

Decision Procedures for Logics of Consequential Implication

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Abstract The paper introduces a new kind of implication named “consequential implication”, which is a variant of traditional connexive implication lacking a certain monotonicity property and allowing the distinction between analytic and synthetic conditionals. A system CI.0 for analytical consequential implication is proved to be definitionally equivalent to Feys-von Wright system T, and so decidable by the standard tableaux method. Three systems, named CI*0, CI*1, and CI*2, are then introduced as axiomatic and linguistic extensions of CI.0 in which synthetic conditionals are definable. It is shown that these three systems may be translated into certain extensions of T whose language contains new linguistic objects named “quasi-variables”. Since the latter systems are proved to be decidable by the tableaux method, it follows that this method gives a decision procedure also for the related systems of consequential implication.

1 Introduction The aim of this paper is to give practical decision procedures for a family of logics which will be named logics of *consequential implication*. Since the method to be proposed is a refinement of the method of semantic tableaux for propositional modal logic, a familiarity with the first part of Hughes and Cresswell [6] is presupposed.

Logics of consequential implication aim to be a viable alternative to logics of the Stalnaker-Lewis kind. A philosophical framework for the present analysis is given in Pizzi [11]–[13].

The minimal characterizing properties of any relation which we are willing to identify as of consequential implication (CI) are the following:

- (a) If p CI q then it is false that p CI $\neg q$ (Boethius' Thesis)
- (b) It is not a logical truth that $(p \wedge q)$ CI q
- (c) The law of monotonicity—i.e. that p CI q implies $(p \wedge r)$ CI $(q \wedge r)$ —does not hold.
- (d) The logic of analytical consequential implication is not coincident with the logic of synthetic consequential implication.

Points (a) and (b) are strictly interlinked: if we had $(p \wedge q)$ CI q , then we would have both $(p \wedge \neg p)$ CI p and $(p \wedge \neg p)$ CI $\neg p$, which would be a coun-

terexample to (a). For the same reason, Duns Scotus's law $(p \wedge \neg p) \text{ CI } q$ cannot be a theorem for consequential implication. In a two-valued framework, a desirable feature of consequential implication is, therefore, that only a contradiction should consequentially imply another contradiction and be consequentially implied by it.

(a) and (b) are well-known features of so-called connexive implication (see Angell [1] and McCall [8] and [9]), but (c) and (d) are not. As regards (c): the law of monotonicity is an axiom in McCall's systems.¹ But let us suppose that the symbol " \rightarrow " stands for a logical-or-analytical CI-relation. Let us suppose that p stands for "Philip is a bachelor", q for "Philip is a male", and r for "Philip is married". Then a reading of $(p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r))$ is: "if Philip is a bachelor he is male" implies "if Philip is a bachelor and is married he is a married male". But the antecedent here is analytically true (it could be a theorem of any logic extended with meaning postulates) while the consequent is unacceptable from a connexive viewpoint, since a contradiction here consequentially implies a contingent statement. Analytical consequential implication should, therefore, exclude the law of monotonicity, even if it should admit its weakened form $\diamond(p \wedge r) \supset ((p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r)))$.

As regards (c): connexive logics have been formulated with no attention to the role of presuppositions in consequential reasoning, and so they are unfit to render the peculiarities of ordinary synthetic conditionals, especially of the counterfactual kind. In this respect consequential implication is more in the spirit of Reichenbach [14].

2 Definitional equivalence of CI.0 and T Let us call CI.0 the following system of analytical consequential implication, whose axioms are subjoined to the standard propositional calculus PC (" \perp " is an abbreviation for " $p \wedge \neg p$ " and " \top " for " $p \vee \neg p$ "):

- Ax(a)** $((p \rightarrow q) \wedge (q \rightarrow r)) \supset (p \rightarrow r)$
- Ax(b)** $((p \wedge \neg q) \rightarrow \perp) \wedge \neg(p \rightarrow \perp) \wedge \neg(\neg q \rightarrow \perp) \supset (p \rightarrow q)$
- Ax(c)** $\neg((p \wedge r) \rightarrow \perp) \supset ((p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r)))$
- Ax(d)** $(\neg p \rightarrow \neg q) \supset (q \rightarrow p)$
- Ax(e)** $(p \rightarrow \perp) \supset (\perp \rightarrow p)$
- Ax(f)** $(\perp \rightarrow p) \supset (p \rightarrow \perp)$
- Ax(g)** $(p \rightarrow q) \supset \neg(p \rightarrow \neg q)$
- Ax(h)** $p \rightarrow p$
- Ax(i)** $(p \rightarrow q) \supset (p \supset q)$.

Rules Modus Ponens (MP) for \supset ; Uniform Substitution (US); Replacement of Proved Material Equivalents (Eq).

Definitions $\Box A =_{df} \top \rightarrow A$; $\Diamond A =_{df} \neg \Box \neg A$; $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$.

A decision procedure for CI.0 can be simply given showing that CI.0 is definitionally equivalent to the well-known Feys-von Wright system T. The definition in question is $A \rightarrow B =_{df} \Box(A \supset B) \wedge (\Diamond B \supset \Diamond A) \wedge (\Box B \supset \Box A)$. Thus the theorem to be proved is the following:

- T1** $T + Def \rightarrow$ is equivalent to CI.0 + $Def \Box$.
- L1** Every thesis of T + $Def \rightarrow$ is a thesis of CI.0 + $Def \Box$.

We have to prove that the axioms and the rules of T are theorems of CI.0. The first axiom of T, $\Box q \supset q$, is proved in the following way:

- (1) $(p \rightarrow q) \supset (p \supset q)$ Ax(i)
- (2) $(\top \rightarrow q) \supset (\top \supset q)$ (1), \top/q
- (3) $\top \supset ((\top \rightarrow q) \supset q)$ (2), PC
- (4) $(\top \rightarrow q) \supset q$ (3), MP
- (5) $\Box q \supset q$ (4), Def \Box .

In order to prove the second axiom, $\Box(p \supset q) \supset (\Box p \supset \Box q)$, we have first to prove $\Box(p \wedge q) \supset \Box q$, namely, $(\top \rightarrow (p \wedge q)) \supset (\top \rightarrow q)$. The proof makes use of $\Box p \supset \Diamond p$, which easily follows from the preceding theorem.

- (1) $(p \wedge \neg q \wedge q) \rightarrow \perp$ $\vdash \perp \rightarrow \perp, \vdash_{PC} q \wedge \neg q \equiv p \wedge \neg q \wedge q$, Eq
- (2) $((1) \wedge \Diamond(p \wedge q) \wedge \neg \Box q) \supset ((p \wedge q) \rightarrow q)$ Ax(b), $p \wedge q/p$
- (3) $(\Diamond(p \wedge q) \wedge \neg \Box q) \supset ((p \wedge q) \rightarrow q)$ (1), (2), MP
- (4) $((\top \rightarrow (p \wedge q)) \wedge (\top \rightarrow q)) \supset ((p \wedge q) \rightarrow q)$ Ax(a), $\vdash_{CI.0} \top \rightarrow p \equiv p \rightarrow \top$
- (5) $(\Box(p \wedge q) \wedge \neg(\top \rightarrow q)) \supset ((p \wedge q) \rightarrow q)$ (3), $\vdash \Box p \supset \Diamond p$
- (6) $(\top \rightarrow (p \wedge q)) \supset ((p \wedge q) \rightarrow q)$ (4), (5), PC
- (7) $\Box(p \wedge q) \supset (((p \wedge q) \rightarrow q) \wedge \Box(p \wedge q))$ (6), PC
- (8) $((\top \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow q)) \supset (\top \rightarrow q)$ Ax(a), $\top/p, p \wedge q/q, q/r$
- (9) $\Box(p \wedge q) \supset \Box q$ (7), (8), PC, Def \Box .

$\Box(p \supset q) \supset (\Box p \supset \Box q)$ then follows by the following proof:

- (1) $\Diamond p \supset ((\top \rightarrow q) \supset (p \rightarrow (p \wedge q)))$ Ax(c), $\top/p, p/r, \vdash_{PC} \top \wedge p \equiv p$
- (2) $(\top \rightarrow p) \supset ((p \rightarrow (p \wedge q)) \supset (\top \rightarrow (p \wedge q)))$ Ax(a), $\top/p, p \wedge q/r, p/q$
- (3) $\Diamond p \supset ((\Box q \wedge \Box p) \supset \Box(p \wedge q))$ (1), (2), PC, Def \Box
- (4) $(\Box p \wedge \Box q) \supset \Box(p \wedge q)$ (3), $\vdash \Box p \supset \Diamond p$, Ax(i), PC
- (5) $(\Box p \wedge \Box(p \supset q)) \supset \Box(p \wedge (p \supset q))$ (4), $p \supset q/q$
- (6) $(\Box p \wedge \Box(p \supset q)) \supset \Box(p \wedge q)$ (5), $\vdash_{PC} (p \wedge (p \supset q)) \equiv p \wedge q$
- (7) $\Box(p \wedge q) \supset \Box q$ Preceding Theorem
- (8) $(\Box p \wedge \Box(p \supset q)) \supset \Box q$ (6), (7), PC
- (9) $\Box(p \supset q) \supset (\Box p \supset \Box q)$ (8), PC.

The proof that rules MP, US, Nec. of T are derived rules of CI.0 is trivial and will be omitted. We also omit the proof of the equivalence $p \rightarrow q \equiv (\top \rightarrow (p \supset q)) \wedge (\neg(\top \rightarrow \neg q) \supset \neg(\top \rightarrow \neg p)) \wedge ((\top \rightarrow q) \supset (\top \rightarrow p))$, corresponding to the definition Def \rightarrow , which can be shown to be a theorem of CI.0.

Lemma 2 *Every thesis of CI.0 + Def \Box is a thesis of T + Def \rightarrow .*

The proof of the lemma makes use of the well-known decision procedure for T. The procedure begins by replacing in the axioms of CI.0 any occurrence of $A \rightarrow B$ with $\Box(A \supset B) \wedge (\Diamond B \supset \Diamond A) \wedge (\Box B \supset \Box A)$, and goes on decomposing the resulting formula in a suitable conjunction, testing the conjuncts by T-tableaux and then applying the PC law known as *Theorema Praeclarum*:

$$(TPr) \quad ((p \supset q) \wedge (r \supset s)) \supset ((p \wedge r) \supset (q \wedge s)).$$

Axiom (d), for instance, may be simply proved by showing that the following wffs are theorems of T:

- (di) $\Box(p \supset q) \supset \Box(\neg q \supset \neg p)$
- (dii) $(\Box q \supset \Box p) \supset (\Diamond \neg p \supset \Diamond \neg q)$
- (diii) $(\Diamond q \supset \Diamond p) \supset (\Box \neg p \supset \Box \neg q)$

and then applying TPr.

Notice that all the axioms of CI.0 with the exception of Ax(f) and Ax(g) are theorems of T if “ \rightarrow ” is replaced by “ \rightarrow ” or “ \supset ”, so that it is not difficult to prove the result for them by TPr. (For the details of these proofs see Pizzi [12].)

We will then restrict ourselves to the proofs of Ax(f) and Ax(g), which are as follows:

Proof of Ax(f): $(\perp \rightarrow p) \supset (p \rightarrow \perp)$ equals, by Def \rightarrow , $(\Box(\perp \supset p) \wedge (\Diamond p \supset \Diamond \perp) \wedge (\Box p \supset \Box \perp)) \supset (\Box(p \supset \perp) \wedge (\Diamond \perp \supset \Diamond p) \wedge (\Box \perp \supset \Box p))$.

By the tableaux method it may be easily checked that the following wffs are T-valid:

- (fi) $\Box(\perp \supset p) \supset (\Diamond \perp \supset \Diamond p)$
- (fii) $(\Diamond p \supset \Diamond \perp) \supset \Box(p \supset \perp)$
- (fiii) $(\Box p \supset \Box \perp) \supset (\Box \perp \supset \Box p)$.

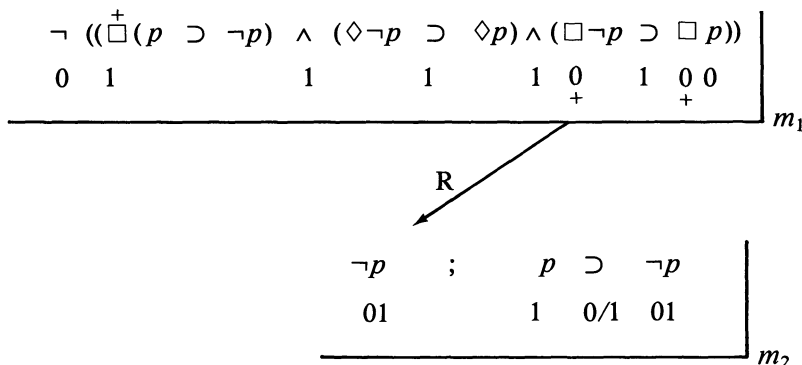
By the completeness of T they are T-provable, and the desired result follows by TPr.

Proof of Ax(g): To simplify the proof we notice that Ax(g), namely Boethius’ thesis, is equivalent to the so-called Aristotle’s thesis $\neg(p \rightarrow \neg p)$, which we will call (g’). That (g’) follows from (g) is simply proved by substituting p for q in (g) and applying Modus Ponens. The derivation of (g) from (g’) is as follows:

- (1) $((p \rightarrow q) \wedge (q \rightarrow \neg p)) \supset (p \rightarrow \neg p)$ Ax(a), $\neg p/r$
- (2) $\neg(p \rightarrow \neg p) \supset ((p \rightarrow q) \supset \neg(p \rightarrow \neg q))$ (1), PC, Ax(d)
- (3) $(p \rightarrow q) \supset \neg(p \rightarrow \neg q)$ (2), (g’).

The proof of the T-validity of (g’) is in two steps. (T-models are here 3-tuples $\langle M, R, V \rangle$ such that M is a nonempty set of possible worlds m_1, m_2, m_3, \dots ; R and V are defined as in Hughes and Cresswell [6]. The rules for tableaux construction are substantially the same as the ones expounded in this book.)

(a) If $V(p, m_1) = 0$ the following tableau closes:



- (b) If $V(p, m_1) = 1$, $V((p \supset \neg p), m_1) = 0$, so $V(\Box(p \supset \neg p), m_1)$ cannot be 1, contrary to what follows from the Reductio hypothesis.

To conclude the proof of the lemma we would have to show that the rules of CI.0 are derived rules of T. The proof is trivial and will be omitted. We omit also the simple proof of the equivalence $\Box p \equiv \Box(\top \supset p) \wedge (\Diamond p \supset \Diamond \top) \wedge (\Box p \supset \Box \top)$, corresponding to Def \Box , which turns out to be a thesis of T.

Being equivalent to T, CI.0 is then decidable, and the procedure is simply given by the tableaux method used for T. From the completeness of T and the equivalence result there also follows a completeness result for CI.0.

We list here some theorems and nontheorems of CI.0:

Theorems

- $$\begin{aligned} &(\Diamond(p \wedge q) \wedge \neg \Box q) \supset ((p \wedge q) \rightarrow q) \\ &\Diamond(p \wedge r) \supset ((p \rightarrow q) \supset ((p \wedge r) \rightarrow q)) \\ &(\top \rightarrow p) \supset (p \rightarrow \top) \\ &(\Diamond q \wedge \neg \Box(p \vee q)) \supset (q \rightarrow (p \vee q)) \\ &(\Diamond q \wedge \neg \Box(p \vee q)) \supset (((p \vee q) \rightarrow r) \supset (p \rightarrow r)) \\ &(\Box q \wedge \Box p) \supset (p \rightarrow q) \\ &(p \rightarrow q) \supset \neg(\neg p \rightarrow q). \end{aligned}$$

Nontheorems

- $$\begin{aligned} &(p \rightarrow q) \supset ((p \wedge r) \rightarrow (q \wedge r)) \\ &(p \wedge q) \rightarrow p \\ &p \rightarrow (p \vee q) \\ &\perp \rightarrow p \\ &p \rightarrow \top. \end{aligned}$$

3 The logic of synthetic consequential implication We have now to move from a logic of analytic conditionals to a logic of synthetic conditionals. To achieve this goal we can adopt a device introduced by Åqvist in [2]. (For a defense of this approach see also Humberstone [7].) Åqvist's proposal is to introduce a "circumstantial" operator for the notion of *ceteris paribus*, whose symbol is "*". This will allow us to express the basic idea that *A* consequentially implies *B* iff *B* follows from *A* *ceteris paribus*. We could also introduce a "dual" circumstantial operator, defined as $\Diamond A =_{df} \neg * \neg A$.

The first thing to be noted is that nonmonotonicity is held to be a highly desirable property for the logic of synthetic conditionals (see Ginsberg [4]). This suggests that nonmonotonicity could be reached forbidding a rule which allows us to infer $\vdash *A \supset *B$ from $\vdash A \supset B$ and retaining the simple $\vdash A \equiv B \Rightarrow \vdash *A \equiv *B$, which would follow having Eq as a primitive rule. The drawback of such a choice lies, however, in the fact that it is an obstacle to proving such intuitive laws as $*(p \wedge q) \supset *q$. Luckily, this restriction is not strictly necessary insofar as we are already taking for granted a logic for an implicative relation which is nonmonotonic: we already know in fact that $(A \rightarrow B) \supset ((A \wedge C) \rightarrow (B \wedge C))$ is not a thesis for the logic of " \rightarrow ", even if a weakened version of it is a thesis. So we can hope to reach the desired result without introducing further nonmonotonicity principles.

To begin, let us take into consideration the system we shall call CI*0. This system is obtained by adding to CI.0, which we already know to be equivalent to T, two axioms for “*”, namely

CI*0.1 $*p \supset p$

CI*0.2 $\diamond p \supset \diamond *p$

and the rule:

R*0 $\vdash A \supset B \Rightarrow \vdash *A \supset *B.$

The rule of replacement Eq is easily obtained in this system thanks to (R*0). The primitive rules MP, US, Nec are now intended to apply not to theorems of CI.0 but to theorems of CI*0.

This approach to conditional logic is interesting because of the fact that several definitions of conditional operators are possible. *Prima facie*, the most plausible is

$$A >^0 B =_{df} *A \rightarrow B,$$

which is of course equivalent to

$$A >^0 B =_{df} \square (*A \supset B) \wedge (\diamond B \supset \diamond *A) \wedge (\square B \supset \square *A).$$

For the reasons which will be explained on p. 634, however, it is better in the present paper to restrict ourselves to the following weaker definition:

$$A > B =_{df} \square (*A \supset B) \wedge (\diamond B \supset \diamond *A).^2$$

Notice that since the converse of CI*0.2 it an easily proved theorem of CI*0, we reach the equivalence between $\diamond A$ and $\diamond *A$, so that the definition of “>” turns out to be equivalent to the definition

$$A > B =_{df} \square (*A \supset B) \wedge (\diamond B \supset \diamond A).$$

We will now define a system, named T^w, which is different from T in being a particular linguistic extension of it. In fact we assume that beyond the propositional variables its language contains a countable number of symbols of form $w, w^A, w^B \dots w^{\neg A}, w^{\neg B} \dots w^{A \wedge B}, w^{A \vee B} \dots$, where the exponents A, B , etc., if any, are wffs of T^w. The definition of a wff must be extended in this way:

1. $p, q, r, \dots w$ are wffs
2. if A is a wff, $\neg A, \square A, w^A$ are wffs
3. if A and B are wffs, $A \vee B$ is a wff.

We shall call the new linguistic objects w, w^A, \dots , etc. *quasi-variables*: they are not in fact variables since, although they can be uniformly substituted, as any other wff, for propositional variables, we cannot uniformly substitute wffs for them; substitution is, however, allowed for the atomic variables which are part of the exponents.

The *circumstantial degree* (gr) of a wff of CI*0 is defined in this way:

If all the connectives are reduced to $\neg, \square, *, \vee$:

1. If A is a wff of CI.0(=T), $gr(A) = 0$
2. If $gr(A) = n$, $gr(\neg A) = gr(\square A) = n$

3. If $\text{gr}(A) = n$ and $\text{gr}(B) = m$ then $\text{gr}(A \vee B) = \max\{\text{gr}(A), \text{gr}(B)\}$
4. If $\text{gr}(A) = n$, $\text{gr}(*A) = n + 1$.

A parallel definition of circumstantial degree can be given for wffs of T:

1. If A is a wff of CI.0(=T) or A is \mathbf{w} , $\text{gr}(A) = 0$.
- 2 and 3 are as in the preceding definition.
4. If $\text{gr}(A) = n$, $\text{gr}(\mathbf{w}^A) = n + 1$.

The rule of Uniform Substitution in $T^{\mathbf{w}}$ is not different from the one in T. It may, however, be reformulated in more precise terms, specifying that Uniform Substitution of wffs for atomic variables may be applied not only to variables in standard position but also to variables which are exponents of quasi-variables.

$T^{\mathbf{w}}$ includes all the theorems of T along with all the substitution-instances of them containing quasi-variables. In the semantic tableaux for $T^{\mathbf{w}}$ we have only an application of the rules given for T. In other terms quasi-variables are semantically treated on a par with atomic variables, and quasi-variables having identical exponents are treated as identical atomic variables.

This simple semantics for $T^{\mathbf{w}}$ excludes that in this system quasi-variables having logically equivalent exponents turn out to be equivalent wffs: \mathbf{w}^A and $\mathbf{w}^{\neg\neg A}$ are, for instance, nonequivalent wffs and they may receive different values at the same world in a given model.

Let us now introduce an extension of $T^{\mathbf{w}}$, which we shall call $T^{\mathbf{w}0}$, which will be used to give a representation of the theorems of CI*0. $T^{\mathbf{w}0}$ will be the same as $T^{\mathbf{w}}$ with the addition of the axiom:

$$\mathbf{T}^{\mathbf{w}0} \quad \diamond p \supset \diamond (\mathbf{w}^p \wedge p)$$

and of the rule

$$\mathbf{R}^{\mathbf{w}0} \quad \vdash A \supset B \Rightarrow \vdash \mathbf{w}^A \supset \mathbf{w}^B.$$

Rules US, MP, Nec are now intended to apply not to $T^{\mathbf{w}}$ -theorems but to $T^{\mathbf{w}0}$ -theorems.

Remark Thanks to $\mathbf{R}^{\mathbf{w}0}$ it is straightforward to derive the following rule:

$$\mathbf{Rw}0' \quad \vdash A \equiv B \Rightarrow \vdash \mathbf{w}^A \equiv \mathbf{w}^B$$

and, since Eq is a valid rule for $T^{\mathbf{w}}$, it is simple to obtain the following extended version of it which holds for $T^{\mathbf{w}0}$:

(Eqw) If $A \equiv B$ is a $T^{\mathbf{w}0}$ thesis, then A and B may be interchanged in every occurrence, both in standard position and in exponent position.

4 Properties of $T^{\mathbf{w}0}$ The logical properties of $T^{\mathbf{w}0}$ can easily be proved by a suitable extension of the methods employed for $T^{\mathbf{w}}$. $T^{\mathbf{w}}$ -models are 3-tuples $\langle M, R, V \rangle$ with the same properties of T-models as defined in Hughes and Cresswell [6].

The definition of a $T^{\mathbf{w}0}$ -model needs, however, an extension both in the set of accessibility relations and in the definition of the valuation function. A $T^{\mathbf{w}0}$ -model is in fact a 4-tuple $\langle M, R, R'', V \rangle$ such that:

1. M is a nonempty set of possible worlds m_1, m_2, m_3, \dots
2. R is a reflexive dyadic relation over M

3. R^w is a dyadic relation over M
4. V is a valuation function which is defined as in Hughes and Cresswell [6] plus the following clauses:

VR1 If some m_j exists such $m_i R m_j$ and $V(A, m_j) = 1$, then some m_l exists such that $m_i R m_l$ and $V(A, m_l) = V(w^A, m_l) = 1$.

VR2 If $(w^A, m_i) = 1$ and $V(w^B, m_i) = 0$, then some m_j exists such that $m_i R^w m_j$ and $V(A \supset B, m_j) = 0$.

A wff A is said to be T^w0 -valid if $V(A, m_i) = 1$ for every world m_i of every T^w0 -model.

4.1 Soundness The key steps to prove soundness of T^w are the following: (i) The axioms of T^w turn out to be T^w -valid by a proof which is the same as the standard one for T ; (ii) Axiom T^w0 turns out to be valid thanks to VR1; (iii) Rule R^w0 preserves validity. Let us in fact suppose by Reductio that $A \supset B$ is T^w0 -valid, and that some world m_j exists such that $V(w^A, m_j) = 1$ and $V(w^B, m_j) = 0$. Then by VR2 some R^w -accessible world m_j exists such that $V(A \supset B, m_j) = 0$, which is contrary to the initial supposition.

4.2 Consistency This follows from soundness along standard lines

4.3 Decidability T^w0 is decidable. The decision procedure is an extension of the procedure for T given in Chapter V of Hughes and Cresswell [6]. Beyond the rules given there for the tableaux construction we have to introduce two further rules:

R1 Let m_j be a rectangle which is R -accessible to a rectangle m_i and such that A is a wff with value 1 inside m_j (so possibly $m_i = m_j$). Then, provided some operator in m_i has sign “+” on it, we have to: (a) build a rectangle m_l such that both A and w^A in it have value 1, and (b) reproduce in m_l the arguments of the modal operators having sign “+” on them in m_i . By converse, if w^A or $w^A \wedge A$ receives value 0 in all rectangles m_j which are R -accessible to m_i , we introduce in them A with value 0.

R2 Whenever we have w^A with assignment 1 and w^B with assignment 0 in the same rectangle m_i , we have to build a new rectangle m_j in which $A \supset B$ must be assigned value 0.

It is simple to see that the procedure which has been so described must always have an end in a finite time. Let us now call R -accessible to m_i the rectangles which are either such in the sense of Hughes and Cresswell [6] or are built applying R1, and R^w -accessible the rectangles which are built applying R2. Then rectangles of both these kinds include wffs which have a modal and/or circumstantial degree which is lower than the one of the wffs included in the rectangles to which they are R -accessible or R^w -accessible. So, unless it ends earlier, the procedure leads us to evaluate wffs which have zero circumstantial degree and zero modal degree.

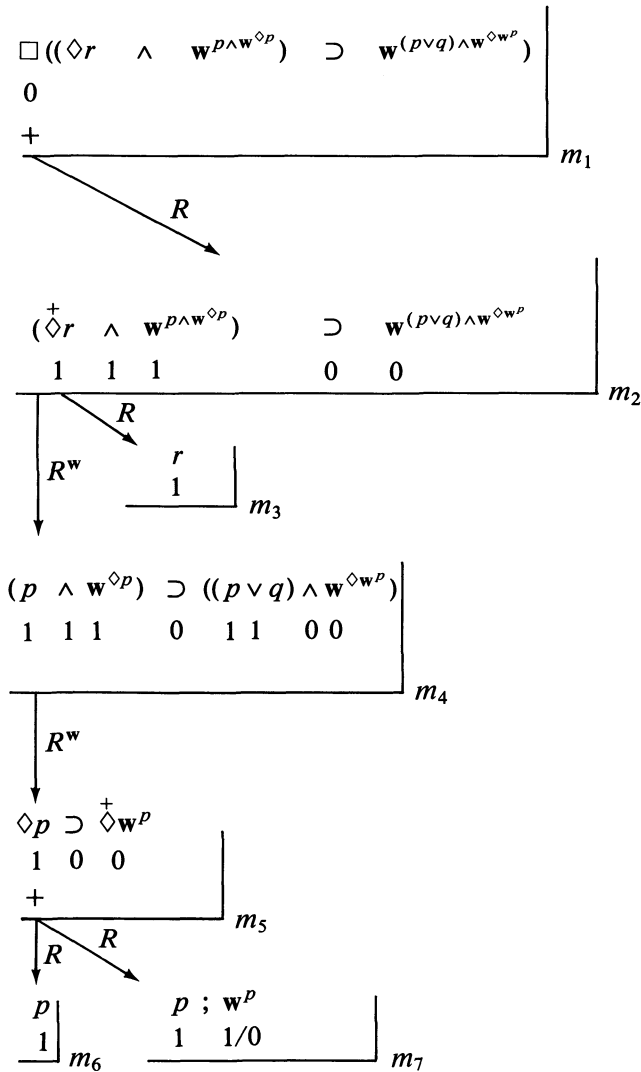
Notice that to simplify the procedure we may make use of the equivalence $\diamond p \equiv \diamond (w^p \wedge p)$ and replace every occurrence of $\diamond (w^A \wedge A)$ by an occurrence

of $\diamond A$. The resulting wff may then be tested in place of the original one since it is equivalent to it.

A sufficiently complicated example will illustrate the method. Let us suppose that the wff to be tested is

$$D = \Box((\diamond(w^r \wedge r) \wedge w^{p \wedge w^{\diamond p}}) \supset w^{(p \vee q) \wedge w^{\diamond w^p}}).$$

This wff is, to begin with, simplified by replacing $\diamond(w^r \wedge r)$ by $\diamond r$. The relevant tableaux system is the one drawn below. Notice that we omit introducing new rectangles as an application of R1, if no argument of modal operators having sign “+” on them in some preceding rectangle has to be reproduced.



[Since we find no inconsistency in m_2 and m_3 , and two quasi-variables receive assignment 1 and 0 respectively in m_2 , we open an auxiliary (R^w -accessible) rec-

tangle m_4 applying R2. Since $w^{\diamond p}$ and $w^{\diamond w^p}$ have assignment 1 and 0 respectively in m_4 , we have to introduce a further (R^w -accessible) rectangle m_5 containing $\diamond p \supset \diamond w^p$ with value 0, and this yields an inconsistency via R1. Notice that from a practical viewpoint m_7 might be built simply as an extension of m_6 .]

4.4 Completeness It is possible to transform a closed system of tableaux for any wff A in a proof for A using a method which is an appropriate extension of the one expounded in Hughes and Cresswell [6] (pp. 98–100). (For sake of simplicity we will omit treating what Hughes and Cresswell call alternatives.) Let us call *associated* to a rectangle m_i the wff \mathbf{m}_i which is so defined along the Hughes–Cresswell lines: if any modal operator occurring in a wff is definitionally reduced to the simple \square , \mathbf{m}_i is the wff which is the disjunction of the only wff in m_i whose initial assignment is 0 and of the negation of all the wffs in m_i whose initial assignment is 1. We have then to distinguish two cases:

Case i. The inconsistent rectangle m_i is R -accessible to some rectangle m_{i-1} . The inconsistency may follow by applying rule R1 or not. In the second case the method of constructing a proof of the wff which is associated to the explicitly inconsistent rectangle is the one formulated by Hughes and Cresswell: if $\gamma_1 \dots \gamma_k$ are well-formed subformulas of m_i such that both $\square \gamma_1 \dots \square \gamma_k$ and $\gamma_1 \dots \gamma_k$ have received an assignment, then some wff β exists such that

$$\mathbf{m}'_i \quad ((\square \gamma_i \supset \gamma_i) \wedge \dots \wedge (\square \gamma_k \supset \gamma_k)) \supset ((\beta \supset \mathbf{m}_i) \wedge (\neg \beta \supset \mathbf{m}_i))$$

turns out to be PC-valid, and then by the completeness of PC also T^w -provable. Then \mathbf{m}_i is also proved in T^w0 by repeated applications of modus ponens and PC-theorems.

Let us however suppose that at the end of the procedure the inconsistency is found by applying Rule R1. Let us write $\bigwedge^{m_i} (\diamond B \supset \diamond (w^B \wedge B))$ to indicate the conjunction of wffs of form $(\diamond B \supset \diamond (w^B \wedge B))$ such that B and w^B are in m_i and receive a value in some rectangle m_j , R -accessible to m_i , by an application of R1. Let us suppose that some m_j turns out to be explicitly inconsistent. Then it follows that

$$\mathbf{m}''_i : \bigwedge^{m_i} (\diamond B \supset \diamond (w^B \wedge B)) \supset \mathbf{m}'_i$$

is PC-valid or T^w -valid and, by the completeness of T^w , T^w -provable and therefore T^w0 -provable. By repeated applications of modus ponens and PC-theorems then \mathbf{m}'_i and \mathbf{m}_i are also proved in T^w0 .

Since by hypothesis m_i is R -accessible to m_{i-1} , if \mathbf{m}_{i-1} is the wff which is associated to m_{i-1} , \mathbf{m}_{i-1} is provable adapting the method in Hughes and Cresswell ([6], p. 100).

Case ii. Let us suppose that the inconsistent rectangle m_i is R^w -accessible to m_{i-1} . The wff in m_i has form $A \supset B$ and, by a standard argument, it turns out to be T^w -provable and then T^w0 -provable. We know that the rectangle m_{i-1} is such that in it $V(w^A) = 1$ and $V(w^B) = 0$. But this means that $V(w^A \supset w^B) = 0$ for the value assignments in m_{i-1} , and that $(w^A \supset w^B) \supset \mathbf{m}_{i-1}$ is T^w -valid; so, by the completeness of T^w , it is T^w -provable and T^w0 -provable. Since $w^A \supset w^B$ turns out also to be T^w0 -provable applying Rule R^w0 to theorem $A \supset B$, \mathbf{m}_{i-1} turns out to be a T^w0 -theorem applying modus ponens.

To conclude, since every rectangle beyond the first is either R -accessible or R^w -accessible, we have a tool to move in any possible case from the wff associated to a rectangle to a wff associated to the preceding one in any given tableaux system. So, in a finite number of steps, we reach a proof of the wff contained in the first rectangle, which is the wff under test.

5 Decidability and other properties of CI*0 We are now interested in transforming the decision procedure for T^w0 in a decision procedure for CI*0. With this aim we introduce a mapping from wffs of CI*0 to wffs of T^w0 . Let Tr be the function which maps wffs of CI*0 into wffs of T^w0 (which we will call Tr -images) and is defined by:

- If p is any atomic wff, $\text{Tr}(p) = p$
 If $A, B, C \dots$ are wffs of CI*0:
 $\text{Tr}(\neg A) = \neg \text{Tr}(A)$
 $\text{Tr}(A \wedge B) = \text{Tr}(A) \wedge \text{Tr}(B)$
 $\text{Tr}(\Box A) = \Box(\text{Tr}(A))$
 $\text{Tr}(*A) = w^{\text{Tr}(A)} \wedge \text{Tr}(A).$

The representation theorem is the following:

(RT) A is a CI*0 thesis if and only if $\text{Tr}(A)$ is a T^w0 -thesis.

(RT) follows from Lemmas L1 and L2:

L1 If A is a CI*0-thesis, $\text{Tr}(A)$ is a T^w0 -thesis.

The proof is by induction on the length of the proofs in CI*0.

(a) L1 holds trivially for the axioms with no circumstantial operators.

Axiom CI*01, i.e. $*p \supset p$, has as a Tr -image $(w^p \wedge p) \supset p$, which is an obvious T^w0 -thesis.

Axiom CI*02, i.e. $\diamond p \supset \diamond *p$, has as a Tr -image $\diamond p \supset \diamond (w^p \wedge p)$, i.e. Axiom T^w1 .

(b) *Induction Step:* Let us suppose that the theorem holds for all the n rows of a proof in CI*0. We shall prove that it holds also for row $n + 1$. Being Eq a derived rule in CI*0, the rules which can be applied to move from a row to the following one in the proof are MP, Nec, R^* , US:

- (MP)** Let us suppose that the property holds for rows of form A and $A \supset B$, which means that in T^w0 $\text{Tr}(A)$ and $\text{Tr}(A \supset B)$ are theses. Since $\text{Tr}(A \supset B) = \text{Tr}(A) \supset \text{Tr}(B)$, by modus ponens $\text{Tr}(B)$ will also be a thesis. But $\text{Tr}(B)$ is the Tr -image of B , which is also derived in CI*0 via MP by A and $A \supset B$.
- (Nec)** $\vdash A \Rightarrow \vdash \Box A$ is a rule both in CI*0 and in T^w0 . So in T^w0 we have by Nec that $\vdash \Box(\text{Tr}(A))$ follows from $\vdash \text{Tr}(A)$. But $\Box(\text{Tr}(A))$ equals $\text{Tr}(\Box A)$, which is the Tr -image of $\Box A$.
- (R*)** In T^w0 we have as a derived rule R^w0' : $\vdash A \supset B \Rightarrow \vdash (w^A \wedge A) \supset (w^B \wedge B)$ which follows from R^w0 by TPr. The conclusion of the rule equals $\text{Tr}(*A \supset *B)$, and we have in CI*0 $\vdash A \supset B \Rightarrow \vdash *A \supset *B$; so any application of R^w0 leads from Tr -images of CI*0-theorems to Tr -images of other CI*0-theorems obtained via R^* .

(US) Let us take into account only the case of wffs which, after an application of US, come to contain at least one circumstantial operator. Let us symbolize by $A[C]$ any wff A containing one or more occurrences of C and by $A[C/p]$ a wff A which is obtained by substituting C for every occurrence of p . Then we may have the following steps:

- (1) From $A[p]$ by US of $C[*B]$ for p we obtain $A[C[*B]/p]$.
- (2) From $A[*p]$ by US of C for p we obtain $A[C/p]$.

These steps in T^w0 are mirrored by

- (1') From $\text{Tr}(A[p])$ by US of $\text{Tr}(C[*B])$ for p we obtain $\text{Tr}(A[\text{Tr}(C[*B])/p])$.
- (2') From $\text{Tr}(A[*p])$ by US of $\text{Tr}(C)$ for p we obtain $\text{Tr}(A[\text{Tr}(C)/p])$.

It is straightforward to see that the results of the substitutions in (1') and (2') are equivalent to $\text{Tr}(A[C[*B]/p])$ and to $\text{Tr}(A[C/p])$, and this ends the proof.

L2 *If $\text{Tr}(A)$ is a T^w0 -thesis then A is a $CI*0$ -thesis.*

The proof is by induction on the length of the proofs of T^w0 .

(a) The theorem holds trivially for the axioms of T^w0 shorn of quasi-variables and for T^w1 .

(b) Let us suppose, by Induction hypothesis, that the theorem holds for all the n rows of a proof which are Tr -images of $CI*0$ -wffs. We have to prove that it holds also for row $n + 1$. Let us suppose that row $n + 1$ is a Tr -image of B and let us prove that B is a $CI*0$ -thesis. We will omit the trivial case in which $\text{Tr}(B) = B$, so we will take into consideration only rows $n + 1$ whose form is $\text{Tr}(B) [w^A \wedge A]$.

We have to consider the possibility that $\text{Tr}(B)$ is derived from preceding rows thanks to US, MP, Nec.

(US) We have to take into account two cases:

Case a: Row $n + 1$ is derived, *via* US, from some preceding row which is a Tr -image. There are three subcases to be considered since $\text{Tr}(B) [w^A \wedge A]$ may be obtained *via* US from some preceding row which has one of these forms: (1) $\text{Tr}(B) [w^r \wedge r]$; (2) $\text{Tr}(B) [p]$; and (3) $\text{Tr}(B) [p \wedge q]$. By Induction hypothesis each one of them is a Tr -image. So in Subcase 1 we have in $CI*0$ a counterimage-theorem of form $B[*r]$: from this we may obtain by Uniform Substitution of A to r $B[*A]$, whose Tr -image is just $\text{Tr}(B) [w^A \wedge A]$. In Subcases (2) and (3) we have in $CI*0$ counterimage rows of form $B[p]$ and $B[p \wedge q]$. In Subcase (2) we obtain the same result as before by Uniform Substitution of $*A$ for p . In Subcase (3) we put $*A$ in place of p and A in place of q . So the result in the third subcase is a row of form $B[*A \wedge A]$. But $*A \wedge A \equiv *A$ is a simple $CI*0$ -theorem, so that by Eq we again obtain obtain $B[*A]$ as a theorem.

Case b: Row $n + 1$ is derived from some preceding row which is not a Tr -image. This means that at least one quasi-variable occurring in this row is not conjoined with a wff which is its exponent. We shall call *free* the quasi-variables lacking these properties. So in this row we will have wffs of form $B[w^A \wedge C]$ or $B[w^C \wedge A]$ which are such that substituting inside

the wffs in parentheses we reach as row $n + 1$ a theorem of form $B[\mathbf{w}^A \wedge A]$, shorn of free quasi-variables.

Notice however that in any such case we may construct an alternative proof of $B[\mathbf{w}^A \wedge A]$ in which US is applied not to $B[\mathbf{w}^C \wedge A]$ or to $B[\mathbf{w}^A \wedge C]$ but to their variants $B[\mathbf{w}^C \wedge C] \wedge A]$ or $B[(\mathbf{w}^A \wedge A) \wedge C]$. From the latter we obtain in fact $B[(\mathbf{w}^A \wedge A) \wedge A]$, hence $B[\mathbf{w}^A \wedge A]$. We have to consider that in the proof of $B[\mathbf{w}^A \wedge C]$ derived from the tableaux construction the free quasi-variable \mathbf{w}^A is introduced either by Substitution of \mathbf{w}^A for a propositional variable or by Rule $\mathbf{R}^{\mathbf{w}0}$. We may however obtain the desired variant either by substituting $\mathbf{w}^A \wedge A$ for p or by employing the derived rule $\vdash A \supset B \Rightarrow \vdash (\mathbf{w}^A \wedge A) \supset (\mathbf{w}^B \wedge B)$. An analogous argument works of course for $B[\mathbf{w}^C \wedge A]$. In this new proof the row to which US is applied is then a Tr-image, and by Induction Hypothesis we assume that its counterimage is a CI*0 thesis. But from $B[*A \wedge C]$ or $B[*C \wedge A]$ we obtain in CI*0 $B[*A \wedge A]$, and from the latter, by Eq, $B[*A]$, which is the counterimage of $B[\mathbf{w}^A \wedge A]$.

(MP) Here too we have two possible cases:

Case a: $\text{Tr}(B)$ is derived from two preceding rows which are Tr-images, for instance $\text{Tr}(A)$ and $\text{Tr}(A) \supset \text{Tr}(B)$. But $\text{Tr}(A) \supset \text{Tr}(B) = \text{Tr}(A \supset B)$, so by Induction Hypothesis and modus ponens this means that in CI*0 B is also a thesis.

Case b: $\text{Tr}(B)$ is derived from two rows A and $A \supset \text{Tr}(B)$, where A is not a Tr-image. Here the argument parallels the one given for US. Both A and $A \supset \text{Tr}(B)$ have two variants shorn of free quasi-variables, A' and $A' \supset \text{Tr}(B)'$, whose counterimages by Induction Hypothesis are also CI*0-theses. Then by suitable substitutions and applications of modus ponens we reach B as a theorem of CI*0.

(Nec) The rule of Necessitation is part of the axiomatic basis of both CI*0 and $\mathbf{T}^{\mathbf{w}0}$. It cannot lead from a theorem which is not a Tr-image to another which is such, so the argument is simply the converse of the one for Nec in Lemma L1 (see p. 628).

As regards rule $\mathbf{R}^{\mathbf{w}0}$, this rule leads from theorems of form $\vdash A \supset B$ to theorems of form $\vdash \mathbf{w}^A \supset \mathbf{w}^B$, which are not Tr-images even in the case in which A and B are Tr-images, so it must not be taken into account in the proof of the lemma. This ends the proof.

After proving the preceding Representation Theorem it is very simple to formulate the decision procedure for CI*0. If A is any CI*0-wff, one must calculate its Tr-image $\text{Tr}(A)$ and test it by the tableaux-method. If $\text{Tr}(A)$ turns out to be $\mathbf{T}^{\mathbf{w}0}$ -valid it is also, by the completeness of $\mathbf{T}^{\mathbf{w}0}$, $\mathbf{T}^{\mathbf{w}0}$ -provable: so, by the Representation Theorem, its counterimage A is CI*0-provable. If the test gives a negative result, $\text{Tr}(A)$ is not $\mathbf{T}^{\mathbf{w}0}$ -provable and so neither is A CI*0-provable.

5.1 Completeness A simple corollary of the Representation Theorem is also the completeness of CI*0. We may define in fact CI*0-validity in this way: a wff A is CI*0-valid iff its Tr-image is $\mathbf{T}^{\mathbf{w}0}$ -valid. If A is CI*0-valid, its Tr-image is so by definition $\mathbf{T}^{\mathbf{w}0}$ -valid, and by the completeness of $\mathbf{T}^{\mathbf{w}0}$ it is provable in $\mathbf{T}^{\mathbf{w}0}$. By the Representation Theorem this implies that A is CI*0-provable. Con-

versely, if A is CI*0-provable, by the Representation Theorem $\text{Tr}(A)$ is T^w0-provable, so by the completeness of T^w0 it is CI*0-valid. So A is CI*0-provable iff it is CI*0-valid.

5.2 Non-triviality of the circumstantial operator It is enough to prove that $p \supset *p$ is not a CI*0-theorem. If it were, $\text{Tr}(p \supset *p)$ would be a T^w0-thesis. But $p \supset (\mathbf{w}^p \wedge p)$ is not T^w0-valid, and this concludes the proof.

6 The system CI*1 We are now going to introduce an extension of CI*0 which we will call CI*1. CI*1 is the result of adjoining to CI*0 the axiom

$$\text{CI*1} \quad (*p \wedge *q) \supset *(p \wedge q)$$

and of course we shall also introduce a parallel extension of T^w0, which will be named T^w1. T^w1 is the result of adding to T^w0 the axiom

$$\text{T}^w1 \quad (\mathbf{w}^p \wedge \mathbf{w}^q) \supset \mathbf{w}^{p \wedge q}.$$

The equivalence $(*p \wedge *q) \equiv *(p \wedge q)$ is an immediate consequence of CI*1, while $(\mathbf{w}^p \wedge \mathbf{w}^q) \equiv \mathbf{w}^{p \wedge q}$ is an easily obtained law of T^w1. T^w1 has obviously among its theorems $((\mathbf{w}^p \wedge p) \wedge (\mathbf{w}^q \wedge q)) \supset (\mathbf{w}^{p \wedge q} \wedge (p \wedge q))$, which is the Tr-image of CI*1.

The properties of T^w1 are proved by simple extensions of the proofs for T^w0. One has obviously to add that T^w1-models are like T^w0-models (see p. 624) with the further clause

$$\text{VR3} \quad \text{If } V(\mathbf{w}^A, m_i) = 1 \text{ and } V(\mathbf{w}^B, m_i) = 1, V(\mathbf{w}^{A \wedge B}, m_i) = 1.$$

(Alternatively: if $V(\mathbf{w}^{A \wedge B}, m_i) = 0$ either $V(\mathbf{w}^A, m_i) = 0$ or $V(\mathbf{w}^B, m_i) = 0$.) The properties of CI*1 follow by simple extensions of the ones of CI*0. In fact we have:

6.1 Soundness Soundness and consistency follow by a suitable extension of the proof given for CI*0.

6.2 Decidability Since we now have at our disposal the equivalence between $\mathbf{w}^{p \wedge q}$ and $\mathbf{w}^p \wedge \mathbf{w}^q$, we may “normally” simplify the procedure with suitable replacements of $\mathbf{w}^{p \wedge q}$ by $\mathbf{w}^p \wedge \mathbf{w}^q$ or *vice versa*. The full procedure is however an extension of the procedure for CI*0 mirroring the introduction of VR3. Thus decidability, completeness, and nontriviality follow from the corresponding results for CI*0 by suitably extended arguments.

7 The system CI*2 A third system of consequential implication, which we will name CI*2, is the result of adding to CI*0

$$\text{CI*2} \quad (*p \wedge q) \supset (*q \wedge p).$$

Axiom CI*1 is easily seen to be a theorem of this system, which hence properly includes CI*1. Substituting $p \wedge q$ for q we also easily obtain from CI*2 $(*p \wedge q) \supset *(p \wedge q)$, which mirrors the simple idea that the *ceteris paribus* clause is something invariant in respect of the members of any conjunction having a *-formula as a conjunct.

The latter fact simplifies, to a certain extent, the definition of the representation of CI*2 on the extension of T^w which may be chosen as a tool for this aim.

We may in fact define a new system, to be named T^w2, adding to T^w0 the axiom

$$\mathbf{T}^w2 \quad \diamond(\mathbf{w}^p \wedge p) \supset \diamond(\mathbf{w} \wedge p)$$

which by T^w0 and Eq equals $\diamond p \supset \diamond(\mathbf{w} \wedge p)$. But in this extended system the rule of Uniform Substitution must be restricted in the following way. Let us call *w-formulas* the wffs \mathbf{w} , $\neg\mathbf{w}$, $\Box\mathbf{w}$, $\Box\neg\mathbf{w}$ and anyone of these having as a prefix an arbitrary combination of negation signs and modal operators.

Then the Restricted Rule of Substitution (RUS) may be formulated as follows:

(RUS) Uniformly substituting a wff for any atomic variable in a T^w2 thesis we obtain a T^w2-thesis, provided the result of the substitution does not contain any subformula of form $\alpha \circ \beta$, where “ \circ ” is any truth-functional operator and α and β are *w-formulas*.

The reason for this restriction is to avoid that the simple \mathbf{w} may be proved as a theorem, for instance by the following proof (which violates the restriction at step (2))

- | | |
|-----------------------------------------------------------------------------------------|--------------------------------------------------------------------|
| (1) $\diamond p \supset \diamond(\mathbf{w} \wedge p)$ | T ^w 2, PC |
| (2) $\diamond\neg\mathbf{w} \supset \diamond(\mathbf{w} \wedge \neg\mathbf{w})$ | (1), $\neg\mathbf{w}/p$ |
| (3) $\neg\diamond(\mathbf{w} \wedge \neg\mathbf{w}) \supset \neg\diamond\neg\mathbf{w}$ | (2), PC |
| (4) $\Box\mathbf{w}$ | (3), $\vdash_T\neg\diamond(\mathbf{w} \wedge \neg\mathbf{w})$, MP |
| (5) \mathbf{w} | (4), $\vdash\Box p \supset p$. |

The mapping from CI*2 to T^w2 is different from the one defined for CI*0 since it may now be simplified by putting $\text{Tr}(*A) = \mathbf{w} \wedge A$. By this new definition the axiom T^w2 is not a Tr-image, while $\diamond p \supset \diamond(\mathbf{w} \wedge p)$ is such: indeed, it is the Tr-image of Axiom CI*1. The Tr-image of Axiom CI*2 becomes now $((\mathbf{w} \wedge p) \wedge q) \supset ((\mathbf{w} \wedge q) \wedge p)$, which is a substitution instance of a PC-theorem. Notice that CI*2 needs no restriction on substitution which is parallel to the one of T^w2, since no wff containing a truth-functional compound of *w-formulas* is a Tr-images of any CI*2-wff.

7.1 Properties of CI*2 A T^w2-model is like a T^w0-model (see p. 624) with the addition of the following clause:

VR3 If some m_j exists such that $m_i R m_j$ and $V(A, m_j) = V(\mathbf{w}^A, m_j) = 1$ then some m_l exists such that $m_i R m_l$ and $V(A, m_l) = V(\mathbf{w}, m_l) = 1$.

The proof of the validity of the axioms of T^w0 is easily extended to Axiom T^w2. The only complication concerns the rules, since we have now to prove that Uniform Substitution is validity-preserving only with the required restriction.

We may reason as follows. Since T^w2-models are like T^w0-models except for VR3, RUS would lead to a nonvalid substitution instance only if, substituting \mathbf{w} , \mathbf{w}^A , or A inside a T^w2-valid wff, we were to obtain a wff semantically imply-

ing that at least one m_i exists such that $V(\diamond(w^A \wedge A), m_i) = 1$ and $V(\diamond(w \wedge A), m_i) = 0$. But the latter assignment would be possible only if at every possible world m_j such that $m_i R m_j$, $V(w \wedge A, m_j) = 0$. This might happen only for one of two different reasons: (1) A is a T_2^w -contradiction. This is, however, to be excluded, otherwise $\diamond(w^A \wedge A)$ should also have everywhere value 0. (2) $A = \neg w$. This is, however, also impossible, since the substitution-instance which would be so obtained is a w -formula, and this is excluded by the restriction on the Substitution. It is so proved that Restricted Uniform Substitution is validity-preserving, while unrestricted Substitution is not.

Consistency follows from soundness along standard lines.

Since the Representation Theorem may be reformulated, and indeed simplified in the light of the new definition of the mapping from $CI*2$ to T^w2 , the soundness of $CI*2$ is simply proved along the lines followed for $CI*0$. Decidability, completeness, and nontriviality of the circumstantial operator are also proved along the same lines.

8 Theorems and nontheorems of $CI*0$, $CI*1$, $CI*2$ Here is a list of theorems and nontheorems containing the $*$ -operator of $CI*0$, $CI*1$, $CI*2$. Since we know that every system properly includes the preceding one in the given order, the nontheorems are given only for $CI*2$, being understood that the theorems listed under every system are nontheorems in the preceding one in the given order.

$CI*0$

- $p > p$
- $p \supset \diamond * p$
- $(p \rightarrow q) \supset (p > q)$
- $\neg(p > \neg p)$
- $(p > q) \supset \neg(p > \neg q)$
- $\diamond(p \wedge q) \supset ((p \wedge q) > q)$
- $\diamond * p \supset (p > (p \vee q))$
- $\diamond p \supset (((p \vee q) > r) \supset (p > r))$
- $*p \supset \diamond p$
- $(p > q) \supset (*p \supset q)$
- $\diamond(*p \wedge *q) \supset (((p > r) \supset ((p \wedge q) > r))$
- $\diamond(p \wedge q) \supset ((p > r) \supset ((p \wedge q) > r))$
- $(\neg \diamond p \wedge \neg \diamond q) \supset (p > q)$
- $\Box * p \supset \Box p$
- $\diamond * T$
- $\diamond T.$

$CI*1$

- $((p \wedge q) > r) \supset ((\Box * p \wedge \Box * q) \supset \Box r)$
- $((p \wedge q) > q) \supset ((*p \wedge *q) \rightarrow q)$
- $\Box(*p \wedge *q) \supset \Box((p \wedge q) > r).$

$CI*2$

- $((p > q) \wedge (q > r)) \supset (p > r)$
- $*p \equiv **p.$

Nontheorems

$$(p \wedge q) > p$$

$$p > (p \vee q)$$

$$(p > q) \supset (\neg q > \neg p)$$

$$(p > r) \supset ((p \wedge q) > r)$$

$$(p \rightarrow q) \supset *(p > q)$$

$$p > \top$$

$$\perp > p$$

$$(p > \top) \supset (\top > p)$$

$$*\top.$$

Remark 1 CI*2 allows us to prove the transitivity of conditionals, even if the logic of the conditional operator is essentially a nonmonotonic one.

Remark 2 The main drawback in adopting $A >^0 B =_{df} *A \rightarrow B$ as a definition of the synthetic conditional is that $*A \rightarrow A$, i.e. $A >^0 A$, does not turn out to be a thesis unless we introduce $\Box A \supset \Box *A$ as an additional axiom. However, CI*2 cannot be extended in this way. If it were, in fact, from $*\top \rightarrow \top$ we would derive the equivalence $\top \leftrightarrow *\top$ (by a theorem listed on p. 622), so $\top \equiv *\top$. But thanks to the law $(p \wedge *q) \supset (*p \wedge q)$, it would follow $(p \wedge *\top) \supset (*p \wedge \top)$, hence $(p \wedge \top) \supset (*p \wedge \top)$, hence $p \supset *p$.

9 Final remarks A feature of CI*0, CI*1, CI*2 which could be an object of criticism is that $\Diamond p$ in them collapses to $\Diamond *p$, so that $\Diamond(p \wedge q)$ collapses to $\Diamond*(p \wedge q)$. This result amounts to saying that the notion of consistency becomes indistinguishable from the one of “consistency with what is presupposed”. It would be easy to reply by observing that the distinction is not so clear, as a matter of fact, in many cases: if one says “Bizet is French” and “Bizet is Italian”, these two statements are inconsistent insofar as we know that Italy and France are independent countries, but they would both be true in a possible world where Italy is a part of France or France is a part of Italy.

If someone is under the impression that the notion of possibility which is so obtained is too comprehensive, we may suggest that a solution could be reached subjoining the axioms for “*” not to system T but to Burks’ system of “causal” modalities (see [2]). As is well known, this system is obtained by adding to system T the following axioms for a new modal operator \Box :

$$\mathbf{Ax(1)} \quad \Box p \supset \Box p$$

$$\mathbf{Ax(2)} \quad \Box p \supset p$$

$$\mathbf{Ax(3)} \quad \Box(p \supset q) \supset (\Box p \supset \Box q).$$

To the Axiom CI*0.1 we could now add, in place of CI*02,

$$\mathbf{(CI*02')} \quad \Box p \supset \Box *p$$

and, since the converse implication also becomes a theorem, $\Diamond p$ takes the meaning of “ p is logically and physically compatible with the presuppositions”, and $\Diamond(p \wedge q)$ could be read as “ p and q are logically and physically compatible with the presuppositions”, or “ p and q are cotenable”, in the sense of cotenability devised in Goodman [5].

Let us then define a new connective “ \gg ” in this way:

$$A \gg B =_{df} \Box (*A \supset B) \wedge (\Diamond B \supset \Diamond *A).$$

Since all the theorems in \Box now have a duplication given by theorems in \Box , we will have a duplication of the theorems for “ $>$ ” in terms of “ \gg ”. In particular, we will derive

$$\Diamond(p \wedge r) \supset ((p \gg q) \supset ((p \wedge r) \gg q))$$

and its equivalent with $\Diamond*(p \wedge r)$ in place of $\Diamond(p \wedge r)$, but not $\Diamond(p \wedge r) \supset ((p \gg q) \supset ((p \wedge r) \gg q))$.

The models for Burks’ system are 4-tuples $\langle M, R, R^c, V \rangle$ where M, R, V are like the ones for T and R^c is an accessibility relation such that $R^c \subseteq R$. The decision procedure for this new system, and so for wffs containing “ \gg ”, is easily reached by suitable modifications of the one given for CI*0. The details of this construction lie, however, beyond the scope of the present paper.

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NOTES

1. See for instance McCall’s system CFL in [9], p. 442. The system presented in Pizzi [10] includes the law of monotonicity, and so is intermediate between logics of connexive implication and logics of consequential implication.
2. This abridged definition allows us to avoid vacuous truth but cannot avoid that a tautology turns out to be conditionally implied by any consistent statement. In fact $\Diamond *A$ implies (by PC) $(\Diamond B \supset \Diamond *A)$, and $\Box B$ implies $\Box (*A \supset B)$; so, by TPr, $\Diamond *A \wedge \Box B$ implies $A > B$. $\Diamond A \supset (A > \top)$ also turns out to be a theorem. These seem to be paradoxical results for the corner operator, even if they are compatible with Boethius’ Thesis. Readers who find that a logic of synthetic conditionals has to avoid this kind of behavior are justified in thinking that only “ $>^0$ ” is the correct connective for synthetic conditionals. A treatment of the logic of “ $>^0$ ” is outside the scope of the present paper, even if it may easily be obtained using the tools introduced in Section 3. See however Remark 2 on p. 634.

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