifying the proof of Lemma 3, if the following form of the Peirce rule is adopted instead (see [1], p. 193):

$$\frac{\neg A, \Gamma \to A}{\Gamma \to A}$$

We will call this version of the Peirce rule P*. Let LJP* be LJ together with P*. If $\Theta$ is the sequence of formulas $B_1, \ldots, B_n$ let $\neg\Theta$ be the sequence $\neg B_1, \ldots, \neg B_n$. Lemma 3 can now be restated:

**Lemma 4** *If* $\Gamma \to \Theta$ *is provable in LK then* $\neg\Theta, \Gamma \to$ *is provable in LJP*. If* $\Gamma \to \Theta$ *is provable in LK without use of the cut inference then* $\neg\Theta, \Gamma \to$ *is provable in LJP* without use of the cut inference. The transformation of the proof of* $\Gamma \to \Theta$ *to the proof of* $\neg\Theta, \Gamma \to$ *is linear.*

*Proof:* The proof is similar to the one for Lemma 3 except that formulas of the form $A \supset F$ are replaced by $\neg A$. We will repeat just case $k = 0$ and Subcase $\to\supset$:

*Case* $k = 0$. $\Gamma \to \Theta$ is $A \to A$.

$$\frac{A \to A}{\neg A, A \to}$$

*Case* $k > 0$. *Subcase* $\to\supset$. The last inference of the proof of $\Gamma \to \Theta$ in LK is of the form

$$\frac{A, \Gamma \to \Theta', B}{\Gamma \to \Theta', A \supset B}$$

where $\Gamma \to \Theta$ is $\Gamma \to \Theta', A \supset B$. By the inductive hypothesis $\neg\Theta', \neg B, A, \Gamma \to$ is provable in LJP*. We have:

$$\cfrac{\cfrac{\cfrac{\cfrac{\neg\Theta', \neg B, A, \Gamma \to}{\neg\Theta', \neg B, A, \Gamma \to B}}{\neg\Theta', A, \Gamma \to B}\,\mathrm{P}^*}{\neg\Theta', \Gamma \to A \supset B}}{\neg\Theta', \neg(A \supset B), \Gamma \to}$$

Likewise, versions of Theorem 1 and Corollary 1 can be proven for LJP*:

**Theorem 2** *If $\Gamma \to A$ is provable in LK then $\Gamma \to A$ is provable in LJP*. If $\Gamma \to A$ is provable without the cut inference in LK then $\Gamma \to A$ is provable without the cut inference in LJP*. The transformation of the proof of $\Gamma \to A$ in LK to the one in LJP* is linear.*

**Corollary 2** *There is a primitive recursive operator which transforms a derivation of $\Gamma \to A$ in LJP* into a cut-free derivation of $\Gamma \to A$ in LJP*.*

We conclude with an example of a transformation of a theorem of LK into one of LJP* using Lemma 4 and Theorem 2. Consider the law of the excluded middle $\to A \vee \neg A$, which is provable in LK but not in LJ (see, for example, [5], pp. 483–484):

$$\begin{array}{cc}
A \to A & (1)\\ \hline
\to A, \neg A & (2)\\ \hline
\to A, A \vee \neg A & (3)\\ \hline
\to A \vee \neg A, A \vee \neg A & (4)\\ \hline
\to A \vee \neg A & (5)
\end{array}$$

Using Lemma 4, we transform each sequent $(n)$ of the form $\Gamma \to \Theta$ in the proof above into a sequent $(n)'$ of the form $\neg\Theta, \Gamma \to$ provable in LJP*:

$$\begin{array}{ccc}
A \to A & & \\ \hline
\neg A, A \to & & (1)'\\ \hline
\neg A \to \neg A & & \\ \hline
\neg\neg A, \neg A \to & & (2)'\\ \hline
\neg\neg A, \neg A \to \neg A & & \\ \hline
\neg A \to \neg A & \text{P*} & \\ \hline
\neg A \to A \vee \neg A & & \\ \hline
\neg(A \vee \neg A), \neg A \to & & (3)'\\ \hline
\neg(A \vee \neg A), \neg A \to A & & \\ \hline
\neg(A \vee \neg A) \to A & \text{P*} & \\ \hline
\neg(A \vee \neg A) \to A \vee \neg A & & \\ \hline
\neg(A \vee \neg A), \neg(A \vee \neg A) \to & & (4)'\\ \hline
\neg(A \vee \neg A) \to & & (5)'\\ \hline
\neg(A \vee \neg A) \to A \vee \neg A & & \\ \hline
\to A \vee \neg A & \text{P*} &
\end{array}$$

Of course the proofs obtained in LJP* by using Lemma 4 are not optimal. For example, several of the inferences above (such as the last three) could have been omitted.

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