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Pure Second-Order Logic

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Abstract Pure second-order predicate calculus is a predicate calculus where the only variables are predicate variables. In it, logical truth is decidable, and semantic consequence is compact. Pure second-order functional calculus is a functional calculus where the only variables are function variables. In it, semantic consequence is not compact, and there is no complete proof procedure for logical truth.

The language of the pure second-order predicate calculus consists of those formulas of the second-order predicate calculus whose only variables are predicate variables. A statement of its semantics will help to elucidate my notational conventions.

A model $\mathfrak{M}$ is a pair $\langle\mathfrak{D},\mathfrak{R}\rangle$, where $\mathfrak{D}$ is a domain of individuals, and $\mathfrak{R}$ is a function such that:

1. for any name n, $\mathfrak{R}(n) \in \mathfrak{D}$;
2. for any k-adic function sign f, $\mathfrak{R}(f)$ is a k-adic operation on $\mathfrak{D}$;
3. for any k-adic predicate F, $\mathfrak{R}(F) \subseteq \mathfrak{D}^k$.

Let an S be a function such that for each k-adic predicate variable ϕ, $S(\phi) \subseteq \mathfrak{D}^k$. Let $S\langle s/\phi\rangle$ be just like S, save that $S\langle s/\phi\rangle$ assigns s, a subset of $\mathfrak{D}^k$, to ϕ. For each S, let $S[\ ]$ be such that:

1. for any name n, $S[n] = \mathfrak{R}(n)$;
2. for any function sign f, $S[f(t_1,\ldots,t_k)] = \mathfrak{R}(f)(S[t_1],\ldots,S[t_k])$;
3. for any predicate F, $S[F] = \mathfrak{R}(F)$;
4. for any predicate variable ϕ, $S[\phi] = S(\phi)$.

We say that in $\mathfrak{M}$ S satisfies:

1. an atomic formula $\mathfrak{f}t_1,\ldots,t_k$ iff $\langle S[t_1],\ldots,S[t_k]\rangle \in S[\mathfrak{f}]$;

2.1. $\neg A$ iff S does not satisfy A;

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2.2. A & B iff S satisfies A and S satisfies B; and so on in the obvious way for other truth-functional connectives;
3.1. $\exists\phi A$ iff for some s, $S\langle s/\phi\rangle$ satisfies A;
3.2. $\forall\phi A$ iff for any s, $S\langle s/\phi\rangle$ satisfies A.

A is true in $\mathfrak{M}$ iff A is a closed formula satisfied in $\mathfrak{M}$ by every S. And A is a logical truth iff A is true in every $\mathfrak{M}$.

Theorem *Logical truth for the pure second-order predicate calculus is decidable.*

Proof: For any A, $\mathfrak{M} = \langle\mathfrak{D},\mathfrak{R}\rangle$, and S, we give the following definitions of $\mathfrak{M}^A = \langle\mathfrak{D}^A,\mathfrak{R}^A\rangle$ and S^A, the A-condensations of $\mathfrak{M}$ and of S:

1. $\mathfrak{a} \in \mathfrak{D}^A$ iff for some singular term t used in A, $\mathfrak{a} = S[t]$;
2.1. if n is a proper name used in A, then $\mathfrak{R}^A(n) = \mathfrak{R}(n)$;
2.2. if f is a k-adic function sign, then $\mathfrak{R}^A(f) = (\mathfrak{D}^A)^{k+1} \cap \mathfrak{R}(f)$.
2.3. if F is a k-adic predicate, then $\mathfrak{R}^A(F) = (\mathfrak{D}^A)^k \cap \mathfrak{R}(F)$.
3. if ϕ is a k-adic predicate variable, then $S^A(\phi) = (\mathfrak{D}^A)^k \cap S(\phi)$.

We can now give an inductive proof that S satisfies A in $\mathfrak{M}$ iff S^A satisfies A in $\mathfrak{M}^A$. In virtue of the definitions of $\mathfrak{M}^A$ and S^A, this obviously holds for atomic A. As for the inductive step, it may be instructive to give some details for two of the cases.

Case &. Suppose that B & C is satisfied in $\mathfrak{M}$ by S. Then B too is satisfied in $\mathfrak{M}$ by S. Hence B is satisfied in $\mathfrak{M}^B$ by S^B. But since any terms that occur in B occur also in B & C, $\mathfrak{M}^B$ and S^B are the B-condensations not only of $\mathfrak{M}$ and S but also of $\mathfrak{M}^{B\&C}$ and $S^{B\&C}$. Hence B is satisfied in $\mathfrak{M}^{B\&C}$ by $S^{B\&C}$. By similar reasoning, so is C. Hence if B & C is satisfied in $\mathfrak{M}$ by S, it is satisfied in $\mathfrak{M}^{B\&C}$ by $S^{B\&C}$. To show the converse is now easy.

Case $\exists$. Suppose that $\exists\phi B$ is satisfied in $\mathfrak{M}$ by S, and that ϕ is k-adic. Then for some s, $s \subseteq \mathfrak{D}^k$ and B is satisfied in $\mathfrak{M}$ by $S\langle s/\phi\rangle$. Then B is satisfied in $\mathfrak{M}^B$ by $S\langle s/\phi\rangle^B$. But $\mathfrak{M}^B = \mathfrak{M}^{\exists\phi B}$, and $S\langle s/\phi\rangle^B = S\langle s/\phi\rangle^{\exists\phi B}$. So B is satisfied in $\mathfrak{M}^{\exists\phi B}$ by $S\langle s/\phi\rangle^{\exists\phi B}$. So for some s', namely $s \cap (\mathfrak{D}^{\exists\phi B})^k$, B is satisfied in $\mathfrak{M}^{\exists\phi B}$ by $S^{\exists\phi B}\langle s'/\phi\rangle$. Therefore $\exists\phi B$ is satisfied in $\mathfrak{M}^{\exists\phi B}$ by $S^{\exists\phi B}$. Suppose now that $\exists\phi B$ is satisfied in $\mathfrak{M}^{\exists\phi B}$ by $S^{\exists\phi B}$. Then for some s, $s \subseteq (\mathfrak{D}^{\exists\phi B})^k$ and B is satisfied in $\mathfrak{M}^{\exists\phi B}$ by $S^{\exists\phi B}\langle s/\phi\rangle$. But $\mathfrak{M}^{\exists\phi B} = \mathfrak{M}^B$. And $S^{\exists\phi B}\langle s/\phi\rangle = S\langle s/\phi\rangle^{\exists\phi B} = S\langle s/\phi\rangle^B$. So B is satisfied in $\mathfrak{M}^B$ by $S\langle s/\phi\rangle^B$. So B is satisfied in $\mathfrak{M}$ by $S\langle s/\phi\rangle$. Therefore $\exists\phi B$ is satisfied in $\mathfrak{M}$ by S.

The reader can now be left to complete the inductive proof and infer that if A is false in a model, it is false in the A-condensation of that model. Now the A-condensation of a model has a domain with no more individuals than there are singular terms used in A. Thus by counting the finite number of singular terms used in A, we can determine a finite set of finite models such that, if A is false in any model at all, it is false in one of them. Thus given any formula, we can institute a systematic search, and finitely far into our search we will reach a stage at which we know that if we have not already found a model in which the formula is false, the formula is true in all models. Logical truth for the pure second-order predicate calculus is therefore decidable.

Theorem *If each finite subset of γ has a model, then γ is a subset of some maximal set Γ which has a model.*

Proof: Expand the language of γ by adding, for each k, as many new k-adic predicates as there are finite sets of ordered k-tuples of singular terms in the language of γ. Let an $\mathbf{n}$ be an ordered k-tuple of singular terms; if $\mathbf{m}$ is the k-tuple of singular terms $a_1, \ldots, a_k$, and $\mathbf{n}$ is $b_1, \ldots, b_k$, let $\mathbf{m} = \mathbf{n}$ be the formula $a_1 = b_1 \,\&\ldots\&\, a_k = b_k$. For each finite set of ordered k-tuples of singular terms used in the language of γ, take a distinct k-adic predicate, F, from those that we have just added to the language of γ. If that finite set of ordered k-tuples is empty, form for every $\mathbf{n}$ the formula $\neg F\mathbf{n}$. If on the other hand that finite set of ordered k-tuples is $\{\mathbf{n}_i, \ldots, \mathbf{n}_j\}$, then form the formula $F\mathbf{n}_i \,\&\ldots\&\, F\mathbf{n}_j$, and for each $\mathbf{m}$ which is not in $\{\mathbf{n}_i, \ldots, \mathbf{n}_j\}$, form the formula $F\mathbf{m} \Rightarrow (\mathbf{m} = \mathbf{n}_i \vee \ldots \vee \mathbf{m} = \mathbf{n}_j)$. Let all the formulas we thus form from our new vocabulary be added to γ and call the result Γo.

Now assume that each finite subset of γ has a model. It follows that the same is true of Γo. For suppose some finite subset of Γo has no model. Then that subset will have in turn a subset δ such that δ has no model and such that every proper subset of δ does have a model. Suppose that one of our new predicates—let it be F—is used in δ. Let δ^F be those formulas in δ that use F; and let $\neg\&\delta^F$ be the negation of the conjunction of the members of δ^F. Now $\neg\&\delta^F$ will have either the form $\neg(\neg F\mathbf{m}_a \,\&\ldots\&\, \neg F\mathbf{m}_z)$ or the form $\neg[\{F\mathbf{n}_i \,\&\ldots\&\, F\mathbf{n}_j\} \,\&\, \{(F\mathbf{m}_a \Rightarrow (\mathbf{m}_a = \mathbf{n}_i \vee \ldots \vee \mathbf{m}_a = \mathbf{n}_j)) \,\&\ldots\} \,\&\, \{F\mathbf{m}_z \Rightarrow (\mathbf{m}_z = \mathbf{n}_i \vee \ldots \vee \mathbf{m}_z = \mathbf{n}_j)\}]$; or some form which results from this by omitting one or two of the subformulas enclosed by $\{\;\}$. Whatever its precise form, $\neg\&\delta^F$ is evidently contingent. Yet $\delta - \delta^F \models \neg\&\delta^F$; and F is not used in $\delta - \delta^F$. Hence $\delta - \delta^F$ can have no model. But it does have a model. Hence δ cannot contain any of our new predicates. Hence δ would have to be a subset of γ; which it cannot be, since each finite subset of γ has a model. Hence every finite subset of Γo has a model after all.

Γo may be expanded into Γ by the following procedure. Let the closed sentences in the language of Γo be well-ordered as $p_1, p_2, \ldots$. For each $n > 0$, let:

$$\Gamma n = \bigcup_{m<n} \Gamma m \cup \{p_n\},$$

if for some finite subset P of $\bigcup_{m<n} \Gamma m$, $P \models p_n$; otherwise, let:

$$\Gamma n = \bigcup_{m<n} \Gamma m \cup \{\neg p_n\}.$$

We now let:

$$\Gamma = \bigcup_{m\geq 0} \Gamma m.$$

The reader can verify, by transfinite induction if need be, that Γ is maximal and that each finite subset of it has a model.

Now consider the model $\mathfrak{M}$. For each singular term t used in Γ, let t' be the equivalence class such that $t_i' = t_k'$ iff $t_i = t_k \in \Gamma$. Let $\mathfrak{D}$ consist of those equivalence classes. For each name n, let $\mathfrak{R}(n) = n'$; for each function sign f, let $t_0' = \mathfrak{R}(f)(t_1', \ldots, t_k')$ iff $t_0 = f(t_1, \ldots, t_k) \in \Gamma$; and for each predicate F, let

$\langle t'_1, \ldots, t'_k \rangle \in \Re(F)$ iff $Ft_1, \ldots, t_k \in \Gamma$. Now a sentence is true in $\mathfrak{M}$ iff that sentence $\in \Gamma$. This can be shown by induction on the complexity of sentences. It obviously holds for atomic sentences. And where we consider truth-functional connectives, the reasoning for the inductive step is trivial. So let us consider the existential quantifier, and show that if every formula with fewer quantifiers than $\exists\phi A$ is true in $\mathfrak{M}$ iff it belongs to Γ, then the same applies to $\exists\phi A$ itself. We will assume that ϕ is k-adic.

Let **I** be the conjunction of every formula in Γ that is either an identity using two terms that occur in $\exists\phi A$, or the negation of such an identity. Now we have so constructed Γ that, for each set j of ordered k-tuples of the terms that occur in $\exists\phi A$, there is at least one predicate F_j such that if **n** is an ordered k-tuple of the terms that occur in $\exists\phi A$, then Γ contains $F_j\mathbf{n}$ if for some $\mathbf{m} \in j$, $\mathbf{I} \vDash \mathbf{m} = \mathbf{n}$, and Γ contains $\neg F_j\mathbf{n}$ if for no $\mathbf{m} \in j$, $\mathbf{I} \vDash \boldsymbol{m} = \mathbf{n}$. For each set j of ordered k-tuples of the terms that occur in $\exists\phi A$, choose one such predicate. Let P_j be the conjunction of those sentences in Γ that use only that predicate and terms in $\exists\phi A$ and are either atoms or the negations of atoms; and let A_j be the result of replacing with that predicate all free occurrences of ϕ in A. Let **P** be the conjunction of each P_j; and let **A** be the disjunction of each A_j.

$(\mathbf{I} \;\&\; \mathbf{P}) \Rightarrow (\exists\phi A \Leftrightarrow \mathbf{A})$ is a logical truth. For if it were false in a model, it would be false in the A-condensation of that model. Now since $\mathbf{A} \Rightarrow \exists\phi A$ is any case a logical truth, $(\mathbf{I} \;\&\; \mathbf{P}) \Rightarrow (\exists\phi A \Leftrightarrow \mathbf{A})$ could be false only if **I**, **P**, and $\exists\phi A$ were all true, while **A** was false. But if **I** and **P** were true in the A-condensation, then each k-adic property in the A-condensation would be the interpretation of some predicate in **P**. So whatever property a sequence assigns to ϕ, if the sequence satisfies A, it will satisfy one of the disjuncts in **A**. So if in the A-condensation $\exists\phi A$ were true, **A** would be true too.

Now since $(\mathbf{I} \;\&\; \mathbf{P}) \Rightarrow (\exists\phi A \Leftrightarrow \mathbf{A})$ is a logical truth, it will both belong to Γ and be true in $\mathfrak{M}$. But **I** & **P** also belongs to Γ; and, since it contains fewer quantifiers than $\exists\phi A$, it is true in $\mathfrak{M}$. Thus $\exists\phi A \Leftrightarrow \mathbf{A}$ both belongs to Γ and is true in $\mathfrak{M}$. Suppose now that $\exists\phi A$ belongs to Γ. Then so does **A**; and so does some disjunct of **A**. But, since it contains fewer quantifiers than $\exists\phi A$, that disjunct of **A** will be true in $\mathfrak{M}$. Hence $\exists\phi A$ will also be true in $\mathfrak{M}$. Suppose instead that $\exists\phi A$ is true in $\mathfrak{M}$. Then **A** is true in $\mathfrak{M}$; and so is some disjunct of **A**. But, since it contains fewer quantifiers than $\exists\phi A$, that disjunct will belong to Γ. And $\exists\phi A$ follows from each such disjunct. Hence $\exists\phi A$ will belong to Γ too.

The reasoning for universally quantified formulas is similar enough for the reader to be left to complete the proof.

A formula is a formula of the pure second-order functional calculus if and only if it is a formula of the second-order functional calculus and its only variables are function variables. Pure second-order functional calculus could not differ more from pure second-order predicate calculus, since:

Theorem *In the pure second-order functional calculus, semantic consequence is not compact, nor is logical truth decidable, nor is there even a complete proof procedure for logical truth.*

Proof: It is possible to simulate first-order quantification by quantifying function variables: for each individual variable v, take a distinct monadic function variable f; replace all occurrences of $\forall v$ by $\forall f$, and of $\exists v$ by $\exists f$; then replace all remaining occurrences of v by $f(0)$. For any formula A, let A^* be its translation according to this scheme. Let C be the conjunction of the standard axioms for succession, addition, and multiplication, together with this version of the second-order induction axiom: $\forall f(\forall x f(x) = f(s(x))) \Rightarrow \forall x f(x) = f(0)$. Now C^* is true in a model iff that model is isomorphic to the standard model of arithmetic. Thus the infinite set of formulas $\{C^*, n \neq 0, n \neq s(0), n \neq s(s(0)), \ldots\}$ has no model; but any finite subset of it has a model, formed from the standard model of arithmetic by extending it to have the name n denote some number whose numeral is not used in our finite subset. Semantic consequence is therefore not compact. Moreover, A is true in the standard model of first-order arithmetic iff $(C \Rightarrow A)^*$ is a logical truth of the pure second-order functional calculus. Thus if we could have a decision procedure for logical truth in the pure second-order functional calculus, we could also have a decision procedure for first-order arithmetical truth, and likewise a complete proof procedure for the former would also be a complete proof procedure for the latter. But there can be no complete proof procedure, and a fortiori no decision procedure, for first-order arithmetical truth. Hence there can be no such things for logical truth in the pure second-order functional calculus either.

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