A Multidimensional Modal Translation for a Formal System Motivated by Situation Semantics

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Abstract The content of this paper is as follows: first a multidimensional quantified modal logic is presented, and soundness, completeness, and compactness are proven for it. Then we propose a translation of the sentences of the language $\mathbb{L}$ presented in J. Barba's "Two formal systems for situation semantics" (to appear in *Notre Dame Journal of Formal Logic*) and prove that it is a suitable translation. Finally, we propose a notion of persistence for the sentences of $\mathbb{L}$ (inspired in the concept of persistence defined in Situation Semantics) and find its modal counterpart in our multidimensional system.

1 A multidimensional modal system

The language $\mathbb{L}(M)$ $\mathbb{L}(M)$ is a multidimensional quantified modal language whose logical symbols are $\wedge$, $\neg$, $\square$, $\forall$, $\mathsf{E}!$ (the existence predicate), $=$, and $\otimes_i$, for each natural number i. We also have predicate, constant, and variable symbols. The set of wffs contains only closed formulas and is defined in the standard way.

The semantics for $\mathbb{L}(M)$ is as follows: $\mathbb{L}(M)$'s models are structures $M = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$, where $\mathbb{W}$ and $\mathbb{U}$ are sets, $\mathbb{W} \neq \varnothing$, $\mathbb{D}$ is a function defined on $\mathbb{W}$ such that $\mathbb{D}(w) \subseteq \mathbb{U}$, I is an interpretation function such that $I(c) \in \mathbb{U}$, for each constant c and $I(P, w) \subseteq \mathbb{U}^n$, for each n-ary predicate P and $w \in \mathbb{W}$. R is a reflexive and transitive binary relation on $\mathbb{W}$. We further stipulate that, for every $v, w \in \mathbb{W}$, if vRw then $\mathbb{D}(v) \subseteq \mathbb{D}(w)$ (nested domains condition).

We shall use $\mathbb{U}$-formulas. A $\mathbb{U}$-formula is the result of substituting elements of $\mathbb{U}$ for some constants in a wff. (The concept of $\mathbb{U}$-formula has been borrowed from Smullyan [6]. Notice, however, that this concept is sound only if we assume that no member of $\mathbb{U}$ is already a constant or a variable of our language. This is a harmless assumption, but it is necessary, as the referee pointed out to me.) A parameter of a $\mathbb{U}$-formula A is either a constant or an element r

of $\mathbb{U}$ in A. We extend the function I in such a way that it is defined for any parameter t, stipulating that for any $r \in \mathbb{U}$, $I(r) = r$.

We can now define a valuation function V which evaluates $\mathbb{U}$-formulas in a model M relative to ω-tuples $\vec{u}$ in $\mathbb{W}$ and an element u_i in $\vec{u}$ as follows:

- For any $\mathbb{U}$-wff A, ω-tuple $\vec{u}$ in $\mathbb{W}$ and u_i $(i \geq 0)$, $V(A, \vec{u}, u_i) \in \{1, 0\}$.
- $V(t = t', \vec{u}, u_i) = 1$ iff $I(t) = I(t')$.
- $V(Pt_1, \ldots, t_n, \vec{u}, u_i) = 1$ iff $\langle I(t_1), \ldots, I(t_n) \rangle \in I(P, u_i)$.
- $V(\mathsf{E}!t, \vec{u}, u_i) = 1$ iff $I(t) \in \mathbb{D}(u_i)$.
- $V(\neg A, \vec{u}, u_i) = 1$ iff $V(A, \vec{u}, u_i) = 0$.
- $V(A \wedge B, \vec{u}, u_i) = 1$ iff $V(A, \vec{u}, u_i) = V(B, \vec{u}, u_i) = 1$.
- $V(\forall x A, \vec{u}, u_i) = 1$ iff $V(A[r/x], \vec{u}, u_i) = 1$ for every $r \in \mathbb{U}$.
- $V(\otimes_j A, \vec{u}, u_i) = 1$ iff $V(A, \vec{u}, u_j) = 1$.
- $V(\square A, \vec{u}, u_i) = 1$ iff $V(A, \vec{u}[v/i], v) = 1$, for every $v \in \mathbb{W}$ such that $u_i R v$, where $\vec{u}[v/i]$ is the result of substituting v for u_i in $\vec{u}$.

As the accessibility relation R is a reflexive and transitive relation, we have a logic whose propositional basis is $S4$. The quantification domain is $\mathbb{U}$, but combining quantifiers with the existence predicate $\mathsf{E}!$ and operators $\otimes_i$ we can quantify over the domain of any u_i in $\vec{u}$ using formulas of the following kind:

$$\forall x(\otimes_i \mathsf{E}!x \rightarrow A(x)).$$

We say that a sentence A is valid in a model M iff for every $\vec{u}$ in M, $V(A, \vec{u}, u_0) = 1$. A is valid iff it is valid in any model M.

Let Σ, Ω be sets of wffs of $\mathbb{L}(M)$. $\langle \Sigma, \Omega \rangle$ is verified by a model M iff for every $\vec{u}$ in M, if $V(A, \vec{u}, u_0) = 1$ for every $A \in \Sigma$, then $V(B, \vec{u}, u_0) = 1$ for some $B \in \Omega$, and otherwise it is refuted by M. $\langle \Sigma, \Omega \rangle$ is valid iff it is verified by every model M. Whenever a pair $\langle \Sigma, \Omega \rangle$ is valid, we write $\Sigma \vDash_M \Omega$. (We omit the index M when no confusion can arise.)

Indices We shall use a certain kind of indices as an auxiliary device in the formulation of a deductive system for $\mathbb{L}(M)$ and in the completeness proof. (Our use of indices is inspired in the ideas contained in Fitting [3], Smullyan [7], and, particularly, Marraud [4] and [5]). We use $a, a', a_1, \ldots$ to represent indices. Each index has the following structure:

$$a = \langle \langle a_0, \ldots, a_i, \ldots \rangle, a_i \rangle$$

(or, more briefly, $a = \langle \vec{a}, a_i \rangle$), where, for each $j \geq 0$ a_j is a finite sequence of natural numbers whose first term is j. Let k be a sequence (perhaps empty) of natural numbers. By $a_j \wedge k$ we mean the result of adding the sequence k on the right of a_j. $a_j \wedge k$ is then an extension of a_j. If $k \neq \varnothing$, the extension is proper. For further use, let $\vec{o} = \langle \langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle \ldots \rangle$, and let o be the index $\langle \vec{o}, \langle 0 \rangle \rangle$.

Index readings An index reading in a model M is a function f defined on the set of all finite sequences of natural number and whose range is a subset of $\mathbb{W}$. Every index reading must satisfy the following restriction:

For any two sequences s and s' such that s' is an extension of s, $f(s) R f(s')$.

We write $f(\vec{a})$ for $\langle f(a_0), \ldots, f(a_i), \ldots \rangle$ and $f(a)$ for $\langle f(\vec{a}), f(a_i) \rangle$.

Valuation of indexed formulas   An indexed formula is a pair $\langle A, a \rangle$, where A is a wff and a is an index. We shall write A_a instead of $\langle A, a \rangle$. We can define a valuation function defined on indexed formulas relative to an index reading f as follows:

$$V(A_a, f) = V(A, f(\vec{a}), f(a_i)).$$

An index reading f verifies an indexed formula A_a in M iff $V(A_a, f) = 1$. Notice that A is valid in a model M iff for every index reading f, $V(A_0, f) = 1$.

The sequent system S(M)   A sequent is a pair $\langle \Gamma, \Delta \rangle$, where Γ and Δ are finite sets of wffs of $\mathbb{L}(M)$. An indexed sequent is a pair $\langle \Gamma, \Delta \rangle$ where Γ, Δ are sets of indexed formulas. All sequents and formulas appearing in the system $S(M)$ are indexed ones, so $S(M)$ is a proof system for indexed sequents. Let $\langle \Gamma, \Delta \rangle$ be an indexed sequent. We write $\Gamma \vdash \Delta$ to mean that $\langle \Gamma, \Delta \rangle$ is provable in $S(M)$.

Let Σ be a set of wffs of $\mathbb{L}(M)$ and let a be any index. Then we stipulate that $\Sigma_a = \{A_a : A \in \Sigma\}$.

Let Γ, Δ be sets of wffs of $\mathbb{L}(M)$ (i.e., they are sets of nonindexed formulas). We define that $\langle \Gamma, \Delta \rangle$ is provable in $S(M)(\Gamma \vdash \Delta)$ iff $\Gamma_o \vdash \Delta_o$, where o is the index specified above.

$S(M)$ has the following axiom schemes and rules:

Axiom schemes

1. $(Pc_1, \ldots, c_n)_a \vdash (Pc_1, \ldots, c_n)_{a'}$, provided that $a = \langle \vec{a}, a_i \rangle$ and $a' = \langle \vec{a}', a_i \rangle$.
2. $(t_1 = s_1)_{a1}, \ldots, (t_n = s_n)_{an}, (Pt_1, \ldots, t_n)_a \vdash (Ps_1, \ldots, s_n)_{a'}$, where a and a' are as in scheme 1.
3. $(c = c')_a \vdash (c = c')_{a'}$, for any indices a and a'.

Rules   (Γ, Δ are sets of indexed formulas).

W $\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$

1a $\dfrac{\Gamma, A_a \vdash \Delta}{\Gamma \vdash \neg A_a, \Delta}$ **1b** $\dfrac{\Gamma \vdash A_a, \Delta}{\Gamma, \neg A_a \vdash \Delta}$

2a $\dfrac{\Gamma, A_a, B_a \vdash \Delta}{\Gamma, A \wedge B_a \vdash \Delta}$ **2b** $\dfrac{\Gamma \vdash A_a, \Delta; \ \Gamma \vdash B_a, \Delta}{\Gamma \vdash A \wedge B_a, \Delta}$

3a $\dfrac{\Gamma, A(t)_a \vdash \Delta}{\Gamma, \forall x A(x)_a \vdash \Delta}$ **3b** $\dfrac{\Gamma \vdash A(t)_a, \Delta}{\Gamma \vdash \forall x A(x)_a, \Delta}$ (*)

(*) provided t does not occur in $\forall x A(x), \Gamma, \Delta$.

4a $\dfrac{\Gamma, A_{a'} \vdash \Delta}{\Gamma, \otimes_i A_a \vdash \Delta}$ (*)   **4b** $\dfrac{\Gamma \vdash A_{a'}, \Delta}{\Delta \vdash \otimes_i A_a, \Delta}$ (*)

(*) provided $a' = \langle \vec{a}, a_i \rangle$ and $a = \langle \vec{a}, a_j \rangle$.

5a $\dfrac{\Gamma, A_{a'} \vdash \Delta}{\Gamma, \Box A_a \vdash \Delta}$ (*)   **5b** $\dfrac{\Gamma \vdash A_{a'}, \Delta}{\Gamma \vdash \Box A_a, \Delta}$ (**)

(*) provided $a = \langle \langle a_0, \ldots, a_i, \ldots \rangle, a_i \rangle$ and $a' = \langle \langle a_0, \ldots, a_i', \ldots \rangle, a_i' \rangle$, where, for some finite sequence of natural numbers k, $a_i' = a_i \wedge k$.

(**) a and a' are as in the former case and, moreover, there is no index in Γ, Δ containing a sequence a_i'' such that, for some sequence k, $a_i'' \wedge k = a_i'$ or $a_i'' = a_i' \wedge k$.

6a $\dfrac{\Gamma, c = c_a' \vdash \Delta}{\Gamma, c' = c_a \vdash \Delta}$ **6b** $\dfrac{\Gamma \vdash c = c_a', \Delta}{\Gamma \vdash c' = c_a, \Delta}$

7 $\dfrac{\Gamma, c = c_a'' \vdash \Delta}{\Gamma, c = c_{a1}', c' = c_{a2}'' \vdash \Delta}$ **8** $\dfrac{\Gamma, c = c_a \vdash \Delta}{\Gamma \vdash \Delta}$

Soundness of S(M)   Let Γ, Δ be (finite or infinite) sets of indexed formulas. We say that $\langle \Gamma, \Delta \rangle$ is valid in a model M iff for every index reading f, if $V(A, f) = 1$ for every $A \in \Gamma$, then for some $B \in \Delta$, $V(B, f) = 1$. Otherwise, $\langle \Gamma, \Delta \rangle$ is refuted by M. $\langle \Gamma, \Delta \rangle$ is valid ($\Gamma \vDash \Delta$) iff it is valid in every M.

Lemma 1   *Let* Γ, Δ *be finite sets of indexed formulas. Then,* $\Gamma \vdash \Delta$ *only if* $\Gamma \vDash \Delta$.

The proof strategy is the usual one. It is immediate that all axioms have the desired property, and it is not difficult to check that rules transmit it. This is obvious for Rule W, and the proof is easy for Rules 1, 2, 3, 4, 6, 7, and 8. So, we shall detail only Rules 5a and 5b.

5a: Let a and a' be as specified. Suppose $\Gamma, A_{a'} \vdash \Delta$. By hypothesis, $\langle \Gamma \cup \{A_{a'}\}, \Delta \rangle$ is valid. Applying 5a we can infer $\Gamma, \Box A_a \vdash \Delta$. Suppose $\langle \Gamma \cup \{\Box A_a\}, \Delta \rangle$ is not valid. Then there are a model M and an index reading f verifying every formula in $\Gamma \cup \{\Box A_a\}$ but none in Δ. Then, $V(\Box A_a, f) = 1$, i.e., $V(A, f(\vec{a}), f(a_i)) = 1$ (where $a = \langle \vec{a}, a_i \rangle$), thus $V(A, f(\vec{a}) [u/i], u)$, for every $u \in \mathbb{W}$ such that $f(a_i)Ru$. a_i' is an extension of a_i, so $f(a_i)Rf(a_i')$. Then, $V(A, f(\vec{a}'), f(a_i')) = 1$, and $V(A_{a'}, f) = 1$. But this means that f refutes $\langle \Gamma \cup \{A_{a'}\}, \Delta \rangle$ in M, contradicting the hypothesis.

5b: Let $\Gamma \vdash A_{a'}, \Delta$, as specified in Rule 5b. Suppose $\langle \Gamma, \{\Box A_a\} \cup \Delta \rangle$ is not valid; that is, for some f in some M, f verifies every formula Γ but none in $\{A_a\} \cup \Delta$. Then, $V(\Box A_a, f) = 0$, i.e., $V(\Box A_a, f(\vec{a}), f(a_i)) = 0$, and for some $u \in \mathbb{W}$ such that $f(a_i)Ru$, $V(A, f(\vec{a}) [u/i], u) = 0$. Let g be an index reading defined as below:

a. $g(a_i') = u$.

b. $g(a_i) = f(a_i)$.

c. for every sequence of natural numbers b which is a proper extension of a_i and such that a_i' is an extension of b or such that b is an extension of a_i', $g(b) = u$.

d. $g(b) = f(b)$ for any sequence b not considered above.

g satisfies the conditions imposed on index readings: for any extension k of a_i of which a_i' is an extension, $g(a_i)Rg(k)$, because $g(k) = u$ and $g(a_i) = f(a_i)Ru$.

Moreover, as R is a reflexive relation, uRu, and so $g(k)Rg(a_i')$. By the same reason, for any extension b of a_i', $g(a_i')Rg(b)$. And for any other sequence k, $f(k) = g(k)$. Then, since f is an index reading, so is g.

Now, $g(\vec{a}') = f(\vec{a})\,[u/i]$, and since $g(a_i') = u$, we have that $V(A, g(\vec{a}')$, $g(a_i')) = 0$, and so $V(A_{a'}, g) = 0$. According to the proviso added to Rule 5b, with respect to the indices occurring in $\Gamma \cup \Delta$, g is identical to f, and so g verifies every formula in Γ but none in Δ. Then, g refutes $\langle \Gamma, \{A_{a'}\} \cup \Delta \rangle$, which is not valid. Thus, if this pair is valid, so is $\langle \Gamma, \{\Box A_{a'}\} \cup \Delta \rangle$.

Theorem 2 (Soundness Theorem)   *Let Γ and Δ be finite sets of wffs of $\mathbb{L}(M)$. Then, $\Gamma \vdash \Delta$ only if $\Gamma \vDash \Delta$.*

Proof: By definition, $\Gamma \vdash \Delta$ iff $\Gamma_o \vdash \Delta_o$. By Lemma 1, $\Gamma_o \vdash \Delta_o$ only if $\langle \Gamma_o, \Delta_o \rangle$ is valid, that is, only if for any f in any M if $V(A_o, f) = 1$ for every $A \in \Gamma$, then $V(B_o, f) = 1$ for some $B \in \Delta$. This means that if $V(A, f(\vec{o}), f(0)) = 1$ for every $A \in \Gamma$ then $V(B, f(\vec{o}), f(0)) = 1$ for some $B \in \Delta$. Since this must be so for any index reading f, we have that for any tuple $\vec{u}$ in any M, $V(A, \vec{u}, u_0) = 1$ for every $A \in \Gamma$ only if $V(B, \vec{u}, u_0) = 1$ for some $B \in \Delta$, so that $\Gamma \vDash \Delta$.

Completeness of S(M)   We are going to prove that given any two sets of wffs of $\mathbb{L}(M)$ Σ, Ω, if $\Sigma \vDash \Omega$ then there are finite sets Γ, Δ, $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$. In order to do so we construct a tree for $\langle \Sigma, \Omega \rangle$ and prove that if the tree closes then there exist the above-mentioned sets, and that if the tree has at least an open branch, we can construct a model M which refutes $\langle \Sigma, \Omega \rangle$.

First, we extend the language with a denumerable set of new constants, and suppose fixed enumerations of this set and of $\Sigma \cup \Omega$.

Each node of the tree contains a pair $\langle A_a, i \rangle$, where A_a is an indexed formula of the language extended with the new constants and $i \in \{1, 0\}$. The tree is constructed step by step according to the following instructions:

Step 1: Let A be the first formula in the enumeration of $\Sigma \cup \Omega$. If $A \in \Sigma$, put $\langle A_o, 1 \rangle$ in the topmost node of the tree, and if $A \in \Omega$, put $\langle A_o, 0 \rangle$ in the topmost node.

Step $n + 1$: In each open node consider the nth node of the branch, and proceed according to its form. If the node is:

    1a. $\langle \neg A_a, 1 \rangle$: Add $\langle A_a, 0 \rangle$.
    1b. $\langle \neg A_a, 0 \rangle$: Add $\langle A_a, 1 \rangle$.
    2a. $\langle A \wedge B_a, 1 \rangle$: Add the nodes $\langle A_a, 1 \rangle$ and $\langle B_a, 1 \rangle$.
    2b. $\langle A \wedge B_a, 0 \rangle$: Divide the branch in two new ones. Put $\langle A_a, 0 \rangle$ in the first one and $\langle B_a, 0 \rangle$ in the other.
    3a. $\langle \forall x A(x)_a, 1 \rangle$: Add nodes $\langle A(c)_a, 1 \rangle$, for each constant c occurring in the branch and then repeat $\langle \forall x A(x)_a, 1 \rangle$.
    3b. $\langle \forall x A(x)_a, 0 \rangle$: Add $\langle A(c)_a, 0 \rangle$, for the first new constant c not occurring in the branch.
    4a. $\langle \otimes_i A_a, 1 \rangle$: Add $\langle A_{a'}, 1 \rangle$, where $a = \langle \vec{a}, a_j \rangle$ and $a' = \langle \vec{a}, a_i \rangle$.
    4b. $\langle \otimes_i A_a, 0 \rangle$: Add $\langle A_{a'}, 0 \rangle$, where $a = \langle \vec{a}, a_j \rangle$ and $a' = \langle \vec{a}, a_i \rangle$.

5a. $\langle \Box A_a, 1 \rangle$: Add a node $\langle A_{a'}, 1 \rangle$ for each a' satisfying the condition established below, and finally repeat $\langle \Box A_a, 1 \rangle$. Let $a = \langle\langle a_0, \ldots, a_i, \ldots \rangle a_i \rangle$. Then $a' = \langle\langle a_0, \ldots, a_i', \ldots \rangle, a_i' \rangle$, where a_i' is any extension of a_i occurring in the branch.

5b. $\langle \Box A_a, 0 \rangle$: Add $\langle A_{a'}, 0 \rangle$ where a' is as follows: if $a = \langle\langle a_0, \ldots, a_i, \ldots \rangle a_i \rangle$, then $a' = \langle\langle a_0, \ldots, a_i', \ldots \rangle, a_i' \rangle$, where $a_1' = a_i \wedge k$, and k is the first natural number such that neither a_1' nor any of its extensions occur in the branch.

6a,7. $\langle (c_1 = c_2)_a, 1 \rangle$: First of all, add $\langle (c_2 = c_1)_a, 1 \rangle$. Then for each node $\langle (c_2 = c_3)_{a'}, 1 \rangle$ occurring in the branch add $\langle (c_1 = c_3)_a, 1 \rangle$.

6b. $\langle c = c_a', 0 \rangle$: Add $\langle c' = c_a, 0 \rangle$, if it does not appear in the branch before.

8. Then, extend the branch with all nodes $\langle c = c_o, 1 \rangle$ such that c occurs in the branch but $\langle c = c_a, 1 \rangle$ does not appear in the branch for any a.

Finally, if A is the $n + 1$-th formula of $\Sigma \cup \Omega$, add $\langle A_o, 1 \rangle$ if $A \in \Sigma$, and add $\langle A_o, 0 \rangle$ if $A \in \Omega$. After having performed the instructions above for every open branch in the tree, the $n + 1$-th step is finished.

A branch closes iff it contains a group of nodes of one of the forms detailed below:

1. $\langle (Pc_1, \ldots, c_n)_a, 1 \rangle$, $\langle (Pc_1, \ldots, c_n)_{a'}, 0 \rangle$, for $a = \langle \vec{a}, a_i \rangle$ and $a' = \langle \vec{a}', a_i \rangle$.
2. $\langle (c_1 = d_1)_{a1}, 1 \rangle, \ldots, \langle (c_n = d_n)_{an}, 1 \rangle$, $\langle (Pc_1, \ldots, c_n)_{a'}, 1 \rangle$, $\langle (Pd_1, \ldots, d_n)_{a'}, 0 \rangle$, for a and a' as in 1.
3. $\langle c = c_a', 1 \rangle$, $\langle c = c_{a'}', 0 \rangle$, for any indices a, a'.

A tree closes iff every branch in it closes.

Lemma 3   *If a tree for $\langle \Sigma, \Omega \rangle$ closes, there exist finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$.*

The proof, purely routine, is left to the reader. (Notice that the numbers in front of the instructions for the construction of the tree refer to the rules which justify them.)

Lemma 4   *If a tree for $\langle \Sigma, \Omega \rangle$ does not close, then $\langle \Sigma, \Omega \rangle$ is refutable.*

Proof: Suppose we have constructed a tree for $\langle \Sigma, \Omega \rangle$ and the tree has an open branch. We can construct a model $M = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$ which refutes $\langle \Sigma, \Omega \rangle$. Define M as follows:

$$\mathbb{W} = \{ a_i : a_i \text{ is a sequence occurring in some index in the branch} \}.$$

Let C be the set of all constants appearing in the branch. Define a relation $\approx$ in C as follows: $c \approx c'$ iff $\langle c = c_a', 1 \rangle$ appears in the branch for some a. $\approx$ is an equivalence relation: Instruction 8 ensures reflexivity, while instruction 6a,7 guarantees symmetry and transitivity.

Let $|c|$ be the equivalence class to which c belongs. Then

$$\mathbb{U} = \{ |c| : c \in C \}$$

$$\mathbb{D}(a_i) = \{ |c| : \langle \text{E}! c_a, 1 \rangle \text{ appears in the branch, for } a = \langle \vec{a}, a_i \rangle \}$$

$I(c) = |c|$, for each $c \in C$. For any other constant c, $I(c)$ takes any arbitrary value in $\mathbb{U}$.

$I(P, a_i) = \{\langle |c_1|, \ldots, |c_n| \rangle : \langle (Pc_1, \ldots, c_n)_a, 1 \rangle$ appears in the branch, for $a = \langle \vec{a}, a_i \rangle \}$.

$a_i Ru_{aj}$ iff a_j is an extension (proper or not) of a_i.

Define an index reading f as follows:

$f(a_i) = a_i$, for each a_i in the branch. For any sequence k not appearing in the branch $f(k)$ takes any adequate value in $\mathbb{W}$.

Claim   *For M and f defined as above,*
a. *if $\langle A_a, 1 \rangle$ appears in the branch, $V(A_a, f) = 1$.*
b. *if $\langle A_a, 0 \rangle$ appears in the branch, $V(A_a, f) = 0$.*

The proof, an induction on the complexity of A, is left to the reader.

We can now prove the lemma. Consider the index o as defined above. Let $f(\vec{o}) = \vec{u}$ and $f(i) = u_i$, for each $i \geq 0$. Suppose $A \in \Sigma$. A is the nth formula of $\Sigma \cup \Omega$, for some n, and $\langle A_o, 1 \rangle$ has been introduced in the branch at the nth step, by the claim above, $V(A_o, f) = V(A, \vec{u}, u_0) = 1$. Similarly, for any $B \in \Omega$, $V(B, \vec{u}, u_0) = 0$. So, M refutes $\langle \Sigma, \Omega \rangle$.

Theorem 5 (Completeness theorem for $S(M)$) $\Sigma \vDash \Omega$ *only if there exist finite sets* $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash \Delta$.

Proof: Suppose $\Sigma \vDash \Omega$. A tree for $\langle \Sigma, \Omega \rangle$ closes, because otherwise $\langle \Sigma, \Omega \rangle$ would be refutable, by Lemma 4. But then, by Lemma 3, there exist sets Γ, Δ as required.

Theorem 6 (Compactness theorem)   *Let Σ, Ω be two sets of wffs of $\mathbb{L}(M)$. $\Sigma \vdash \Omega$ iff there are finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \nvDash \Delta$.*

Proof: Suppose $\Sigma \vDash \Omega$. By completeness, there are finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$. By soundness, $\Gamma \vDash \Delta$.

2 A translation from $\mathbb{L}$ to $\mathbb{L}$ (M)   We propose a translation T from $\mathbb{L}$ to $\mathbb{L}(M)$ defined inductively as follows:

1a. $T(Pc_1, \ldots, c_n) = \mathsf{E}!c_1 \wedge \ldots \wedge \mathsf{E}!c_n \wedge \square \lozenge Pc_1, \ldots, c_n$.
1b. $T(\neg Pc_1, \ldots, c_n) = \mathsf{E}!c_1 \wedge \ldots \wedge \mathsf{E}!c_n \wedge \square \lozenge \neg Pc_1, \ldots, c_n$.
2a. $T(c = c') = \mathsf{E}!c \wedge \mathsf{E}!c' \wedge c = c'$.
2b. $T(\neg c = c') = \mathsf{E}!c \wedge \mathsf{E}!c' \wedge \neg c = c'$.
 3. $T(\neg \neg A) = T(A)$.
4a. $T(A \wedge B) = T(A) \wedge T(B)$.
4b. $T(\neg (A \wedge B)) = T(\neg A) \vee T(\neg B)$.
5a. $T(A \rightarrow B) = \neg T(A) \vee T(B)$.
5b. $T(\neg (A \rightarrow B)) = T(A) \wedge T(\neg B)$.
6a. $T(A | B) = T(A) \wedge T(B)$.
6b. $T(\neg (A | B)) = T(A) \wedge T(\neg B)$.
7a. $T(\forall x A) = \forall x T(A)$.

7b. $T(\neg \forall x A) = \neg \forall x \neg T(\neg A) = \exists x T(\neg A)$.

8a. $T(\otimes_i A) = \otimes_i T(A)$.

8b. $T(\neg \otimes_i A) = \otimes_i T(\neg A)$.

Let f be a function such that for any $\mathbb{L}(M)$-model $M = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$ yields an $\mathbb{L}$-model $f(M) = N = \langle W, U, D, [\![\ ]\!], S \rangle$ according to the following clauses:

$W = \mathbb{W}$,

$U = \mathbb{U}$,

$D(v) = \mathbb{D}(v)$,

$[\![c]\!] = I(c)$ and

$S(v) = \{A : A$ is either $Pr_1, \ldots, r_n$ or $\neg Pr_1, \ldots, r_n$ and $V(T(A), \vec{u}, v) = 1\}$, for any $\vec{u}$ such that v is a member of $\vec{u}$. (Notice that $S(v)$ depends only on v, and not on the other members of $\vec{u}$, as $T(A)$ contains no operator $\otimes_j$).

Lemma 7   *Let M be an $\mathbb{L}(M)$-model $M = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$ and N be an $\mathbb{L}$-model $N = \langle W, U, D, [\![\ ]\!], S \rangle$ such that*

  a. *$\mathbb{U} = U$, and*
  b. *$I(c) = [\![c]\!]$,*

for every constant c. Suppose $\vec{u}$ is an ω-tuple $\vec{u} = \langle u_0, .., u_i \ldots \rangle$ such that, for every $i \geq 0$,

  c. *$u_i \in \mathbb{W} \cap W$,*
  d. *$\mathbb{D}(u_i) = D(u_i)$*
  e. *$S(u_i) = \{A : A$ is $(\neg)Pr_1, \ldots, r_n$ and $V(T(A), \vec{u}, u_i) = 1\}$. (As before, $S(u_i)$ depends only on u_i.)*

Then, $V(T(A), \vec{u}, v) = 1$ iff $\vec{u}, v \vDash A$, for every sentence A of $\mathbb{L}$, and any member v of $\vec{u}$.

Proof: By induction on the complexity of A.

Corollary 8   *Let $M = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$ and $f(M) = N = \langle W, U, D, [\![\ ]\!], S \rangle$. Then $V(T(A), \vec{u}, v) = 1$ iff $\vec{u}, v \vDash A$, for any sentence A of $\mathbb{L}$, any ω-tuple $\vec{u}$ in $\mathbb{W}$ and any member v of $\vec{u}$.*

Proof: This corollary follows immediately from Lemma 7 and the definition of f.

Now, let g be a function which, for each $\mathbb{L}$-model $N = \langle W, U, D, [\![\ ]\!], S \rangle$, yields the $\mathbb{L}(M)$-model $M = g(N) = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$ constructed as explained below:

For each $u \in W$ we define:

$$H(u) = \{u\} \cup \{u_{t,1}, u_{t,2}\} \cup \{u_{f,1}, u_{f,2}\}.$$

Then,

$\mathbb{W} = \bigcup_{u \in W} H(u)$.

$\mathbb{U} = U$

$\mathbb{D}(u) = \mathbb{D}(u_{t,i}) = \mathbb{D}(u_{f,i}) = D(u),\ i = 1,2.$

$I(c) = [\![c]\!].$

$I(P, u) = \{\langle r_1, \ldots, r_n \rangle : Pr_1, \ldots, r_n \in S(u)\}.$

$t(u) = \{\langle r_1, \ldots, r_n \rangle : Pr_1, \ldots, r_n \notin S(u) \text{ and } \neg Pr_1, \ldots, r_n \notin S(u)\}.$

$I(P, u_{t,2}) = I(P, u) \cup t(u).$

$I(P, u_{t,1}) = I(P, u_{t,2}) - \{\langle r_1, \ldots, r_n \rangle : \neg Pr_1, \ldots, r_n \in S(u)\}.$

$I(P, u_{f,2}) = I(P, u).$

$I(P, u_{f,1}) = I(P, u) - \{\langle r_1, \ldots, r_n \rangle : \neg Pr_1, \ldots, r_n \in S(u)\}.$

R is the reflexive and transitive closure of the least binary relation in $\mathbb{W}$ which contains all pairs of the following forms:

$\langle u, v \rangle : u, v \in W$ and $u \sqsubseteq v.$

$\langle u, u_{t,i} \rangle,\ \langle u, u_{f,i} \rangle,$ where $u \in W$ and $i = 1,2,$

$\langle u_{t,i}, u_{t,j} \rangle,\ \langle u_{f,i}, u_{f,j} \rangle,$ where $u \in W$ and $i, j = 1,2.$

It can be easily checked that this M is an $\mathbb{L}(M)$-model.

Lemma 9   *Let* $N = \langle W, U, D, [\![\ ]\!], S \rangle$ *and* $g(N) = M = \langle \mathbb{W}, \mathbb{U}, \mathbb{D}, I, R \rangle$. *Let* A *be* $Pc_1, \ldots, c_n$ *or* $\neg Pc_1, \ldots, c_n$. *For any* $u \in W$, $\vec{u}$ *and* u *in* $\vec{u}$, $\vec{u}, u \vDash A$ (*i.e.,* $A \in S(u)$) *iff* $V(\square \lozenge A, \vec{u}, u) = 1.$

Proof:
1. Suppose $\vec{u}, u \vDash A$.
   a. A is $Pc_1, .., c_n$, and so $\langle r_1, \ldots, r_n \rangle \in S(u)$, for $[\![c_i]\!] = r_i$, $1 \le i \le n$. Take any $v \in \mathbb{W}$, uRv.
      i. If v is u, take (for instance) $u_{f,2}$. $\langle r_1, \ldots, r_n \rangle \in I(P, u_{f,2})$, then $V(A, \vec{u}[u_{f,2}/i], u_{f,2}) = 1$ and $V(\lozenge A, \vec{u}, v) = 1.$
      ii. If v is $u_{f,j}$ ($j \in \{1,2\}$), $u_{f,i}Ru_{f,2}$, and, as in 1, $V(A, \vec{u}[u_{f,2}/i], u_{f,2}) = 1$, from which it follows that $V(\lozenge A, \vec{u}[u_{f,j}/i], u_{f,j}) = 1$. The same reasoning works for $u_{t,j}$.
      iii. If v is $w \in W$, $A \in S(w)$, since $u \sqsubseteq w$, and, by the argument developed in 1, $V(\lozenge A, \vec{u}[w/i], w) = 1.$
      iv. If v is $w_{f,j}$ (for $w \in W$), we have that $u \sqsubseteq w$, and, by the argument in 2, $V(\lozenge A \vec{u}[w_{f,j}/i], w_{f,j}) = 1$. The same argument works for $w_{t,j}$.
   For any $v \in \mathbb{W}$ such that uRv, v must be comprehended in one of the cases considered above, thus $V(\lozenge A, \vec{u}[v/i], v) = 1$, and so $V(\square \lozenge A, \vec{u}, u) = 1.$
   b. Suppose now that A is $\neg Pc_1, \ldots, c_n$ and $\neg Pr_1, \ldots, r_n \in S(u)$. Recall that $\langle r_1, \ldots, r_n \rangle \notin I(P, u_{t,1})$ (Idem for $I(P, u_{f,1})$), therefore

   $$V(Pc_1, \ldots, c_n, \vec{u}[u_{t,1}/i], u_{t,1}) = 0, \quad \text{and}$$
   $$V(\neg Pc_1, \ldots, c_n, \vec{u}[u_{t,1}/i], u_{t,1}) = 1.$$

   Using this fact, we can argue cases i–iv as above.
   Regardless, if uRv, then $V(\lozenge A, \vec{u}[v/i], v) = 1$. Thus, $V(\square \lozenge A, \vec{u}, u) = 1.$

2. Suppose $\vec{u}, u \not\Vdash A$:

   a. A is $Pc_1, \ldots, c_n$. Let $[\![c_j]\!] = r_j$, $i \leq j \leq n$. Obviously, $\langle r_1, \ldots, r_n \rangle \notin$ $I(P, u)$ and $\langle r_1, \ldots, r_n \rangle \notin I(P, u_{f,k})$, for $k = 1,2$, so that $V(A, \vec{u}[u_{f,k}/i], u_{f,k}) = 0$. Take $u_{f,1}$, $u_{f,1} R v$ iff v is $u_{f,k}$, for $k = 1,2$; therefore $V(\Diamond A, \vec{u}[u_{f,1}/i], u_{f,1}) = 0$, and, since $u R u_{f,1}$, $V(\Box \Diamond A, \vec{u}, u) = 0$.

   b. A is $\neg Pc_1, \ldots, c_n$, $[\![c_j]\!] = r_j$. We must distinguish two cases:

      i. $Pr_1, \ldots, r_n \in S(u)$. Since $\neg Pr_1, \ldots, r_n \notin S(u)$, $\langle r_1, \ldots, r_n \rangle \in$ $I(P, u_{f,k})$, $k = 1,2$. Thus, $V(Pc_1, \ldots, c_n, \vec{u}[u_{f,k}/i], u_{f,k}) = 1$, and $V(A, \vec{u}[u_{f,k}/i], u_{f,k}) = 0$, and, as in 2a, $V(\Diamond A, \vec{u}, u_{f,1}) = 0$ and $V(\Box \Diamond A, \vec{u}, u) = 0$.

      ii. $Pr_1, \ldots, r_n \notin S(u)$. Then, $\langle r_1, \ldots, r_n \rangle \in I(P, u_{t,k})$ for $k = 1,2$, and $V(Pc_1, \ldots, c_n, \vec{u}[u_{t,k}/i], u_{t,k}) = 1$; therefore $V(A, \vec{u}[u_{t,k}/i], u_{t,k}) = 0$, and, once more, $V(\Box \Diamond A, \vec{u}, u) = 0$.

Corollary 10   *Let N be an $\mathbb{L}$ model and let $g(N)$ be as defined above. Then, for any $\vec{u}$ in W, any member v of $\vec{u}$ and any wff A of $\mathbb{L}$, $V(T(A), \vec{u}, v) = 1$ iff $\vec{u}, v \Vdash A$.*

Proof: Follows from Lemmas 7 and 9.

Theorem 11   *A sentence A is $\mathbb{L}$-valid iff $T(A)$ is $\mathbb{L}(M)$-valid.*

Proof: By the definition of $g(N)$, Conditions 1–4 of Lemma 9 are satisfied. By Lemma 9, Condition 5 is satisfied as well. Applying Lemma 9 we obtain the desired result.

Let $N = \langle W, U, D, [\![\ ]\!], S \rangle$ be an $\mathbb{L}$-model. We define a binary relation $\sqsubseteq$ in W as follows: for any $u, v \in W$, $u \sqsubseteq v$ iff $D(u) \subseteq D(v)$ and $S(u) \subseteq S(v)$. We say that a sentence A of $\mathbb{L}$ is positively persistent relative to i iff for any $N, \vec{u}$ in W and $v \in W$ such that $u_i \sqsubseteq v$, if $\vec{u} \Vdash A$ then $\vec{u}[v/i] \Vdash A$. A is negatively persistent relative to i iff for any N, $\vec{u}$, and $v \in W$, if $\vec{u} \dashv A$ and $u_i \sqsubseteq v$ then $\vec{u}[v/i] \dashv A$. (As in Barba [1], $\vec{u} \Vdash A$ abbreviates $\vec{u}, u_0 \Vdash A$, and similarly for $\dashv$.) A is persistent relative to i iff it is positively and negatively persistent relative to i.

Lemma 12   *Consider any $\mathbb{L}(M)$-model M and the corresponding $f(M)$. For any $u, v \in \mathbb{W}$, $u R v$ only if in $f(M)$, $u \sqsubseteq v$.*

Proof: We must establish two facts:

1. $D(u) \subseteq D(v)$. This is obvious, since $D(u) = \mathbb{D}(u) \subseteq \mathbb{D}(v) = D(v)$.

2. $S(u) \subseteq S(v)$. $S(u) = \{A : A$ is $(\neg)Pr_1, \ldots, r_n$ and $V(T(A), \vec{u}, u) = 1\}$. Let $A \in S(u)$, $T(A)$ is $\mathsf{E}!r_1 \wedge \ldots \wedge \mathsf{E}!r_n \wedge \Box \Diamond A$. Since $V(T(A), \vec{u}, u) = 1$, $V(\mathsf{E}!r_i, \vec{u}, u) = 1$, $1 \leq i \leq n$ and $V(\Box \Diamond A, \vec{u}, u) = 1$. Take any $v \in \mathbb{W}$, $u R v$. Since domains in $\mathbb{L}(M)$-models are nested, $V(\mathsf{E}!r_i, \vec{u}[v/j], v) = 1$ (where u is the jth member of $\vec{u}$), and since the propositional basis of our modal system is $S4$, $V(\Box \Box \Diamond A, \vec{u}, u) = 1$, and so $V(\Box \Diamond A, \vec{u}[v/j], v) = 1$. Thus,

$$V(\mathsf{E}!r_1 \wedge \ldots \wedge \mathsf{E}!r_n \wedge \Box \Diamond A, \vec{u}[v/j], v) = 1,$$

that is, $V(T(A), \vec{u}[v/j], v) = 1$, from which it follows that $A \in S(v)$.

From 1 and 2 conjointly it follows that $u \sqsubseteq v$.

Theorem 13   *A sentence A is positively persistent relative to i iff $T(A) \to \otimes_i \square \otimes_0 T(A)$ is $\mathbb{L}(M)$-valid, and is negatively persistent relative to i iff $T(\neg A) \to \otimes_i \square \otimes_0 T(\neg A)$ is $\mathbb{L}(M)$-valid.*

Proof:

1. Suppose A is not positively persistent. Then there are $N, \vec{u}, v$ such that $u_i \sqsubseteq v$ and $\vec{u}, u_o \vDash A$, but $\vec{u}', u_o' \nvDash A$. Take $g(N)$. Then $V(T(A), \vec{u}, u_0) = 1$, $V(T(A), \vec{u}', u_0') = 0$, by Corollary 10. This means that $V(\otimes_0 T(A), \vec{u}', v) = 0$. Since in $g(N), u_i R v$ (because $u \sqsubseteq v$), $V(\square \otimes_0 T(A), \vec{u}, u_i) = 0$, so that $V(\otimes_i \square \otimes_0 T(A), \vec{u}, u_0) = 0$, and thus $V(T(A) \to \otimes_i \square \otimes_0 T(A), \vec{u}, u_0) = 0$, proving that $T(A) \to \otimes_i \square \otimes_0 T(A)$ is not valid.

2. Now suppose $T(A) \to \otimes_i \square \otimes_0 T(A)$ is not valid. Then there are $M, \vec{u}$ in $\mathbb{W}$ and $v \in \mathbb{W}, u_i R v$ such that $V(T(A), \vec{u}, u_0) = 1$ and $V(\otimes_i \square \otimes_0 T(A), \vec{u}, u_0) = 0$, that is, $V(\square \otimes_0 T(A), \vec{u}, u_i) = 0$. Then, there exists $v \in \mathbb{W}$ such that $u_0 R v$ and $V(\otimes_0 T(A), \vec{u}', v) = 0$, i.e., $V(T(A), \vec{u}', u_0') = 0$. Take $f(M)$. By Corollary 8, $\vec{u}, u_0 \vDash A$ and $\vec{u}', u_0' \nvDash A$. By Lemma 12, $u \sqsubseteq v$, proving that A is not persistent.

For negative persistence, recall that A is positively persistent iff $\neg A$ is negatively persistent, and vice versa.

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