Notre Dame Journal of Formal Logic
Volume 33, Number 2, Spring 1992

End Extensions of Models of Arithmetic

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Abstract A concise proof is presented of Wilkie's Theorem that for every model of Peano Arithmetic there is a diophantine equation having no solution in that model but having a solution in some end extension of that model.

Kaufmann (in [2]) observed that the Completeness Theorem formalized in Peano Arithmetic can be used to give an alternate proof of the MacDowell-Specker Theorem on elementary end extensions of models of PA. (See Theorem 3 for an outline of his proof.) Rabin [4] proved that every model $\mathcal{M}$ of PA has an elementarily equivalent extension which solves a Diophantine equation having coefficients in M but having no solution in M. Gaifman [1] asked whether Rabin's Theorem could be improved by requiring that the extension be an end extension. By Matijasevič's solution to Hilbert's Tenth Problem, which was unavailable to Rabin, Gaifman's question is equivalent to asking whether every model of PA has an elementarily equivalent end extension which is not a Σ_1-extension. After several partial results had been obtained (Manevitz [3], Wilkie [6]), Wilkie [7] proved a comprehensive theorem which yielded an affirmative answer to Gaifman's question. His proof relied heavily on his previously obtained affirmative answer for countable models. In this note we will give a rather quick and direct proof of Wilkie's theorem along the lines of Kaufmann's proof of the MacDowell-Specker Theorem.

Notation and terminology will be that standardly used in the Peano Arithmetic literature. The language L of PA is finite. For $\mathcal{M}$ an L-structure, $L(\mathrm{M})$ is L augmented by constant symbols for elements of M. Note that Σ_n and Π_n are sets of L-formulas (or $L(\mathrm{M})$-formulas) which have a certain syntactic form. For an L-structure $\mathcal{M}$, we let $D(\mathcal{M})$, the elementary diagram of $\mathcal{M}$, be the set of $L(\mathrm{M})$-sentences true in $\mathcal{M}$. We write $\mathcal{M} <_n \mathcal{N}$ to mean that $\mathcal{N}$ is a Σ_n-elementary extension of $\mathcal{M}$ or, equivalently, $D(\mathcal{M}) \cap \Pi_n \subseteq D(\mathcal{N})$. We let $\mathrm{SSy}(\mathcal{M})$ be the standard system of $\mathcal{M}$; and for a complete theory $T \supseteq \mathrm{PA}$, we let $\mathrm{Rep}(T)$ be the standard system of its minimal model.

Given a model $\mathcal{M}$ of PA and an L-structure $\mathcal{N}$, we say that $\mathcal{N}$ is *internal* to

Received March 26, 1990; revised December 15, 1991

$\mathcal{M}$ if there is a subset $D \subseteq M$ which is definable in $\mathcal{M}$ such that $\mathcal{M} \models$ "D is a complete and consistent Henkin theory," and the structure that D determines is isomorphic to $\mathcal{N}$. If $\mathcal{N}$ is internal to $\mathcal{M}$ and is a model of a sufficiently strong (but still very weak) finite fragment of PA, then without loss of generality we can take $\mathcal{N}$ to be an end extension of $\mathcal{M}$.

The following is a version of Wilkie's theorem.

Theorem 1 *Let $\mathcal{M} \models \mathrm{PA}$ be nonstandard and let $T \supseteq PA$ be a complete theory. Then $\mathcal{M}$ has an end extension $\mathcal{N}$ such that $\mathcal{N} \models T$ iff the following two conditions hold*:

(1) $\mathrm{Rep}(T) \subseteq \mathrm{SSy}(\mathcal{M})$;
(2) $T \cap \Pi_1 \subseteq \mathrm{Th}(\mathcal{M})$.

Proof: Suppose $\mathcal{M} \subseteq^{\mathrm{end}} \mathcal{N}$ and $\mathcal{N} \models T$. Clearly, $\mathrm{Rep}(T) \subseteq \mathrm{SSy}(\mathcal{N}) = \mathrm{SSy}(\mathcal{M})$, so that (1) holds. Since Σ_1 sentences persist under end extensions, (2) also holds.

Now suppose (1) and (2) hold. It follows from the Reflection Principle and (2) that $\mathcal{M} \models \mathrm{Con}(\ulcorner \sigma \urcorner)$ for each $\sigma \in T$. Then using (2) and (1), overspill, and the Completeness Theorem formalized in $\mathcal{M}$, we can obtain a model

$$\mathcal{N}_0 \models T \cap \Pi_1$$

which is internal to $\mathcal{M}$ such that $\mathcal{M} \subseteq^{\mathrm{end}} \mathcal{N}_0$. We proceed by constructing a sequence $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots$ of structures, each one of which is internal to $\mathcal{M}$. Each $\mathcal{N}_{i+1}$ should be such that

$$\mathcal{N}_{i+1} \models T \cap \Pi_{i+2}$$

and

$$\mathcal{N}_i \prec_i \mathcal{N}_{i+1}.$$

The existence of each of the $\mathcal{N}_{i+1}$ is easily confirmed by a diagram argument followed by an application of the Completeness Theorem, both taking place inside of $\mathcal{M}$. Clearly, $\mathcal{N} = \bigcup_i \mathcal{N}_i$ is an end extension of $\mathcal{M}$ and $\mathcal{N} \models T$.

In the proof of the next theorem we will use Theorem 1.

Theorem 2 (Wilkie) *For each nonstandard $\mathcal{M}$ there is $\mathcal{N} \equiv \mathcal{M}$ such that $\mathcal{M} \subseteq^{\mathrm{end}} \mathcal{N}$ but $\mathcal{N}$ is not a Σ_1-extension of $\mathcal{M}$.*

Proof: It is enough to find such an $\mathcal{N}$, with $\mathrm{Th}(\mathcal{N}) \cap \Sigma_1 \subseteq \mathrm{Th}(\mathcal{M})$ instead of $\mathcal{N} \equiv \mathcal{M}$, because then Theorem 1 can be applied to it to get the desired model.

Let $\varphi(x)$ be a usual Σ_1 formula defining a simple set as, for example, in Theorem 8.II of Rogers [5]. Thus $\mathcal{M} \models \forall x \exists y(\neg\varphi(y) \wedge x \leq y \leq 2x)$, and for any Σ_1 formula $\psi(x)$ either $\mathcal{M} \models \exists x(\varphi(x) \wedge \psi(x))$ or else there is $n < \omega$ such that $\mathcal{M} \models \forall x(\psi(x) \rightarrow x \leq n)$. Let $a \in M$ be nonstandard such that $\mathcal{M} \models \neg\varphi(a)$. We will obtain $\mathcal{N}$ such that $\mathcal{N} \models \varphi(a)$.

Consider some $\sigma \in \mathrm{Th}(\mathcal{M}) \cap \Pi_1$, and let $\psi(x)$ be the Σ_1 formula

$$(\exists\theta \in D(\mathcal{M}) \cap \Pi_0)\,[\mathrm{length}(\theta) \leq \mathrm{length}(\sigma) \wedge \neg\mathrm{Con}(\ulcorner \sigma \wedge \theta \wedge \varphi(\mathring{x}) \urcorner)].$$

Clearly, it suffices to prove $\mathcal{M} \models \neg\psi(a)$. By the Reflection Principle, $\mathcal{M} \models \forall x(\varphi(x) \rightarrow \neg\psi(x))$. Therefore $\mathcal{M} \models \forall x(\psi(x) \rightarrow x \leq n)$ for some $n < \omega$. Since a is nonstandard, $\mathcal{M} \models \neg\psi(a)$.

The previous theorems have hierarchal versions which we now only state, but for which similar proofs exist. For the remainder of this note fix some $k < \omega$.

Theorem 1.k *Let* $\mathcal{M} \models \mathrm{PA}$ *be nonstandard and let* $T \supseteq \mathrm{PA}$ *be a complete theory. Then* $\mathcal{M}$ *has an end* Σ_k*-extension* $\mathcal{N}$ *such that* $\mathcal{N} \models T$ *iff the following two conditions hold*:

(1) $\mathrm{Rep}(T) \subseteq \mathrm{SSy}(\mathcal{M})$;
(2) $T \cap \Pi_{k+1} \subseteq \mathrm{Th}(\mathcal{M})$.

Theorem 2.k *For each nonstandard* $\mathcal{M}$ *there is* $\mathcal{N} \equiv \mathcal{M}$ *such that* $\mathcal{M} \prec_k^{\mathrm{end}} \mathcal{N}$ *but* $\mathcal{N}$ *is not a* Σ_{k+1}*-extension of* $\mathcal{M}$.

For completeness, we end with a sketch of Kaufmann's proof of the MacDowell–Specker Theorem.

Theorem 3 (MacDowell–Specker) *Every model of* PA *has a proper elementary end extension.*

Proof: Let $\mathcal{M} \models \mathrm{PA}$. Much as we did in the proof of Theorem 1, we construct a sequence $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \ldots$ of structures, each of which is internal to $\mathcal{M}$ such that

$$\mathcal{N}_i \prec_{i+1} \mathcal{N}_{i+1}$$

and

$$\mathcal{M} \prec_{i+2}^{\mathrm{end}} \mathcal{N}_i.$$

Then $\mathcal{N} = \bigcup_i \mathcal{N}_i$ is an elementary end extension of $\mathcal{M}$. Note that $\mathcal{N}_0$ is internal to $\mathcal{M}$, so by Tarski's theorem on the undefinability of truth, $\mathrm{Th}(\mathcal{N}_0) \neq \mathrm{Th}(\mathcal{M})$, and therefore $\mathcal{N}_0$ (and thus also $\mathcal{N}$) is a proper extension of $\mathcal{M}$.

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