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A Note about the Axioms for Branching-Time Logic

ALBERTO ZANARDO

Abstract The axiomatization of branching-time logic presented in S. McCall's paper "The Strong Future Tense" is considered and a counterexample is given to one of the theorems supporting the completeness result. Furthermore, the reason why McCall's method does not work is discussed briefly.

In this note I consider the axiomatization of the *strong future tense logic* given in McCall [3], and I prove that the completeness theorem proved there fails; in particular, Claims 1 and 2 below constitute a counterexample to Theorem 2 (p. 495). The counterexample focuses the inner difficulties connected with the axiomatizations of certain branching-time logics and justifies the axioms adopted in Zanardo [4].

All the definitions of [3] are assumed here, as well as the notation. Thus Claims 1 and 2 can be stated and proved without any preliminaries.

Claim 1 *Let* α *be the formula* α_1 & α_2 & α_3, *where*

$$\alpha_1 = IA \;\&\; SB \;\&\; SC$$

$$\alpha_2 = G[(A \;\&\; B) \to (\sim C \;\&\; \sim SC)]$$

$$\alpha_3 = G[(A \;\&\; FB) \to \sim C]$$

Then $\sim\alpha$ *is valid.*

Proof: Assume that α holds at some *world state* (w.s.) t of the model structure $\mathcal{M} = \langle W, L\rangle$ under the valuation v. By α_1, there is a future branch a relative to t in which A holds everywhere, and there is a w.s. x of a such that $v_{\mathcal{M}}(B,x) = 1$. By α_2, $\sim C$ holds at x, and there is a future branch b relative to x in which $\sim C$ holds everywhere. Consider the set $X = a \cap \{z : Lxz\}$. For every z in X, $v_{\mathcal{M}}(A \;\&\; FB, z) = 1$ and hence, by α_3, $v_{\mathcal{M}}(\sim C, z) = 1$. Then the set $b \cup X \cup \{x\}$ is a

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future branch relative to t in which $\sim C$ holds everywhere, and this contradicts $v_{\mathcal{M}}(SC, t) = 1$.

Theorem 3 in [3] (p. 497) asserts that the **R**-construction of a formula closes whenever its R-construction closes; thus, in order to have a counterexample to Theorem 2, it is enough to prove the following claim.

Claim 2 *Let A, B, and C be propositional variables and let α be as in Claim* 1. *Then the* **R***-construction of α does not close.*

Proof: I show that there is an alternative set T in the **R**-construction of α which does not close. Let α be the initial item of the tableau t. The initial steps of the construction of the set T are specified in 1 to 5 below, where only the main rules that are used are mentioned (within brackets) and tautological equivalences are used freely.

0. $IA, SB, SC, G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$ appear in t
1. [The third alternative of rule **IS2** and rule **G**] : x and y are new tableaux such that Lxt, Lyx and
   1.0. As in 0
   1.1. $A, B, IA, G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$, $(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)$, $(A \,\&\, FB) \rightarrow \sim C$ appear in x
   1.2. $A, C, IA, G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$, $(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)$, $(A \,\&\, FB) \rightarrow \sim C$ appear in y
2. As step 1, except that
   2.1. $\sim C$ and $\sim SC$ are added to x
3. [The third alternative of rule **S1** and rule **G**.] As step 2, except that z is a new tableau such that Lzt, Lxz and
   3.1. $C, G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$, $(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)$, $(A \,\&\, FB) \rightarrow \sim C$ appear in z
4. As step 3, except that
   4.1. $\sim A$ is added to z.
5. [The second alternative of rule **S1**.] As step 4, except that
   5.1. SB is added to z.

Summing up the situation at this point, we have four tableaux t, z, x, y, fulfilling the relationships: Lzt, Lxz, Lyx; the formulas which appear in them are:

t: $IA, SB, SC, G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$
z: $C, SB, \sim A$,
$G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$,
$(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)$, $(A \,\&\, FB) \rightarrow \sim C$
x: $A, B, IA, \sim C, \sim SC$,
$G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$,
$(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)$, $(A \,\&\, FB) \rightarrow \sim C$
y: $A, C, IA, G[(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)], G[(A \,\&\, FB) \rightarrow \sim C]$,
$(A \,\&\, B) \rightarrow (\sim C \,\&\, \sim SC)$, $(A \,\&\, FB) \rightarrow \sim C$.

Let α_x, α_y, and α_z be the conjunctions of the formulas appearing at this stage in x, y, and z respectively. We can observe now that **G**, **H**, and **S1** are the only rules in which more that one tableau of the set at hand is involved, and that, in the situation presented above, the rules **G** and **H** have been applied whenever possible. Furthermore, there are suitable alternatives of rule **S1** which cause no change in that situation. For instance, if we consider the pair $\langle z, x\rangle$, we can apply the first alternative and add B to x, but B already is in x.

Thus in order to conclude that the **R**-construction of α does not close, we have only to prove that the **R**-constructions of α_x, α_y, and α_z do not close. This can be proved directly but can also be viewed as a consequence of Theorem 4 in [3]. Indeed, it can be easily verified that α_x, α_y, and α_z are satisfiable, and hence the negation of none of them is a theorem.

Comments It should be clear that the crucial step of the (not closing) **R**-construction of α is the application of rule **S1** in step 3. We can use the third alternative and let C be the initial item of z because no rule prevents us from adding $\sim A$ in the subsequent step 4; of course, this conflicts with the fact that the sequence $\langle t, x, y\rangle$ was generated by the item IA in t. In other words, when rule **S1** is applied, it should be possible to distinguish the situations in which the tableau t' has been previously generated by means of one of the rules **I**, or **IS**, or **ISn.** In this case, if the corresponding I-item is IA, every tableau between t and t' must contain A.

The logic considered in [3] is very close to Burgess's 'Peircean' branching-time logic (see [1]). The only nontrivial difference is that the structures for Peircean logic are *irreflexive trees*, that is, structures like those considered in [3] with the additional requirements that, for all x, *not-Lxx* (irreflexivity) and that, for all x, y, z, Lxy and $Lxz \Rightarrow y = z$, or Lyz, or Lzy (linearity towards the past). In Burgess [1], Peircean logic is axiomatized by means of a finite set of axioms and rules; one of the rules is an instance of the Gabbay's irreflexivity rule (see Gabbay [2]) which is allowed by the requirement of L being irreflexive. In [4] this rule is eliminated by means of an infinite set of axioms and the relative Henkin-style completeness proof can be easily verified to work also for the structures considered in [3].

The role of the axiom-scheme A7 in [4], which replaces the irreflexivity rule, is just to permit the Henkin construction of a future branch generated by a formula of the form IA, and, actually, the various instances of this axiom-scheme can be viewed as 'constraints' for a correct use of McCall's rule **S1**. The forms of the formulas represented by A7 are rather *ad hoc* for the connected Henkin construction, but their meaning is quite similar to that of the formula $\sim\alpha$ defined in Claim 1. For example, an instance of A7 is

$$(\beta) \qquad IA \,\&\, SB \,\&\, G{\sim}(A \,\&\, FB \,\&\, C) \rightarrow I[A \,\&\, (FB \,\&\, {\sim}SB \,\&\, {\sim}SC \rightarrow P(A \,\&\, B \,\&\, {\sim}SC))],$$

and the similarity between this formula and $\sim\alpha$ appears through the verification that the **R**-construction of $\sim\beta$ does not close.

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Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Belzoni 7
I-35131 Padova Italy