Approximations and Logic

JEAN-PIERRE MARQUIS

Abstract   Scientists work with approximations almost all the time and have to reason with these approximations. Therefore, it is very natural to wonder if approximations influence logical deduction and if so, how. Our goal in this paper is to present a class of lattices which captures some intuitions concerning propositions expressing numerical approximations and consider how they relate to classical logic.

1 Basic assumptions   Our starting point is the relation between approximations and truth values. We assume that a proposition which expresses an approximation to a correct value receives a specific truth value reflecting the accuracy of this approximation. (We have proposed a formalization of this assumption in [8].) This is equivalent to saying that the notion of a partially true proposition makes sense. The next step in this framework consists in extending this attribution of partial truth-values to logically complex propositions. Formally, we want to construct a valuation function from propositions to a multivalent truth structure. In fact, we will do something more general. We will present a whole family of algebraic structures that we believe capture intuitively correct properties of partially true propositions the same way that the class of Boolean algebras captures essential properties of totally true (or false) propositions.

Let us first present an informal motivation that will lead us naturally to a valuation system proposed by Bunge [2]. (A subvaluation system of this system was first presented independently and in a different context by Slupecki [15]. See also Weston [18,19] for some of the results with different proofs and a different point of view.)

The originality and interest of Bunge's valuation system comes from his views about the (semantic) concept of negation (and implication, but the peculiarity of the latter is a consequence of the peculiarity of the former). There are different ways to extend the classical Boolean matrix for negation. After all, for a formal logical operation to qualify as a representation of the concept of negation, it has to satisfy two requirements:

(i) it has to be a unary operator;

(ii) it has to agree with classical negation in the "extreme" cases, i.e. when a proposition is totally true or totally false.

Many of the extensions of the concept have taken the following algebraic form: if we denote the truth value of a proposition p by $V(p)$, where the valuation function V has as target the real unit interval $[0, 1]$, and the negation operator by '$\dot{\neg}$', then $V(\dot{\neg}p) = 1 - V(p)$, where '1' denotes the complete truth. The major advantage of this definition is algebraic. For the equation $V(\dot{\neg}\dot{\neg}p) = V(p)$ is an immediate consequence of it. In other words, truth and falsity are symmetrical. This seems at first reasonable. But is it fair to put a proposition and its negation on a par? If the truth value of a proposition is 0.5, why would the negation also have to be 0.5? It is well known that a negated proposition most of the time "says" less than a nonnegated one and, hence, it is more secure. It is hard to find a nonalgebraic way to justify the above definition. It does not seem to reflect all our uses of negation. Bunge himself, after having accepted it in [2], [3] and [4], rejected it in [5] and [6]. He suggests that we consider negated propositions in a different way.

> Johnny turns 10 years old. His friend Peter believes Johnny to be 11, whereas his friend Jane suspects that he must be 9. Both are in error but not by much: their relative error is 1 in 10, so the truth value of each of their beliefs about Johnny's age can be taken to be $1.0 - 0.1 = 0.9$. Charlie, a third friend of Johnny's, is uncertain about his age and, being a very cautious person, avoids any risky estimates and states "Johnny in *not* 9 years old". He is of course right, and would also be right if Johnny were 8 or 11, 7 or 12, and so on. So, it is a bad mistake to assign his statement the truth value $1.0 - 0.9 = 0.1, \ldots$. (Bunge [5], p. 88)

In fact, the truth value of Charlie's statement should be 1, i.e. he is totally right. Clearly, most scientific propositions we are concerned with, namely, propositions expressing numerical approximations, behave the same way. Given the error involved in all verifiable propositions, we can safely claim that their negation is true. Thus, it is easier to hit on the truth with a negatively charged projectile than with a positive one. The only time a negated proposition is false is when the proposition is totally true. As soon as we move away from the truth, negation brings us right back to it.

This is in a way the most relaxed reading of the negation operator. Notice immediately that even though the above situation seems to be natural in the case of factual propositions, a more restricted reading of negation seems more natural in some areas of mathematics. Indeed, when one claims that a recursive function is not computable, one has to give a proof in the same way as one would have to give a proof, which would undoubtedly look different than the previous one, that the given function is computable. Hence, to establish the truth value of this kind of negated proposition is as hard as establishing the truth value of a positive proposition. This seems to indicate that there might conceivably be many different kinds of negations, as Wittgenstein has already suggested. Moreover, the "strong" interpretation of negation we have just mentioned might be also justifiable in terms of approximations. A topos of sheaves can be thought

of as capturing an approximation of a "part-whole" type. We will explore this possibility elsewhere.

We will now translate the above intuition concerning the behavior of the concept of negation in factual propositions in the following way. First, we will construct a valuation function from a set of propositions to the unit interval [0, 1]. This valuation function will attribute to the atomic propositions an arbitrary value in the interval. This is the given (relative) truth value of the propositions. Then, a negated proposition is equal to 1, i.e. true if and only if the truth value of the proposition is strictly less than 1. When the proposition is fully true, then its negation is false. We will now formalize these facts.

2 First formalization: The unit interval   Following Rasiowa and Sikorski [13] and Rasiowa [12], we define classical propositional logic in a purely syntactic way.

Let **L** denote the structure $\mathbf{L} = \langle \{p_i : i \in \mathbf{N}\}, \neg, \vee, \wedge, \rightarrow \rangle$, where the last four symbols are the usual symbols for the propositional connectives and $\{p_i : i \in \mathbf{N}\}$ is a denumerable set of propositional variables. The formulas of **L** are defined inductively as usual.

Then $L = \langle \mathbf{L}, \vdash_L \rangle$ with the usual axioms is the classical propositional logic, where '$\vdash_L$' is the classical consequence or deducibility relation with its usual properties.

Define $T_L = \{A : \vdash_L A\}$, the set of *theorems* of L and 'A' is a metavariable denoting well-formed formulas of L.

This is all we need as far as the logic is concerned.

Following McKinsey and Tarski [10], we define a valuation system M as an $n + 2$ tuple $\langle M, D, f_1, \ldots, f_n \rangle$, where M is a nonempty set with at least two elements, D is a subset of M with at least one element (usually called the 'designated elements'), and the f_i's are functions from $M^{|(f_i)}$ to M, where $|(f_i)$ is the arity of f_i.

Let M_B be defined by $\langle [0, 1], \{1\}, \max, \min, \neg, \rightarrow \rangle$, where

(i) [0, 1] is the real unit interval;
(ii) max and min are the usual binary functions defined on [0, 1];
(iii) $\neg : [0, 1] \rightarrow \{0, 1\}$ is defined by

$$\neg(x) = \begin{cases} 0 & \text{iff } x = 1 \\ 1 & \text{iff } x < 1, \end{cases}$$

where x is an element of the real unit interval;

(iv) $\rightarrow : [0, 1]^2 \rightarrow [0, 1]$ is defined by

$$\rightarrow(x, y) = \begin{cases} y & \text{iff } x = 1 \\ 1 & \text{iff } x < 1. \end{cases}$$

We should point out that the functions $\neg$ and $\rightarrow$ are interdefinable. Indeed, $\neg(x)$ can be defined as $\rightarrow(x, 0)$ and $\rightarrow(x, y)$ can be defined by $\max(\neg(x), y)$. The following two functions give us the relationship between L and M_B. Let

$\pi = \{ p_i : i \text{ in } N \}$ and $\pi^* \subseteq \pi$ (remember that our evaluation function is a partial function).

Definitions

(i) The function $f : \pi^* \to [0, 1]$ is called an *assignment* function from propositional variables to elements of the unit interval.

(ii) f induces a valuation $\nu_f : WFFS \to [0, 1]$ as follows: Let $A, B, C, \ldots$ denote wffs of L, then

    (1) if A is atomic, i.e. $A = p_i$ for some i, then $\nu_f(A) = f(A) \in [0, 1]$;

    (2) $\nu_f(\neg A) = \neg(\nu_f(A))$;

    (3) $\nu_f(A \wedge B) = \min(\nu_f(A), \nu_f(B))$;

    (4) $\nu_f(A \vee B) = \max(\nu_f(A), \nu_f(B))$;

    (5) $\nu_f(A \to B) = \to(\nu_f(A), \nu_f(B))$.

At this point, we could start proving some facts about the relations between L and the valuation system. However, we will take a step towards abstraction and generalization, for the valuation system is a model of an abstract algebraic system, and thus all the results can be established for a class of models instead of a particular system.

3 Algebraic semantics: Brouwerian lattices and Bunge algebras   In this section, we will consider two classes of algebras which include Boolean algebras and which behave "well" relative to classical logic, in the sense that every member of this class is a model, under the appropriate interpretation, for the tautologies of classical propositional logic. Moreover, our structure above is a member of these classes of algebras. The elements of the first class are essentially sup-complemented distributive lattices with a unit and satisfying both De Morgan's laws. They are the dual of Stone algebras. Because of their particular relationship to classical logic, these algebras deserve a name of their own. We will call them *Bunge algebras*. The second class consists of the standard Brouwerian lattices.

In all the following, $L = \langle |L|, \cup, \cap \rangle$ will denote a lattice, i.e. a set $|L|$ closed under the operations $\cup$ and $\cap$ and satisfying the standard axioms.

Definition 1

(i) An element c in $|L|$ is said to be the *u-complement* or the *sup-complement* of an element a in $|L|$ if c is the least element such that $a \cup c = 1$ (i.e. if c is the least element in the set of all x in $|L|$ such that $a \cup c = 1$ or, equivalently, the $\cup$-complement of a is an element c such that

    (1) $a \cup c = 1$.

    (2) if $a \cup b = 1$ then $c \leq b$).

    Thus, the $\cup$-complement is the first or smallest element of the family of all elements joined to a.

(ii) a distributive lattice L with a unit 1 such that for any a in L there exists the $\cup$-complement, denoted by $\neg a$, of a is called a *Brouwerian lattice*.

Note that in any Brouwerian lattice, the following de Morgan's identity holds:

(1) $\neg(a \cap b) = \neg a \cup \neg b$.

(See Ribenboim [14] for an axiomatization of Brouwerian lattices and a proof of the identity.)

We will now prove a short lemma that will be useful later.

Lemma 1   *In any Brouwerian lattice, the following inequality holds*

$$\neg(a \cup b) \leq \neg a \cap \neg b.$$

Proof: By definition of $\cup$-complement, we have $(a \cup b) \cup \neg(a \cup b) = 1$. But also $(a \cup b) \cup \neg a = 1$, by associativity twice, commutativity, and definition. Thus $\neg(a \cup b) \leq \neg a$ and similarly $\neg(a \cup b) \leq \neg b$ and we get the result by a standard lattice-theoretic fact.

Definition 2   A distributive lattice $L = \langle |L|, \cup, \cap \rangle$ with a unit 1 is called a *Bunge algebra* if the following properties hold:
(1) L is a Brouwerian lattice;
(2) $\neg(a \cup b) = \neg a \cap \neg b$.

Thus a Bunge algebra is a Brouwerian lattice for which the above inequality is a strict equality. Examples abound. Any finite distributive lattice is a Bunge algebra. So is any Boolean algebra, with the Boolean complement being the supcomplement. Of course, our valuation system is also a Bunge algebra, as is easily verified. In fact any linear lattice is a Bunge algebra.

Brouwerian lattices were used early in algebraic logic but with '0' as the designated value (see McKinsey and Tarski [9]). We use '1' as the designated value.

We will now first show that the set of tautologies in an arbitrary Bunge algebra is the same as the set of classical (Boolean) tautologies, provided we interpret the connectives properly.

Definition 3
 (i) A mapping $f: \pi^* \to M$ from the set of propositional variables to the underlying set of a Bunge algebra $\mathbf{B} = \langle M, \cup, \cap, \neg \rangle$ is called an *assignment* function;
(ii) a map $v_f: WFFS \to M$ is a *valuation* if
     (1) if A is atomic, i.e. $A = p_i$ for some i, then $v_f(A) = f(A) \in M$;
     (2) $v_f(\neg A) = \neg(v_f(A))$;
     (3) $v_f(A \wedge B) = v_f(A) \cap v_f(B)$;
     (4) $v_f(A \vee B) = v_f(A) \cup v_f(B)$;
     (5) $v_f(A \to B) = (\neg(v_f(A)) \cup v_f(B)$;
     where $A, B, C, \ldots$ denote wffs.

Definition 4
 (i) A wff A is *valid* in a given Bunge algebra $\mathbf{B}$ under an assignment f if $v_f(A) = 1$ in $\mathbf{B}$;
(ii) A wff A is a *tautology in a given Bunge algebra* $\mathbf{B}$ if it is valid for all valuations v_f in $\mathbf{B}$;
(iii) A wff A is a *tautology* if it is a tautology in all Bunge algebras.

Notation: we will denote the class of valid formulas in a given Bunge algebra $\mathbf{B}$ by '$V_{\mathbf{B}}$' and the class of all tautologies defined above by 'Tt_{Bg}' and the class of all classical tautologies by V_{CPL}.

A crucial property of Bunge algebras is that we can define the notion of a

conjunctive normal form for its polynomials. (See [13] or Balbes and Dwinger [1] for definitions of polynomials in an arbitrary lattice.)

Definition 5   A Bunge polynomial f is said to be in *conjunctive normal form* if it is a meet of joins (i.e. a conjunction of disjunctions) of formulas which are either atomic or negatomic or double negations of atomic formulas.

Lemma 2   *Every Bunge polynomial is equal to a polynomial in a conjunctive normal form.*

Proof: The proof is exaclty as in the case of Boolean algebras, the only difference being that $\neg\neg a \neq a$ in a Bunge algebra and hence we are not allowed to eliminate double negations in front of atomic formulas.

We are now ready to prove our first important result.

Proposition   *Let $\mathbf{B} = \langle M, \cup, \cap, \neg \rangle$ be a Bunge algebra. Then, under the valuation function v_f defined above, $V_{\mathbf{B}} = V_{CPL}$ for any f, i.e. the set of valid formulas over an arbitrary Bunge algebra is the same as the classical two-valued tautologies.*

Proof: Clearly, any formula in V_B is in V_{CPL}, for any Bunge algebra contains at least one Boolean subalgebra, namely the subalgebra consisting of the unit and zero elements and thus if the formula is valid for any Bungean valuation, it is valid for any Boolean valuation. For the opposite direction, we will use conjunctive normal forms. Let A be a classical tautology. If it is already in (Boolean) conjunctive normal form, we do nothing. Since it is a classical tautology, every disjunct contains an atomic variable together with its negatomic counterpart. By Definitions 2 and 3, A will be valid in $\mathbf{B}$. If A is not in (Boolean) c.n.f., we write its Bunge c.n.f. Since, as we have indicated in the proof of the above lemma, the only difference between a classical Boolean c.n.f. and its Bungean counterpart is the presence of doubly negated formulas instead of atomic ones, every disjunct contains either an atomic formula together with its negated counterpart or a negated proposition together with a double-negated proposition and hence it is valid in $\mathbf{B}$.

Corollary $Tt_{Bg} = V_{CPL}$.

Proof: $\mathbf{B}$ was arbitrary in the above proposition.

For the record, we will exhibit a nonlinear Bunge algebra. Consider the following lattice with six elements.

(ii)
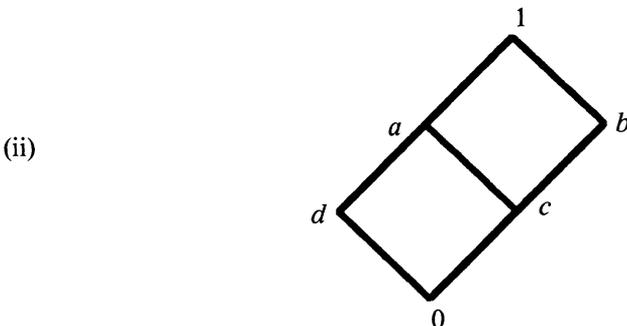
It is obvious from the diagram that $\neg a = b$, $\neg b = d$, $\neg d = b$, $\neg c = 1$. Now, we claim that for all elements of the lattice, $\neg(p \cup q) = \neg p \cap \neg q$. For instance, $\neg(a \cup b) = \neg 1 = 0 = b \cap d = \neg a \cap \neg b$. The other cases are just as easy and are left to the reader. Hence this lattice, when used as a valuation system for classical logic with the above valuation function, yields the classical tautologies.

Surprisingly enough, Bunge algebras happen to be related to a well-known class of algebraic structures, namely the Stone algebras. Suffice it to mention here that a Bunge algebra is the dual of a Stone algebra. This follows from the fact that a Bunge algebra is a dually pseudocomplemented distributive lattice, i.e. a Brouwerian lattice B satisfying the identity:

$$\neg a \cap \neg \neg a = 0.$$

However, in a Stone algebra one considers inf-complementation instead of sup-complementation, and therefore not all tautologies of classical logic are equal to the unit of the algebra in this case. The simple formula $\neg p \cup p$ is not always equal to the unit. Thus, even though Stone algebras and Bunge algebras are structurally dual, it makes a difference which one chooses to work with.

It is natural to wonder whether the condition characterizing Bunge algebras, that is the distributivity of negation over disjunction, is necessary for the above result in hold. In fact, it is not! Instead of considering a Bunge algebra, we can consider a Brouwerian lattice.

Proposition   *Let* **Br** $= \langle N, \cup, \cap, \neg \rangle$ *be a Brouwerian lattice. Then, under the valuation function n_f defined above (same definition), $V_{\mathbf{Br}} = V_{CPL}$ for any f, i.e. the set of valid formulas over an arbitrary Brouwerian lattice is the same as the classical two-valued tautologies.*

Proof: Clearly, $V_{\mathbf{Br}} \subset V_{CPL}$, by the same argument as above. To show the converse, we argue as follows. Let A be a classical tautology. If it is in c.n.f., then it will be valid in **Br,** by definition of a Brouwerian lattice and the definition of our valuation function. If it is not in c.n.f., then we are no longer in a position to transform it, for now $\neg(a \cup b) \leq \neg a \cap \neg b$. However, this is the only thing which might turn out wrong: the Bungean c.n.f. equal to A might be valid whereas A is less than the maximal element in a Brouwerian lattice. So what we have to show is that whenever $\neg a \cap \neg b = 1$, which happens when A is a classical tautology, then $\neg(a \cup b)$ was already equal to the unit. Suppose $\neg(a \cup b) \neq 1$, say $\neg(a \cup b) = c < 1$. By definition we have

$$(a \cup b) \cup \neg(a \cup b) = (a \cup b) \cup c = 1.$$

So, $(a \cup b) \cup c = a \cup (b \cup c) = b \cup (a \cup c) = 1$, by associativity and commutativity and thus $(b \cup c) \geq \neg a = 1$ and $(a \cup c) \geq \neg b = 1$, by definition of the $\cup$-complement again and the fact that $\neg a \cap \neg b = 1$ iff $\neg a = 1$ and $\neg b = 1$. Therefore $(b \cup c) = 1$ and $(a \cup c) = 1$ and hence, once more, $1 = \neg a = \neg b \leq c$ and so $c = 1$, a contradiction.

One last but crucial remark about the interpretation of classical logic in Bunge algebras and Brouwerian lattices: even though we are able to preserve the set of classical tautologies, we *do* lose many things. Firstly, the set of contradictions is not the same. For instance, the formula $\neg A \wedge A$ is not a contradiction

in general. Secondly, we also lose many classical equivalences. This is a consequence of the interplay between the negation operator and the other operators. Here is a sample of formulas which are no longer equivalent in this setup: $A \wedge B$ and $\neg(\neg A \vee \neg B)$; $A \vee B$ and $\neg(\neg A \wedge \neg B)$; $A \rightarrow B$ and $\neg B \rightarrow \neg A$; $\neg\neg A$ and A. Many authors (for instance Weston [18], [19] and Tobar-Arbulu [16]), believe that this is a serious drawback. It is not so clear to us. We will turn to this question after we have considered a crucial modification of our interpretations in Bunge algebras and Brouwerian lattices.

4 Bunge algebras, Brouwerian lattices and implication   Our definition of the implication operator above is a normal extension of the classical Boolean definition. It has the advantage of preserving all classical tautologies. However, we have not at any point given an intuitive justification of this definition. The fact that it is a natural extension of the Boolean definition seems to support it. But there are other ways to define the implication operator, one of which does qualify as being as natural, if not more so, than the one we have presented. Moreover, the alternative fits more closely the semantics, is closer to our intuition of how an implication operator should be formalized, and is now standard in the literature. We will now present this alternative, indicate some of its logical consequences, and briefly discuss its value. We will come back to the question of the choice between the definitions in the next section.

In a Heyting algebra, the implication operation is best defined as a functor, more precisely as the right adjoint to the product functor, which is simply the join operation in a lattice. Formally, this amounts to the following:

($) $\quad a \cap c \leq b \quad$ if and only if $c \leq a \rightarrow b$.

Does this hold for Bunge algebras and Brouwerian lattices? In other words, is a Bunge algebra, considered as a category, cartesian closed? Is a Brouwerian lattice cartesian closed? In a Heyting algebra, the element $a \rightarrow b$ is defined as the supremum of all elements c such that $a \cap c \leq b$. This is well-defined since arbitrary suprema are allowed in a Heyting algebra. We can immediately see that the definition we used in the previous section does not necessarily satisfy ($). For if we set $a \rightarrow b := \neg a \cup b$, then the left-hand side of the equivalence is not necessarily satisfied, since $a \cap \neg a$ is not necessarily equal to 0 in a Bunge algebra or in a Brouwerian lattice. In fact, it is satisfied if and only if the algebra is Boolean, as the reader can immediately check. Thus, it seems necessary to modify the class of lattices considered in order to define the above operation. However, we will take a short cut. Instead of considering Bunge algebras, we consider the intersection of the class of Brouwerian lattices and Heyting algebras. This intersection is not empty, for it contains all Boolean algebras and at least all linear lattices. In other words, it includes at least self-dual lattices. Let us first look at linear lattices, e.g. the real unit interval, first. Given a linear lattice, the implication operator can be defined as follows:

$$\nu_f(A \rightarrow B) = \begin{cases} \nu_f(B) & \text{if } \nu_f(A) > \nu_f(B) \\ 1 & \text{otherwise, i.e. } \nu_f(A) \leq \nu_f(B). \end{cases}$$

With this new definition (the other operators are defined as in the previous section), not all theorems of classical logic are true. The most obvious, and probably the most disturbing, are

$$A \to \neg\neg A \text{ and}$$

$$(\neg A \to B) \to ((\neg A \to \neg B) \to A).$$

The reason is clear: If $\nu_f(A)$ is strictly between 0 and 1, then $\nu_f(\neg\neg A)$ is equal to 0 and therefore $\nu_f(A \to \neg\neg A)$ is also 0. However, if A is Boolean, that is if it takes a Boolean value, which in the case of linear lattices are simply 0 and 1, then $A \to \neg\neg A$ *is* true. It is easy to see that the second formula is bound to a similar fate. Let $0 < \nu_f(B) < 1$ and let $\nu_f(A) \neq 1$, then $\nu_f((\neg A \to B) \to ((\neg A \to \neg B) \to A)) \neq 1$; that is if B is not Boolean, then the formula is not true. What is interesting is that the converse also holds. Thus, $\nu_f((\neg A \to B) \to ((\neg A \to \neg B) \to A))$ is true if and only if $\nu_f(B)$ is Boolean, as the reader can easily check. These remarks might seem trivial at first, but we believe on the contrary that they show that we have not lost as much as we thought. The formula expressing the *reductio* argument might not hold for all possible values anymore, but it still holds for the only operative values, for the foregoing observation says that whenever we can derive a genuine contradiction (and in a Brouwerian lattice this means that $\nu_f(B)$ is Boolean) from a negated premise then we can derive the nonnegated premise. On the other hand, whenever $\nu_f(B)$ is not Boolean, then $\nu_f(B) \cap \nu_f(\neg B)$ is not a contradiction and therefore we cannot use the reductio form. Thus it is tempting to modify the axioms of classical logic as follows:

Ax (i) $A \to (B \to A)$
Ax (ii) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
Ax (iii) $(\neg A \to \neg\neg B) \to ((\neg A \to \neg B) \to A),$

since now we are guaranteed that $\neg B$ will receive a Boolean value in the semantics. But in practice it is enough to know that B does not express an approximation. Needless to say, the above axioms do not constitute a complete list of axioms, since in an arbitrary Bunge algebra the logical connectives are not interdefinable, as should be clear from the previous section. We will give a simple proof of (ii), using the adjunction ($) and leave the remaining axioms to the reader. First observe that by substituting $a \to b$ for c in ($) above, we get

(!) $a \cap (a \to b) \leq b$ if and only if $a \to b \leq a \to b$.

Since the right-hand side is always true in a lattice, we get the left-hand side. Now, we start from a true proposition in any lattice and apply the previous observation and the adjunction as follows:

(1) $c \cap d \leq c$ *is always true*;
(2) $(b \cap (b \to c)) \cap d \leq c$ by (!), (1), and transitivity
(3) $((a \cap (a \to b)) \cap (b \to c)) \cap d \leq c$ by (!) again
(4) $((a \cap (a \to b)) \cap ((a \cap (a \to (b \to c)))) \cap d \leq c$ by (!) again
(5) $(((a \to b) \cap a) \cap ((a \cap (a \to (b \to c)))) \cap d \leq c$ by commutativity of $\cap$
(6) $((a \to b) \cap (a \cap a) \cap (a \to (b \to c))) \cap d \leq c$ by associativity of $\cap$
(7) $(((a \to b) \cap a) \cap (a \to (b \to c))) \cap d \leq c$ by $a \cap a = a$

(8) $((a \cap (a \to b)) \cap (a \to (b \to c))) \cap d \le c$ by commutativity of $\cap$

(9) $a \cap ((a \to b) \cap (a \to (b \to c)) \cap d) \le c$ by associativity

(10) $(a \to b) \cap (a \to (b \to c)) \cap d \le a \to c$ by ($)

(11) $d \le (a \to (b \to c)) \to ((a \to b) \to (a \to c))$ by ($) twice,

and since d is arbitrary, the right-hand side has to be the unit and we are done.

The observation preceding the proof immediately shows that modus ponens also holds under this new definition. Furthermore, the deduction theorem also holds in this framework. For now $A \to B$ is a theorem if and only if $A \le B$, so we simply have to interpret the entailment relation as the order relation in the lattices, as usual. Observe also that we can still prove the proposition $\neg\neg A \to A$, since its proof in classical logic depends on our Axiom (iii). However, we cannot prove the converse $A \to \neg\neg A$, since the proof depends on the Boolean version of Axiom (iii). For similar reasons, the formula $(\neg B \to \neg A) \to (A \to B)$ cannot be proved in general but the converse $(A \to B) \to (\neg B \to \neg A)$ can. (See Mendelson ([11], pp. 33–35) for standard proofs in classical propositional logic.) We will close this section with a conjecture: if one defines the relation of logical equivalence in the standard way, that is A and B are logically equivalent if and only if $A \to B$ and $B \to A$ are theorems of the system, then the Lindenbaum–Tarski algebra of the system of axioms above is a Brouwerian lattice.[1] If our conjecture is correct, then a completeness proof becomes elementary, but we prefer to ignore the last problem for the moment.

5 Concluding remarks   We now seem to be facing an alternative: either we pick the definitions which preserve all classical tautologies but destroy the classical logical equivalences in the semantics or we choose the definitions which allow what seems to be a reasonably good fit between the syntax and the semantics but shrink the class of (genuine) tautologies. At this stage, we believe that the second choice is preferable. Our argument is simple: we have not really lost anything by moving to this new system. In fact, the new semantics might allow a finer classification of propositions than the classical one. One might want to consider quasi-tautologies, i.e., formulas that are tautologies if one of their variables is Boolean. Similarly, we might consider quasi-contradictions. Be that as it may, at the operative level, we are still working with a "full-blown" classical logic.

One last word about logical equivalence and the results of Section 3. We have pointed out at the end of Section 3 that even though all classical tautologies are valid in a Bunge algebra under the appropriate interpretation of the implication operator, we lose many classical equivalences. Tobar-Arbulu [16] claimed that these inequalities are contradictions. We have already replied to these claims and therefore we will not go over this issue again (see Marquis [7] and Tobar-Arbulu's reply in Tobar-Arbulu [17]. The real issue is whether or not this loss of equivalence is crucial. According to Weston [18], who has independently established the fact that the valuation system presented in the second section determines the same tautologies as classical logic, it is essential. He even raises it to the level of a principle and rejects the valuation system because it violates it. He formulates his principle as follows:

(EP)   If φ and ψ are logically equivalent, then $\| \varphi \| = \| \psi \|$,

$\| - \|$ denotes a valuation function from a deductive system to the unit interval [0, 1]. (We have to point out that this is one form of his principle. It is also enunciated in terms of approximate truth. But the latter is more complex and is essentially the same as the above.)

Should we accept this principle? If so, on what grounds?[2] The motivation behind the equivalence principle seems reasonable: "it is highly desirable to assure that a statement's degree of accuracy does not depend on details of how it is formulated" ([18], p. 206). The whole question here depends on the words "on details of how it is formulated". Does it refer to details of the grammatical form? The mathematical formulation? The logical formulation? It seems to us that what is at stake here is the type of "linguistic" transformations allowed and the properties preserved by these transformations. It is not clear to us that the property of "being logically equivalent" constitutes the correct set of transformations and even less clear that "being classically logically equivalent" is appropriate. It might be too general or simply insensitive to the degree of accuracy of a formula.

Notice that in some cases the way a proposition is formulated *is* crucial with respect to some properties, particularly in the sciences.

> . . . experimental accuracy depends not only upon the quality of the instrumental, e.g. the laboratory equipment, but also upon the mathematical form of the formulas being checked — which form is to some extent conventional. A simple example will suffice to make this point. Suppose the task is to check a theoretical formula of the form "$y = ax/(b + x)$". This task is complicated by the fact that a is the asymptotic value of y,.... To facilitate the task, one performs the simple trick of transforming the given formula into a linear equation by means of the variable changes $x = 1/u$ and $y = 1/v$, which ensues in "$v = (1/a) + (b/a)u$". Now a is the reciprocal of the ordinate value at the origin, usually an accessible number. (Bunge [6], p. 127)

One way to justify the equivalence principle is as follows: if the meaning of a proposition is given by the class of propositions which entail it and all propositions it entails, then two propositions which are deducible from one another must mean the same thing. Therefore, they must always have the same truth value and hence must be equivalent. In our framework, this cannot be acceptable, for our valuation function is a partial function. Hence it is possible to have logically equivalent formulas such that one of them will get a truth value while the other or others will not.

Why should there be such a close relationship between the equivalence classes induced by the deducibility relation and the equivalence classes induced by the relation "having the same truth value"? The answer is well known: because the consequence relation preserves truth values. Given a proposition P with truth value t, if you deduce Q from P, then you can be sure that Q has also the truth value t. It is the very essence of the consequence relation to do that. We even *define* the consequence relation in this way: "a proposition follows from a set of propositions" means that if the premises are true then the conclusion is necessarily true. Therefore, when there is only one premise and moreover when this premise can be deduced from the conclusion alone, we see that in a bivalent context both propositions have to have the same truth value. It is still not entirely clear to us that this has to be extended to the nonbivalent case.

One last word about the relations between truth and logic. It seems to us reasonable to assume that scientists do in fact use classical logic. This is reasonable since they use classical mathematics all the time, and classical logic is built into it. Moreover, when a scientist reasons, she does not know all the truth values of the propositions she uses in her reasonings. In some cases, she does not know the truth value of any of the propositions involved, e.g. when she is investigating a new theory and trying to deduce a verifiable consequence. When she does that, we can presume that she assumes her premises to be fully true. In other words, the actual truth value of a proposition is irrelevant in logic. The standard definition of the consequence relation is almost a counterfactual: in some cases we do know the truth value of the propositions involved, but in many cases, at least in the factual sciences, we do not have the faintest idea of the actual truth value of the propositions. But when we assume that a proposition is true, we simply fall back in the classical Boolean territory.

NOTES

1. Two points have to be made. Firstly, how does the logical system have to be modified such that the resulting Lindenbaum–Tarski is a Bunge algebra? Secondly, this opens a possibility which might be interesting. In some toposes, the subobject classifier is such that we can define a "nonstandard" negation, namely the dual of the standard morphism. We then get a new logic. The question is then obvious: How does this "new" logic influence the kind of mathematics which can be done in such a framework? We will consider this question elsewhere.

2. We have to point out that Weston does come up with a new logical system. But his system is at odds with scientific practice, since it does not preserve modus ponens. We find this hard to accept.

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Département de Philosophie
Université de Montréal
Québec, Canada
H3C 3J7