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The Admissibility of γ in R4

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Abstract The logic **NR** of Meyer's "Entailment and relevant implication" is extended to include the axiom scheme $\Box(A \vee B) \rightarrow (\Diamond A \vee \Box B)$ to create the logic **R4**, so named because it is a conservative extension of **S4**. It has been an open problem since the writing of "Entailment and relevant implication" whether Ackermann's rule γ is admissible in **R4**. In this paper, we close this problem by proving that γ is admissible in this system.

In [4], Meyer formulates the system **NR**, which combines the axioms governing implication from Anderson and Belnap's system **R** with the axioms governing the behavior of necessity from **S4**. **NR**, however, does not contain **S4** on a direct translation. For example, **NR** does not contain the scheme $\sim\Box(A \vee B) \vee (\Diamond A \vee \Box B)$. To overcome this deficiency, Belnap and Meyer have suggested adding the postulate

$$\Box(A \vee B) \rightarrow (\Diamond A \vee \Box B)$$

to the axioms of **NR** (see Routley and Meyer [6], p. 70). We call the system that results from the addition of this new axiom scheme **R4** (to signify the fact that it contains all of **S4**). **R4** has not yet been adopted as *the* system of modality and relevance, at least to a large extent, because it has not been shown to be complete over the semantics suggested in [6], and Ackermann's rule γ (from $\vdash\sim A \vee B$ and $\vdash A$ infer $\vdash B$) had not been shown to be admissible in it. The purpose of this paper is to remove the latter difficulty. That is, we show that γ is admissible in **R4**.

Our argument uses a modified version of Meyer's method of metavaluations (following, e.g., Meyer [5]). We build a structure of regular, prime **R4** theories that mimics, for the most part, a Kripke model for **S4**. We impose a binary accessibility relation on this structure. We then show, using a version of the method of metavaluations, that each of the theories in our structure can be reduced to a theory that is prime, regular, and consistent, while retaining the same accessibility relation between these reduced theories. As a corollary of this construc-

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tion we show that γ is admissible in **R4**. We also present a brief argument that **R4** is a conservative extension of **S4**.

1 We begin with a standard modal sentential language $\mathcal{L}$ that contains propositional constants $p_1, p_2, p_3, \ldots$, connectives $\wedge$, $\sim$, $\rightarrow$, and the modal operator $\Box$. The language has the usual formation rules. We call the set of formulas of $\mathcal{L}$ wff($\mathcal{L}$) or merely wff. In our metalanguage, we use lower case letters from the latter half of the Roman alphabet to range over propositional constants and capital letters from the early part of the Roman alphabet to range over formulas in general. We also make use of two defined connectives, viz.,

D$\vee$ $A \vee B =_{df} \sim(\sim A \wedge \sim B)$
D$\Diamond$ $\Diamond A =_{df} \sim\Box\sim A$.

R4 has the following axioms schemes:

A0 $A \rightarrow A$
A1 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
A2 $A \rightarrow ((A \rightarrow B) \rightarrow B)$
A3 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
A4 $(A \wedge B) \rightarrow A$
A5 $(A \wedge B) \rightarrow B$
A6 $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
A7 $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
A8 $\sim\sim A \rightarrow A$
A9 $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
A10 $\Box A \rightarrow A$
A11 $\Box A \rightarrow \Box\Box A$
A12 $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
A13 $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$
A14 $\Box(A \vee B) \rightarrow (\Diamond A \vee \Box B)$
A15 If A is an axiom, $\Box A$ is an axiom.

And it has the rules:

$\wedge$I From A and B, infer $A \wedge B$
$\rightarrow$E From $A \rightarrow B$ and A, infer B.

Anderson and Belnap's **R** contains A0–A9 and the rules $\wedge$I and $\rightarrow$E (for the motivation of **R**, see Anderson and Belnap [1]). The system **NR** is made up of A0–A13 + A15 with $\wedge$I and $\rightarrow$E. Moreover, the following rule can be easily derived:

$\Box$I From A, infer $\Box A$.

We leave the derivation of this rule to the interested reader.

It is also easy to prove the following lemma:

Lemma 1 *The following are theorems of* **R4**:

$\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
$(\Box A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$
$A \rightarrow \Diamond A$.

2 Now that we have set out **R4**, we go on to show that γ is admissible in it. Our proof relies heavily on a priming lemma due originally to Belnap (and, independently, to Gabbay—see [3]). Before we can state the lemma, we need to state a few definitions. For formulas A and B, $A \vdash_{\mathbf{R4}} B$ iff $\vdash A \rightarrow B$. Where Δ and Γ are sets of formulas, $\Delta \vdash_{\mathbf{R4}} \Gamma$ iff for some $A_1, \ldots A_n \in \Delta$ $(n > 0)$ and some $B_1, \ldots B_m \in \Gamma$ $(m > 0)$, $A_1 \wedge \ldots \wedge A_n \vdash_{\mathbf{R4}} B_1 \vee \ldots \vee B_m$. If it is not the case that $\Delta \vdash_{\mathbf{R4}} \Gamma$, we write $\Delta \dashv_{\mathbf{R4}} \Gamma$. Where Γ has only one member B, we can write $\Delta \vdash_{\mathbf{R4}} B$ instead of $\Delta \vdash_{\mathbf{R4}} \Gamma$ or $\Delta \vdash_{\mathbf{R4}} \{B\}$. A set of sentences Δ is said to be an *R4 theory* (or in the context of this paper just *theory*) iff for any formula B of $\mathcal{L}$, if $\Delta \vdash_{\mathbf{R4}} B$, then $B \in \Delta$. Note that every theory is closed under $\wedge$I and $\rightarrow$E. A theory Δ is *prime* iff for all formulas A and B, if $A \vee B \in \Delta$, $A \in \Delta$ or $B \in \Delta$. A theory is *regular* if it is a theory and contains all theorems of **R4** and it is *consistent* if it does not contain both A and $\sim A$ for any wff A. A theory is *normal* if it is regular, prime, and consistent.

For sets of formulas Δ and Γ, (Δ,Γ) is said to be a *consistent R4 pair* iff $\Delta \dashv_{\mathbf{R4}} \Gamma$. (Δ,Γ) is an *inconsistent R4 pair* otherwise. We can now state the priming lemma.

Lemma 2 (Priming) *Let* (Δ,Γ) *be a consistent* **R4** *pair. Then* Δ *can be extended to a prime theory* Δ' *such that* (Δ',Γ) *is a consistent* **R4** *pair.*

For a nice presentation of the proof of this lemma, see Dunn [2].

Now we take an arbitrary prime regular theory and show that it contains a normal theory. Given the above priming lemma, this will imply that for any non-theorem A there is a prime regular theory that does not contain A. To see that this is so, just replace Δ in the statement of the above lemma with the set of theorems of **R4** expressible in $\mathcal{L}$ and Γ with $\{A\}$.

Before we can perform our reduction, we need to state a few more definitions. Let T be an arbitrary prime theory, then

D$\Diamond^{-1}$ $\Diamond^{-1}T =_{df} \{A : \Diamond A \in T\}$
D$\Box^{-1}$ $\Box^{-1}T =_{df} \{A : \Box A \in T\}$
D$\mathbf{K_T}$ $K_T =_{df} \{T_i$: T_i is a prime theory and $\Box^{-1}T \subseteq T_i \subseteq \Diamond^{-1}T\}$.
DS For $T_i, T_j \in K_T$, ST_iT_j iff $\Box^{-1}T \subseteq T_j \subseteq \Diamond^{-1}T_i$.

We call $\Diamond^{-1}T$ the *depossibilitation of* T and $\Box^{-1}T$ the *denecessitation of* T.

Let us take an arbitrary prime regular theory T. Let T_i be some member of K_T. It can be readily seen that T_i is also prime and regular. Moreover, if $\Box A \in T$, then, by A11 and $\rightarrow$E, $\Box\Box A \in T$, so, by DK$_{\mathrm{T}}$, $\Box A \in T_i$. And, if $\Diamond A \in T_i$, then, by A10, A9, and $\rightarrow$E, $\Diamond\Diamond A \in T_i$. So, by DK$_{\mathrm{T}}$, $\Diamond A \in T$. This implies that $K_{T_i} \subseteq K_T$.

We now need to show that the relation S is adequate for our purposes. We do so by proving the following two lemmas.

Lemma 3 *Let* T_i *be an arbitrary prime theory and let* $A \in \Diamond^{-1}T_i$. *Then there is a prime theory* T_j *such that* $A \in T_j$ *and* $\Box^{-1}T_i \subseteq T_j \subseteq \Diamond^{-1}T_i$.

Proof: Suppose that $A \in \Diamond^{-1}T_i$. Suppose also, for the sake of a reductio, that for some $B \in \Box^{-1}T_i$. $\vdash B \wedge A \rightarrow C$, for some C not in $\Diamond^{-1}T_i$.[1] By Lemma 1, $\vdash\Diamond(B \wedge A) \rightarrow \Diamond C$. Since $A \in \Diamond^{-1}T_i$, $\Diamond A \in T_i$, and since $B \in \Box^{-1}T_i$, $\Box B \in T_i$. Since T_i is closed under $\wedge$I, $\Box B \wedge \Diamond A \in T_i$, whence, by Lemma 1, $\Diamond(B \wedge A) \in$

T_i, and so $\Diamond C \in T_i$. Thus by the definition of $\Diamond^{-1} T_i$, $C \in \Diamond^{-1} T_i$, contra hypothesis. This shows that $((\Box^{-1} T_i \cup \{A\}), \text{wff} - (\Diamond^{-1} T_i))$ is a consistent **R4** pair. So by the priming lemma, there is a prime theory T_j such that $A \in T_j$ and $\Box^{-1} T_i \subseteq T_j \subseteq \Diamond^{-1} T_i$, concluding the proof of the lemma.

We use the preceding lemma to prove the following important lemma:

Lemma 4 *For an arbitrary prime theory T_i,*

(i) $\Box A \in T_i$ *iff* $\forall T_j (T_j \in K_{T_i} \Rightarrow A \in T_j)$
(ii) $\Diamond A \in T_i$ *iff* $\exists T_j (T_j \in K_{T_i}$ & $A \in T_j)$.

Proof: (i) First, suppose that $\Box A \in T_i$. Then $A \in \Box^{-1} T_i$. Thus by the construction of K_{T_i}, for all $T_j \in K_{T_i}$, $A \in T_j$. For the converse, suppose that for all $T_j \in K_{T_i}$, $A \in T_j$ and, for the sake of a reductio, suppose that $\Box A \notin T_i$. We show that $\Box^{-1} T_i \nvdash_{\mathbf{R4}} ((\text{wff} - \Diamond^{-1} T_i) \cup \{A\})$. Suppose otherwise. Then there is some $B \in \Box^{-1} T_i$ and some $C \in (\text{wff} - \Diamond^{-1} T_i)$ such that $B \vdash_{\mathbf{R4}} C \vee A$. So $\Box B \vdash_{\mathbf{R4}} \Box(C \vee A)$, by $\Box$I, A12, and $\to$E. By A1, A14, and $\to$E, $\Box B \vdash_{\mathbf{R4}} \Diamond C \vee \Box A$. But T_i is a theory and $\Box B \in T_i$, so $\Diamond C \vee \Box A \in T_i$. Since T_i is prime, either $\Diamond C \in T_i$ or $\Box A \in T_i$. By assumption, $\Diamond C \notin T_i$, whence $\Box A \in T_i$, contradicting the hypothesis. So by the priming lemma, there is a prime theory T_k such that $\Box^{-1} T_i \subseteq T_k \subseteq \Diamond^{-1} T_i$ and $A \notin T_k$. By $\mathrm{DK_T}$ there is a theory T_k in K_{T_i} such that $A \notin T_k$, contrary to the hypothesis of the reductio.

(ii) Suppose that $\Diamond A \in T_i$. Then, by Lemma 3, there is some T_j in K_{T_i} such that $A \in T_j$. For the converse, suppose that $A \in T_j$ for some $T_j \in K_{T_i}$. It follows from the construction of K_{T_j} that $\Diamond A \in T_i$, thus concluding the proof of the lemma.

Note that by DS and $\mathrm{DK_T}$, ST_iT_j iff $T_j \in K_{T_i}$. So, by the proof of the preceding lemma, it is clear that $\Box A \in T_i$ iff $\forall T_j (ST_iT_j \Rightarrow A \in T_j)$ and $\Diamond A \in T_i$ iff $\exists T_j (ST_iT_j$ & $A \in T_j)$. We can also show that S is reflexive and transitive, just like the accessibility relation in Kripke's model for **S4**. That it is reflexive follows immediately from the fact that if $A \in T_i$, then $\Diamond A \in T_i$ and if $\Box A \in T_i$, then $A \in T_i$. To show that it is transitive, suppose that ST_iT_j and ST_jT_k. Thus if $A \in T_k$, then $\Diamond A \in T_j$, and so $\Diamond\Diamond A \in T_i$. But, since it is a theorem of **R4** that $\Diamond\Diamond A \to \Diamond A$, $\Diamond A \in T_i$. Furthermore, if $\Box A \in T_i$, then, because $\Box A \to \Box\Box A$ is a theorem of **R4**, $\Box\Box A \in T_i$. So $\Box A \in T_j$. But this implies that $A \in T_k$. In other words, $\Box^{-1} T_i \subseteq T_k \subseteq \Diamond^{-1} T_i$; i.e. ST_iT_k, whence we have shown that S is transitive.

We define * (*the Routley star operator*) in the customary way. That is, for a prime theory T, $T^* = \{A : {\sim}A \notin T\}$. It can be easily shown that T^* is also a prime theory and that $T^{**} = T$. We also define the set $K_{T^*} = \{T_i^* : T_i \in K_T\}$. Furthermore, following our treatment of K_T, we define a binary relation S_* on K_{T^*} such that $S_*T_i^*T_j^*$ iff $\Box^{-1} T_i^* \subseteq T_j^* \subseteq \Diamond^{-1} T_i^*$.

As the following lemma shows, there is a tidy correlation between S and S_*.

Lemma 5 *For all $T_i, T_j \in K_T$, ST_iT_j iff $S_*T_i^*T_j^*$.*

Proof: $\Rightarrow$ Suppose that ST_iT_j. Also suppose that $\Box A \in T_i^*$. Assume, for the sake of a reductio, that $A \notin T_j^*$. By the definition of *, ${\sim}A \in T_j$. By the definition of S, $\Diamond{\sim}A \in T_i$; i.e. ${\sim}\Box A \in T_i$. By the definition of *, this implies that

$\Box A \notin T_i^*$, contradicting our hypothesis. Now we must show that if $A \in T_j^*$, then $\Diamond A \in T_i^*$. Suppose that $A \in T_j^*$. Also assume that $\Diamond A \notin T_i^*$. By the definition of *, $\sim\Diamond A \in T_i$; i.e. $\Box\sim A \in T_i$. By ST_iT_j and the definition of S, $\sim A \in T_j$; whence, by the definition of *, $A \notin T_j$, contradicting our hypothesis. Thus we have shown that, if ST_iT_j, then $\Box^{-1}T_i^* \subseteq T_j^* \subseteq \Diamond^{-1}T_i^*$; i.e. $S_*T_i^*T_j^*$.

$\Leftarrow$ Suppose now that $S_*T_i^*T_j^*$. First, suppose that $\Box A \in T_i$. We must show that $A \in T_j$. For the sake of a reductio, assume that it does not. Then, by the definition of * and the fact that $T_j^{**} = T_j$, $\sim A \in T_j^*$. By the definition of S_*, $\Diamond\sim A \in T_i^*$, whence $\sim\Box A \in T_i^*$, since T_i^* is a theory. By virtue of the fact that $T_i^{**} = T_i$ and the definition of *, $\Box A \notin T_i$, contradicting our hypothesis. Now let us show that if $A \in T_j$, $\Diamond A \in T_i$. For the sake of a reductio, assume that $A \in T_j$ and $\Diamond A \notin T_i$. By $T_i^{**} = T_i$ and the definition of *, $\sim\Diamond A \in T_i^*$, so, by the fact that T_i^* is a theory, $\Box\sim A \in T_i^*$. By the definition of S_*, $\sim A \in T_j^*$. By the definition of * and $T_j^{**} = T_j$, $A \notin T_j$, contradicting our hypothesis. Thus we have shown that $\Box^{-1}T_i \subseteq T_j \subseteq \Diamond^{-1}T_i$ (i.e. ST_iT_j) concluding the proof of the lemma.

For each $T_i \in K_T$, we define a metavaluation v_i. v_i is a total function from formulas into the set $\{T,F\}$. In addition, for each v_i the following conditions hold:

Tr p $v_i(p) = \mathrm{T}$ iff $p \in T_i$
Tr∧ $v_i(A \wedge B) = \mathrm{T}$ iff $v_i(A) = \mathrm{T}$ and $v_i(B) = \mathrm{T}$
Tr∼ $v_i(\sim A) = \mathrm{T}$ iff $\sim A \in T_i$ and $v_i(A) = \mathrm{F}$
Tr→ $v_i(A \to B) = \mathrm{T}$ iff $A \to B \in T_i$ and either $v_i(A) = \mathrm{F}$ or $v_i(B) = \mathrm{T}$
Tr□ $v_i(\Box A) = \mathrm{T}$ iff $\forall j(ST_iT_j \Rightarrow v_j(A) = \mathrm{T})$
DTr $Tr_i = \{A : v_i(A) = \mathrm{T}\}$.

Tr p, Tr∧, Tr→, and Tr∼ are equivalent to the conditions that Meyer uses to prove that γ is admissible in R. The novelty of the approach of the present paper is in its use of Tr□.

We also define a binary relation S_{Tr} on K_{Tr}, such that for $Tr_i, Tr_j \in K_{Tr}$, $S_{Tr}Tr_iTr_j$ iff ST_iT_j. By the definition of S_{Tr} and Lemma 5, ST_iT_j iff $S_{Tr}Tr_iTr_j$ iff $S_*T_i^*T_j^*$, for all $T_i, T_j \in K_T$. Since the three accessibility relations coincide, we say that Sij iff ST_iT_j (iff $S_{Tr}Tr_iTr_j$ iff $S_*T_i^*T_j^*$).

Note that Tr→ forces each Tr_i to be closed under $\to E$. For suppose $A \to B \in Tr_i$ and $A \in Tr_i$. By DTr, $v_i(A) \neq F$. By Tr→, $v_i(B) = T$, and so, by DTr, $B \in Tr_i$. In other words, Tr_i is closed under $\to E$. Moreover, each Tr_i is closed under $\wedge$I. For suppose $A \in Tr_i$ and $B \in Tr_i$. By DTr, $v_i(A) = T$ and $v_i(B) = T$. So, by Tr∧, $v_i(A \wedge B) = T$, whence, by DTr, $A \wedge B \in Tr_i$ (i.e. Tr_i is closed under $\wedge$I).

Following the now standard argument, we show that, given a prime regular theory T, for each $T_i \in K_T$, Tr_i is normal. That is, Tr_i is prime, regular, and consistent. The following lemma proves this latter point and is a key step in proving that Tr_i is regular. (Throughout the remainder of the paper we assume that T, hence every member of K_T, is prime and regular.)

Lemma 6 *For all $T_i \in K_T$, $T_i^* \subseteq Tr_i \subseteq T_i$.*

Proof: For the most part, showing that $Tr_i \subseteq T_i$ is trivial. The induction case for $\Box$, however, is somewhat tricky so we do it here. Suppose that $\Box A \in Tr_i$.

By Tr$\Box$, for all Tr_j such that Sij, $A \in Tr_j$. By the inductive hypothesis, $A \in T_j$. So, for all T_j such that Sij, $A \in T_j$. By the definition of S, then, for all T_j such that $\Box^{-1}T_i \subseteq T_j \subseteq \Diamond^{-1}T_i$, $A \in T_j$. So, by Lemma 4, $\Box A \in T_i$. We must now show that $T_i^* \subseteq Tr_i$. Suppose $A \in T_i^*$. We show by an induction on the complexity of A that $A \in Tr_i$. We may use the fact that $T_i^* \subseteq T_i$, which is straightforward to show.

Case 1. $A = p$. The atomic case follows directly from Tr p.

Case 2. $A = B \wedge C$. Follows from Tr$\wedge$ and the inductive hypothesis.

Case 3. $A = {\sim}B$. Suppose ${\sim}B \in T_i^*$. So ${\sim}B \in T_i$. Assume for the sake of a reductio that ${\sim}B \notin Tr_i$. By Tr$\sim$, $B \in Tr_i$. By the inductive hypothesis, $B \in T_i$. But, by the definition of $*$, this in turn implies that ${\sim}B \notin T_i^*$, contradicting the hypothesis of Case 3.

Case 4. $A = B \to C$. Suppose that $B \to C \in T_i^*$. Thus $B \to C \in T_i$. Moreover, suppose that $B \in Tr_i$. Then, by the inductive hypothesis, $B \in T_i$. Since T_i is a theory, it is closed under $\to$E, so $C \in T_i$. Assume for the sake of a reductio that $C \notin Tr_i$. Thus by the inductive hypothesis, $C \notin T_i^*$, hence, by the definition of $*$, ${\sim}C \in T_i$. Since T_i is closed under &I, $B \wedge {\sim}C \in T_i$, and so since T_i is an **R4** theory and $\vdash_{\mathbf{R4}} (B \wedge {\sim}C) \to {\sim}(B \to C)$, ${\sim}(B \to C) \in T_i$. By the definition of $*$, this in turn implies that $B \to C \notin T_i^*$, contradicting our assumption and concluding the proof of Case 4.

Case 5. $A = \Box B$. Suppose that $\Box B \in T_i^*$. Also suppose that $S_* T_i^* T_j^*$ (hence, S_{ij}). Then, by the definition of S_*, $B \in T_j^*$. And so by the inductive hypothesis $B \in Tr_j$, for all Tr_j such that Sij. Thus by Tr$\Box$, $\Box B \in Tr_i$, concluding the proof of the case and the lemma.

Lemma 7 *For all $Tr_i \in K_{Tr}$, $A \in Tr_i$ or ${\sim}A \in Tr_i$.*

Proof: Suppose that $A \notin Tr_i$. Then, by Lemma 6, $A \notin T_i^*$. By the definition of $*$, ${\sim}A \in T_i$. By Tr$\sim$, ${\sim}A \in Tr_i$.

From Lemma 7 it follows that ${\sim}A \in Tr_i$ iff $v_i(A) = F$. It is also a consequence of Lemma 7 that each Tr_i is prime. For suppose that $A \vee B \in Tr_i$. By D$\vee$, ${\sim}({\sim}A \wedge {\sim}B) \in Tr_i$. So ${\sim}A \wedge {\sim}B \notin Tr_i$. By Tr$\wedge$, either ${\sim}A \notin Tr_i$, in which case $A \in Tr_i$, or ${\sim}B \notin Tr_i$, in which case $B \in Tr_i$. It is equally straightforward to show that if $A \in Tr_i$ or $B \in Tr_i$, then $A \vee B \in Tr_i$. Thus we can state the following derived condition on v_i:

Tr$\vee$ $v_i(A \vee B) = \mathrm{T}$ iff $v_i(A) = \mathrm{T}$ or $v_i(B) = T$.

Moreover, each Tr_i is consistent. For suppose that $A \in Tr_i$. Then, by DTr, $v_i(A) = T$. Therefore, by Tr$\sim$, $v_i({\sim}A) = F$, whence, by DTr, ${\sim}A \notin Tr_i$. So, if we show that every Tr_i is a regular theory, we will have shown that they are normal. First we show that every instance of the axiom schemes are in each of the Tr_i.

We use Lemma 7 to show that all instances of Axioms A8–A9 are members of each Tr_i.

Lemma 8 *If A is an instance of* A8–A9, *then $A \in Tr_i$ for all $Tr_i \in K_{Tr}$.*

Proof: These proofs are very straightforward and well-known. So we prove only the case for A8, leaving A9 as an exercise for the reader. Suppose A is an instance of A8; i.e. A is of the form $\sim\sim B \rightarrow B$. By Tr$\rightarrow$, we must show (α) $\sim\sim B \rightarrow B \in T_i$ and (β) if $\sim\sim B \in Tr_i$, then $B \in Tr_i$. (α) follows immediately from the fact that T_i is regular. To show (β), suppose that $\sim\sim B \in Tr_i$. Suppose, for the sake of a reductio, that $B \notin Tr_i$. By Lemma 7, $\sim B \in Tr_i$. But then, by Tr$\sim$, $v_i(\sim\sim B) = F$; i.e. $\sim\sim B \notin Tr_i$, contra hypothesis.

To show that the relation S_{Tr} is an adequate accessibility relation, we prove the following:

Lemma 9 $\Diamond A \in Tr_i$ *iff* $\exists Tr_j(Sij \;\&\; A \in Tr_j)$.

Proof: $\Rightarrow$ Suppose that $\Diamond A \in Tr_i$. By D$\Diamond$, $\sim\Box\sim A \in Tr_i$. By Tr$\sim$, $\Box\sim A \notin Tr_i$. By Tr$\Box$, there exists a Tr_j such that Sij and $\sim A \notin Tr_j$. By Lemma 7, $A \in Tr_j$.
$\Leftarrow$ Let Tr_j be such that Sij and $A \in Tr_j$. By Lemma 6, $A \in T_j$. By Lemma 4, $\Diamond A \in T_i$; i.e. $\sim\Box\sim A \in T_i$. By Tr$\sim$, we must show that $\Box A \notin Tr_i$. Assume the opposite. By Tr$\Box$, $A \in Tr_j$. But, Tr_j is consistent, so $\sim A \notin Tr_j$, contradicting our assumption and concluding the proof of the lemma.

We continue the proof that each Tr_i is regular by proving the following lemma:

Lemma 10 *Let A be an instance of* A10, A11, A12, A13, *or* A14, *and let Tr_i be an arbitrary member of K_{Tr}. Then A is in Tr_i.*

Proof:

Case 1. A is an instance of A10; i.e. $A = \Box B \rightarrow B$. Trivial.

Case 2. A is an instance of A11; i.e. $A = \Box B \rightarrow \Box\Box B$. (i) That $A \in T_i$ follows from the fact that T_i is regular. (ii) Suppose $\Box B \in Tr_i$. It suffices to show that for all x such that Six, $B \in Tr_x$. Also assume that Sij and Sjk. We need to show that $B \in Tr_k$, as required. Since S is transitive, Sik. So $B \in Tr_k$, as required.

Case 3. A is an instance of A12; i.e. $A = \Box(B \rightarrow C) \rightarrow (\Box B \rightarrow \Box C)$. (i) Since T_i is regular, $A \in T_i$. (ii) Assume that $\Box(B \rightarrow C) \in Tr_i$. We show that $(\Box B \rightarrow \Box C) \in Tr_i$. To show this we first prove that $(\Box B \rightarrow \Box C) \in T_i$. By Tr$\rightarrow$ and the assumption, $\Box(B \rightarrow C) \in T_i$. Since T_i (like all regular theories) is closed under $\rightarrow$E, and $A \in T_i$, $(\Box B \rightarrow \Box C) \in T_i$, as required. Now we show that either $\Box B \notin Tr_i$ or $\Box C \in Tr_i$. Suppose that $\Box B \in Tr_i$. We show that $\Box C \in Tr_i$. Suppose that Tr_j is such that Sij. By the assumption of (ii) and Tr$\Box$, $B \rightarrow C \in Tr_j$. Moreover, by assumption and Tr$\Box$, $B \in Tr_j$. Thus, by Tr$\rightarrow$, $C \in Tr_j$, as required. Thus, by Tr$\rightarrow$, $\Box B \rightarrow \Box C \in Tr_i$, concluding the proof of Case 3.

Case 4. A is an instance of A13; i.e. $A = (\Box B \wedge \Box C) \rightarrow \Box(B \wedge C)$. Trivial.

Case 5. A is an instance of A14; i.e. $A = \Box(B \vee C) \rightarrow (\Diamond B \vee \Box C)$. (i) $A \in T_i$, since T_i is regular. (ii) Suppose $\Box(B \vee C) \in Tr_i$. By DS, for all Tr_j such that Sij, $B \vee C \in Tr_j$. By Tr$\vee$, either $B \in Tr_j$ or $C \in Tr_j$, for every such Tr_j. Suppose $B \in Tr_j$. Then, by Lemma 9, $\Diamond B \in Tr_i$, whence, by Tr$\vee$, $\Diamond B \vee \Box C \in Tr_i$. On the other hand, suppose that for no such Tr_j does $B \in Tr_j$. Then, $C \in Tr_j$ for every Tr_j such that Sij. by Tr$\Box$, $\Box C \in Tr_i$. By Tr$\vee$, $\Diamond B \vee \Box C \in Tr_i$. By (i), (ii), and Tr$\rightarrow$, $A \in Tr_i$, concluding the proof of the lemma.

To conclude our argument that each of the members of K_{Tr} are regular, we must show that all instances of Axioms A0–A7 and A15 are in each $Tr_i \in K_{Tr}$. The following two lemmas do exactly this.

Lemma 11 *For all $Tr_i \in K_{Tr}$, if A is an instance of any of axioms* A0–A7, *then $A \in Tr_i$.*

Proof: As usual.

Lemma 12 *If A is an axiom of* **R4**, *then $\Box A \in Tr_i$ for all $Tr_i \in K_{Tr}$.*

Proof: Suppose A is an instance of one of the axiom schemes A1–A14. Then, by the preceding lemmas, $A \in Tr_j$ for each Tr_j in K_{Tr}. Thus, by Tr$\Box$, $\Box A \in Tr_i$ for arbitrary Tr_i in K_{Tr}. By the same reasoning, for any axiom A, $\Box\Box A \in Tr_i$, $\Box\Box\Box A \in Tr_i$, and so on. Generalizing, we can say that if A is an axiom of **R4**, $\Box A \in Tr_i$, concluding the proof of Lemma 12.

Lemmas 1–12 enable us to prove the following corollary:

Corollary 13 *Every Tr_i in K_{Tr} is normal.*

Proof: Let Tr_i be an arbitrary member of K_{Tr}. We have already observed that Tr_i is prime and consistent. By Lemmas 8, 10, 11, and 12, every instance of the axiom schemes is in Tr_i. We have also observed that Tr_i is closed under $\rightarrow$E and $\wedge$I. Thus Tr_i is a normal theory.

In other words, every prime, regular **R4** theory contains a normal **R4** theory as a subtheory. It is now very easy to prove the main theorem of the paper, namely:

Theorem 14 *γ is admissible in* **R4**.

Proof: Suppose $\vdash \sim A \vee B$ and $\vdash A$. Choose an arbitrary prime regular theory T. Following the construction we have outlined, build a set of theories K_T. Using our metavaluation technique, we have shown that we can reduce K_T to a set of normal theories K_{Tr} such that for each T_i in K_T there is a Tr_i in K_{Tr} such that $Tr_i \subseteq T_i$. Let Tr be such that $Tr \in K_{Tr}$ and $Tr \subseteq T$. Since Tr is regular, $\sim A \vee B \in Tr$ and $A \in Tr$. Since Tr is normal, it is consistent, whence $\sim A \notin Tr$. But Tr is also prime, so $B \in Tr$. And so, $B \in T$, that is to say, B belongs to every prime regular theory. By the priming lemma, if a formula is in every prime regular theory, then it is a theorem, whence $\vdash B$.

3 We now go on to give a simple proof that **R4** conservatively extends **S4**. Before we can do so, we need to state a few definitions. First we give a standard definition of the material conditional, *viz.*,

D$\supset$ $A \supset B =_{df} \sim A \vee B$.

Moreover, we say that a formula A is *in the modal vocabulary* iff the only primitive connectives that occur in A are $\wedge$, $\sim$, and $\Box$.

Before we can show that **S4** is a subsystem of **R4**, we need to state the following fact:

Fact 15 The following is a theorem of **R4**:

$(A \rightarrow B) \rightarrow (A \supset B)$.

Corollary 16 *Let A be a theorem of* **S4** *in the modal vocabulary. Then A is a theorem of* **R4**.

Proof: First, we note that it is shown in [1] that the $\wedge$ and $\sim$ fragment of classical propositional logic is a fragment of **E** (see [1], §24.1.2). Since **E** is a subsystem of **R4**, every substitution instance of a classical tautology is valid in **R4**. Moreover, by the axioms of **R4** and Fact 15 above, $\vdash \Box (A \vee B) \supset (\Diamond A \vee \Box B)$, $\vdash \Box A \supset A$, and $\vdash \Box A \supset \Box\Box A$. Furthermore, by the main theorem of this paper, if $\vdash A$ and $\vdash A \supset B$, then $\vdash B$. Moreover, as we have said above, if $\vdash A$, then $\vdash \Box A$. Therefore, all the theorems of **S4** are theorems of **R4**.

Lemma 17 *If A is in the modal vocabulary and a theorem of* **R4**, *then A is a theorem of* **S4**.

Proof: Formulate **S4** in the full vocabulary of $\mathcal{L}$, identifying $\supset$ and $\rightarrow$ (i.e., interchanging them freely). It is clear that every theorem of **R4** is a theorem of **S4** thus formulated. So, restricting our attention to the modal fragment of this system, every formula in the modal vocabulary that is a theorem of **R4** is a theorem of **S4**.

Theorem 18 R4 *is a conservative extension of* **S4** *in the modal vocabulary.*

Proof: Follows directly from Corollary 16 and Lemma 17.

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NOTE

1. Note that $\Box^{-1} T_i$ is closed under $\wedge$I and wff $- \Diamond^{-1} T_i$ is closed under disjunction (i.e. if $A \in (\text{wff} - \Diamond^{-1} T_i)$ and $B \in (\text{wff} - \Diamond^{-1} T_i)$, then $A \vee B \in (\text{wff} - \Diamond^{-1} T_i)$). For this reason we use single a single formula from each of $\Box^{-1} T_i$ and wff $- \Diamond^{-1} T_i$ instead of a conjunction from the former and a disjunction from the latter.

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