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Ancestral Kripke Models and Nonhereditary Kripke Models for the Heyting Propositional Calculus

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Abstract This is a sequel to two previous papers, where it was shown that, for the Heyting propositional calculus H, we can give Kripke-style models whose accessibility relation R need not be a quasi-ordering relation, provided we have:

$$x \vDash A \Leftrightarrow \forall y(xRy \Rightarrow y \vDash A).$$

From left to right, this is the heredity condition of standard Kripke models for H, but, since R need not be reflexive, the converse is not automatically satisfied. These Kripke-style models for H were called "rudimentary Kripke models". This paper introduces a kind of dual of rudimentary Kripke models, where the equivalence above is replaced by:

$$x \vDash A \Leftrightarrow \exists y(yRx \;\&\; y \vDash A).$$

From right to left, this is again the usual heredity condition, but the converse, which is automatically satisfied if R is reflexive, yields a proper subclass of rudimentary Kripke models, whose members are called "ancestral Kripke models". In all that, the semantic clauses for the connectives are as in standard Kripke models for H. The propositional calculus H is strongly sound and complete with respect to ancestral Kripke models.

The remainder of the paper is devoted to Kripke-style models for H where the semantic clauses for the connectives are changed so that we need not assume any of the heredity conditions above. The resulting Kripke-style models are called "nonhereditary Kripke models". These models are inspired by some particular embeddings of H into $S4$ and a somewhat weaker normal modal logic. With respect to a notion of quasi-ordered nonhereditary Kripke model, we can prove a certain form of strong soundness and completeness of H. With respect to another notion, where quasi-ordering is not assumed, we can only prove the ordinary soundness and completeness of H.

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The standard conception of Kripke models for Heyting's logic is based on the following assumptions:

(I) valuations must be defined inductively, starting from atomic formulas;
(II) the accessibility relation R must be at least a quasi-ordering relation, i.e., it must be reflexive and transitive;
(III) heredity must be satisfied, in the sense that if a formula A holds at a point x of a model, which we write $x \in v(A)$, then $\forall y(xRy \Rightarrow y \in v(A))$.

In [2] and [3], one can find the beginning of an investigation of Kripke-style models for Heyting's logic where these assumptions are changed in various ways and the following is established.

First, if we reject (I), we also may reject (II), and get models where R must be only serial (i.e., $\forall x \exists y(xRy)$), provided we have assumed in addition to (III) the following *converse heredity* condition:

$$\forall y(xRy \Rightarrow y \in v(A)) \Rightarrow x \in v(A).$$

This condition is trivially satisfied if R is reflexive, but our R need not be reflexive any more. The resulting models, with respect to which we have strong soundness and completeness of the Heyting propositional calculus, were called *rudimentary Kripke models.*

Even if we keep (I), but work with both heredity and converse heredity, (II) is not necessary. Instead of a quasi-ordering, we will have a more general, and somewhat more involved, type of ordering. The resulting models, with respect to which we have again strong soundness and completeness of the Heyting propositional calculus, were called *inductive Kripke models.* In all that, the conditions for valuations, and in particular the clauses concerning connectives, are as in the standard conception. We have only introduced converse heredity, which is anyway satisfied in standard Kripke models.

In this paper, we repeat the pattern of these changes with a condition that is in a certain sense dual to converse heredity. Heredity may alternatively be written as:

$$\exists y(yRx \ \& \ y \in v(A)) \Rightarrow x \in v(A),$$

and our dual condition, which we call *ancestrality*, is the converse implication:

$$x \in v(A) \Rightarrow \exists y(yRx \ \& \ y \in v(A)).$$

This condition is trivially satisfied if R is reflexive, but, as before, our R need not be reflexive.

If we reject (I), we also may reject (II), provided we have assumed heredity and ancestrality. We call the resulting models *ancestral Kripke models.* These models make a proper subclass of rudimentary Kripke models, and, in them, both R and the inverse relation must be serial. Nothing else need be assumed about R.

Next, even if we keep (I), but work with heredity and ancestrality, (II) is again not necessary, and we get analogs of inductive Kripke models that we call *inductive ancestral Kripke models.* As before, the conditions for valuations, and, in particular, the clauses concerning connectives, are as in the standard conception. We have only introduced ancestrality, which is anyway satisfied in standard Kripke models.

In the first section, we investigate ancestral Kripke models in general, and obtain strong soundness and completeness of the Heyting propositional calculus with respect to these models as a corollary of results of [2]. In the second section, we investigate inductive ancestral Kripke models.

In the third section, we make another type of change. We reject (III), as well as all other similar conditions, like converse heredity and ancestrality. This will entail a change in the standard conditions for valuations concerning connectives, which up to now we did not touch. We call the resulting models *nonhereditary Kripke models*. With nonhereditary Kripke models, for questions we consider, it is not interesting whether (I) is kept or not, because there is nothing like heredity that our inductively defined valuations must inherit from atomic formulas. It may be more interesting to see whether (II) must be kept. With our main notion of nonhereditary Kripke model, which seems to be the simplest one could think of, it must. With another notion of nonhereditary Kripke model, briefly envisaged at the end, (II) can be replaced by a somewhat more general assumption.

As ordinary Kripke models for Heyting's logic are related to an embedding of Heyting's logic into the modal logic $S4$, so nonhereditary Kripke models are related to some particular embeddings of Heyting's logic into $S4$ and a somewhat weaker normal modal logic. We shall see that some, but not every, form of strong soundness and completeness can be established for the Heyting propositional calculus with respect to our main notion of nonhereditary Kripke model. With respect to the notion where (II) is rejected, we will obtain only the ordinary soundness and completeness.

The various types of Kripke-style models investigated in [2], [3], and here are meant to be primarily an instrument for the analysis of the inner mechanism of Kripke models. We do not propose rudimentary Kripke models as something to replace standard Kripke models for the technical investigation of Heyting's logic. For the time being, we want these new models to be only an instrument that will help us to understand better the standard models.

It is not clear whether our new Kripke-style models also have a philosophical significance. Our intuition cannot remain the same if valuations are not defined inductively, or if the accessibility relation is not reflexive and transitive, or if heredity is not satisfied, or if the semantic clauses for connectives are changed. However, our aim is not to work for a single new intuition to replace the old one. It is rather an attempt to delimit a field within which we can look for new models without ever leaving standard Kripke models too far behind.

Kripke models for intuitionistic logic are often taken as a paradigm when we try to model other nonclassical logics. So, it might be worth knowing all the possibilities inherent in this paradigm, lest we should be stranded by a too narrow imitation of features that are perhaps accidental. (In [4] one can find models for logics based on weak implications, like relevant and linear implication, that are in some respect analogous to rudimentary Kripke models.)

This paper is a sequel to [2] and [3], and a full understanding of a number of things we want to say here presupposes an acquaintance with these earlier papers, especially [2]. However, we shall try to make this paper as self-contained as possible, short of repeating rather straightforward things. For some basic results (for example, soundness with respect to rudimentary Kripke models), we must rely on [2] or on the acuity of the reader. As in the earlier papers, we concentrate on propositional logic and leave aside a possible extension of our approach to predicate logic.

1 Ancestral Kripke models Our propositional language has infinitely many propositional variables, the binary connectives $\rightarrow$, $\wedge$, and $\vee$, and the propositional constant $\bot$. We use the following metavariables, i.e. schematic letters:

propositional variables: $p, q, r, \ldots$
formulas: $A, B, C, \ldots$
sets of formulas: $\Gamma, \Delta, \Theta, \ldots$

possibly with subscripts or superscripts. As usual, $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$, and $\neg A$ as $A \rightarrow \bot$. The set of all formulas is called L. In the metalanguage, we use $\Rightarrow$, $\Leftrightarrow$, &, *or*, *not*, $\forall$, $\exists$, and set-theoretical symbols with the usual meaning they have in classical logic.

The Heyting propositional calculus H in L is axiomatized by the following usual axiom-schemata:

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)),\ A \rightarrow (B \rightarrow A),$$

$$(C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow (C \rightarrow (A \wedge B))),\ (A \wedge B) \rightarrow A,\ (A \wedge B) \rightarrow B,$$

$$A \rightarrow (A \vee B),\ B \rightarrow (A \vee B),\ (A \vee B) \rightarrow ((A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow C)),$$

$$\bot \rightarrow A,$$

and the primitive rule modus ponens.

A *frame* is a pair $\langle W, R \rangle$ where W is a nonempty set and R a binary relation on W. We use the following metavariables:

members of W: $x, y, z, \ldots$
subsets of W: $X, Y, Z, \ldots$

possibly with subscripts or superscripts; PW is the power set of W.

We define R^k, where k is a natural number, by the following inductive clauses:

$$xR^0y \Leftrightarrow x = y,$$

$$xR^{k+1}y \Leftrightarrow \exists z(xR^kz \,\&\, zRy).$$

It is clear that $xR^1y \Leftrightarrow xRy$. We define R^{-k} and R^* by:

$$xR^{-k}y \Leftrightarrow yR^kx,$$

$$xR^*y \Leftrightarrow (\exists k \geq 0)xR^ky.$$

The relation R^{-1} is the relation inverse to R and R^* is the reflexive and transitive closure of R.

On PW, we define the operations Cone^i, where i is an integer, Cone and $\mathrm{Cone}_{\mathrm{inv}}$ by:

$$\mathrm{Cone}^i X = \{x : \exists y(yR^i x \ \&\ y \in X)\},$$

$$\mathrm{Cone}\, X = \{x : \exists y(yR^* x \ \&\ y \in X)\},$$

$$\mathrm{Cone}_{\mathrm{inv}} X = \{x : \exists y(xR^* y \ \&\ y \in X)\}.$$

In this section, we use only Cone^1 and Cone^{-1}, whereas Cone and $\mathrm{Cone}_{\mathrm{inv}}$, which are closure operations (see [10], Chap. I, §8) will be used in the next section.

For every $X \subseteq W$, let $-X$ be $W - X$, i.e. the complement of X with respect to W. A set X is *hereditary* iff $X \subseteq -\mathrm{Cone}^{-1} -X$, i.e., for every x:

$$x \in X \Rightarrow \forall y(xRy \Rightarrow y \in X).$$

A set X is *conversely hereditary* iff $-\mathrm{Cone}^{-1} -X \subseteq X$, i.e., for every x:

$$\forall y(xRy \Rightarrow y \in X) \Rightarrow x \in X.$$

It is easy to verify that X is hereditary iff $\mathrm{Cone}^1 X \subseteq X$, i.e., for every x:

$$\exists y(yRx \ \&\ y \in X) \Rightarrow x \in X.$$

A set X is *ancestral* iff $X \subseteq \mathrm{Cone}^1 X$, i.e., for every x:

$$x \in X \Rightarrow \exists y(yRx \ \&\ y \in X).$$

Note that, if R is reflexive, every X is conversely hereditary and ancestral.

A hereditary and conversely hereditary X satisfies for every x:

$$x \in X \Leftrightarrow \forall y(xRy \Rightarrow y \in X),$$

whereas a hereditary and ancestral X satisfies for every x:

$$x \in X \Leftrightarrow \exists y(yRx \ \&\ y \in X).$$

Note that the first equivalence is analogous to the schema:

$$\varphi(x) \Leftrightarrow \forall y(x = y \Rightarrow \varphi(y)),$$

which can serve to axiomatize equality in the predicate calculus, whereas the second equivalence is analogous to the schema:

$$\varphi(x) \Leftrightarrow \exists y(y = x \ \&\ \varphi(y)),$$

which can also serve to axiomatize equality in the predicate calculus (cf. [5], §6).

A *pseudo-valuation* v on a frame $\langle W, R\rangle$ is a function from L into PW that satisfies the following conditions for every $A, B \in L$:

($v\!\rightarrow$) $v(A \rightarrow B) = -\mathrm{Cone}^{-1} -(-v(A) \cup v(B))$
$= \{x : \forall y(xRy \Rightarrow (y \in v(A) \Rightarrow y \in v(B)))\}$,
($v\wedge$) $v(A \wedge B) = v(A) \cap v(B)$,
($v\vee$) $v(A \vee B) = v(A) \cup v(B)$,
($v\bot$) $v(\bot) = \varnothing$.

We use $X \rightarrow_R Y$ as an abbreviation for $-\mathrm{Cone}^{-1} -(-X \cup Y)$; i.e., $\rightarrow_R$ is the binary operation involved in ($v\!\rightarrow$). The conditions ($v\!\rightarrow$), ($v\wedge$), ($v\vee$), and ($v\bot$)

are exactly like the usual semantic clauses for connectives in Kripke models for Heyting's logic.

A *valuation* v on a frame $\langle W, R\rangle$ is a pseudo-valuation that satisfies:

(A-Heredity)	for every formula A, the set $v(A)$ is hereditary;
(Converse A-Heredity)	for every formula A, the set $v(A)$ is conversely hereditary.

In other words, for a valuation v, we have $v(A) = -\text{Cone}^{-1} - v(A)$. A *pseudo-Kripke model* is a triple $\langle W, R, v\rangle$ where $\langle W, R\rangle$ is a frame and v a pseudo-valuation on $\langle W, R\rangle$. A *rudimentary Kripke model* is a pseudo-Kripke model $\langle W, R, v\rangle$ where v is a valuation. Ordinary Kripke models are quasi-ordered rudimentary Kripke models, i.e. rudimentary Kripke models $\langle W, R, v\rangle$ where R is reflexive and transitive. A formula A *holds* in $\langle W, R, v\rangle$ iff $v(A) = W$.

Consider now the following condition:

(A-Ancestrality) for every formula A, the set $v(A)$ is ancestral,

for which we can easily prove:

Proposition 1 *Every pseudo-valuation that satisfies A-Ancestrality satisfies Converse A-Heredity.*

Proof: Suppose the pseudo-valuation v on $\langle W, R\rangle$ satisfies A-Ancestrality, and suppose $\forall y(xRy \Rightarrow y \in v(A))$. It follows that $x \in v((B \to B) \to A)$. Then, by A-Ancestrality, we have $\exists y(yRx \;\&\; y \in v((B \to B) \to A))$, which implies $\exists y(yRx \;\&\; \forall u(yRu \Rightarrow u \in v(A)))$, and this implies $x \in v(A)$.

So, every pseudo-valuation that satisfies A-Heredity and A-Ancestrality is a valuation. We shall call such valuations *ancestral valuations*. Rudimentary Kripke models $\langle W, R, v\rangle$ where v is ancestral will be called *ancestral Kripke models*.

Converse A-Heredity and $v(\bot) = \varnothing$ imply that, for every rudimentary Kripke model, the relation R is *serial*, i.e., $\forall x \exists y(xRy)$. Analogously, A-Ancestrality and $v(B \to B) = W$ imply that, for every ancestral Kripke model, R^{-1} is serial, i.e., $\forall x \exists y(yRx)$. So, in ancestral Kripke models, both R and R^{-1} must be serial.

We need not assume anything besides the seriality of R and R^{-1} for frames of ancestral Kripke models. For, if both R and R^{-1} are serial, then, by letting $v(p)$ be either W or $\varnothing$, and by using $(v\!\to)$, $(v\wedge)$, $(v\vee)$, and $(v\bot)$ as clauses in an inductive definition, we obtain an ancestral valuation v such that $v(A)$ is either W or $\varnothing$ (cf. Proposition 2 of [2]).

Since on every frame where R is serial, and where R^{-1} is not necessarily serial, we can define a valuation v by letting $v(p)$ be either W or $\varnothing$ as above (cf. Proposition 2 of [2]), it is clear that not every valuation is ancestral, and, hence, that not every rudimentary Kripke model is ancestral. A less trivial counterexample is provided by the canonical rudimentary Kripke model (defined in [2], §2). This is the model $\langle W_c, R_c, v_c\rangle$, where W_c is the set of prime theories (i.e., consistent and deductively closed sets of formulas that have the disjunction property), R_c is defined on W_c by:

$$\Gamma R_c \Delta \Leftrightarrow ((\Gamma = \Delta \ \& \ \exists A (A \notin \Gamma \ \& \ \forall B (B \in \Gamma \ or \ B \rightarrow A \in \Gamma)))$$

or

$$(\Gamma \neq \Delta \ \& \ \Gamma \subseteq \Delta))$$

and $v_c(A) = \{\Gamma \in W_c : A \in \Gamma\}$. Then we can check that v_c is a valuation (as in Proposition 12 of [2]). However, v_c is not ancestral, since, for a theorem A of H and for the set Γ_H of theorems of H, we have:

$$\Gamma_H \in v_c(A) \ \& \ not \ \exists \Delta (\Delta R_c \Gamma_H \ \& \ \Delta \in v_c(A)).$$

That $\Delta R_c \Gamma_H$ is impossible is shown as follows. The prime theory Δ must be different from Γ_H since we do not have $\Gamma_H R_c \Gamma_H$. But $\Delta \neq \Gamma_H$ & $\Delta \subseteq \Gamma_H$ is equally impossible, since that would mean that a theorem of H is missing from a prime theory.

The condition of A-Ancestrality is trivially satisfied if, in $\langle W, R, v \rangle$, the relation R is reflexive. However, A-Ancestrality may be satisfied without R being reflexive, as we have shown in the penultimate paragraph. This is also shown by the following proposition, analogous to Proposition 7 of [2]:

Proposition 2 *For every quasi-ordered rudimentary Kripke model $\langle W, R, v \rangle$, there is an ancestral Kripke model $\langle W', R', v' \rangle$ where R' is neither reflexive nor transitive such that, for every A, we have $v(A) = W$ iff $v'(A) = W'$.*

Proof: Let W_1, W_2, and W_3 be mutually disjoint copies of W, each in *one-one* correspondence with W. The point x from W corresponds to x_1 from W_1, x_2 from W_2, and x_3 from W_3. On W_1 and W_3, we define, respectively, R_1 and R_3 by:

$$(\forall x, y \in W)(x_1 R_1 y_1 \Leftrightarrow xRy),$$

$$(\forall x, y \in W)(x_3 R_3 y_3 \Leftrightarrow xRy),$$

and, on W_2, we define an irreflexive relation R_2 by:

$$(\forall x, y \in W)(x_2 R_2 y_2 \Leftrightarrow (xRy \ \& \ x \neq y)).$$

Let now $W' = W_1 \cup W_2 \cup W_3$, and let R' be defined on W' by:

$$R' = R_1 \cup R_2 \cup R_3 \cup \{\langle x_1, x_2 \rangle : x \in W\} \cup \{\langle x_2, x_3 \rangle : x \in W\}.$$

The relation R' is not reflexive because R_2 is not reflexive, and it is not transitive because $\langle x_1, x_3 \rangle \notin R'$. Next, for $i \in \{1,2,3\}$, let $v_i(A) = \{x_i \in W_i : x \in v(A)\}$, and let $v'(A) = v_1(A) \cup v_2(A) \cup v_3(A)$. It remains to check that v' is an ancestral valuation on $\langle W', R' \rangle$.

Note that the mapping $f: W' \rightarrow W$ defined by $f(x_i) = x$ is a pseudo-epimorphism from $\langle W', R' \rangle$ onto $\langle W, R \rangle$, since we have:

$$(\forall x_i, y_j \in W')(x_i R' y_j \Rightarrow f(x_i) R f(y_j)),$$

$$(\forall x_i \in W')(\forall y \in W)(f(x_i) R y \Rightarrow (\exists y_j \in W')(f(y_j) = y \ \& \ x_i R' y_j))$$

(cf. [8], pp. 70–75). We also have for every $x_i \in W'$ and every A:

$$x_i \in v'(A) \Leftrightarrow f(x_i) \in v(A).$$

If, in $\langle W', R', v' \rangle$ of the proof above, we omit W_1 from W', and hence also R_1 from R' and $v_1(A)$ from $v'(A)$, we obtain another nonreflexive rudimentary Kripke model, but this rudimentary Kripke model is not ancestral.

It follows easily from results in [2], §1 that H is strongly sound and complete with respect to ancestral Kripke models. Let $\Gamma \vdash A$ mean as usual that there is a proof of A in H from hypotheses in Γ, i.e., there is a sequence of formulas terminating with A, each of which is either in Γ, or a theorem of H, or obtained by modus ponens from formulas preceding it in the sequence. Then our strong soundness and completeness may be stated as follows:

Proposition 3 *For every $\Gamma \subseteq L$ and every $A \in L$:*

$$\Gamma \vdash A \Leftrightarrow (*) \textit{ for every } \langle W, R, v \rangle,\ \textstyle\bigcap_{C \in \Gamma} v(C) = W \Rightarrow v(A) = W,$$

$$\Leftrightarrow (**) \textit{ for every } \langle W, R, v \rangle,\ \textstyle\bigcap_{C \in \Gamma} v(C) \subseteq v(A),$$

where $\langle W, R, v \rangle$ ranges over ancestral Kripke models.

For soundness, we just use the fact that every ancestral Kripke model is a rudimentary Kripke model, and, for completeness, the fact that every ordinary quasi-ordered Kripke model is an ancestral Kripke model. Of course, this strong soundness and completeness imply the ordinary soundness and completeness of H with respect to ancestral Kripke models; namely, A is provable in H iff A holds in every ancestral Kripke model. Proposition 2 guarantees that, for H, we can also prove strong soundness and completeness, in both senses of Proposition 3, as well as ordinary soundness and completeness, with respect to ancestral Kripke models that are neither reflexive nor transitive (the proof of Proposition 7 of [2] guarantees the same with respect to rudimentary Kripke models that are neither reflexive nor transitive).

2 Inductive ancestral Kripke models In the third section of [2], we have investigated the maximal class of frames such that for every pseudo-valuation v the conjunction of the conditions:

(p-Heredity) for every propositional variable p, the set $v(p)$ is hereditary;

(Converse p-Heredity) for every propositional variable p, the set $v(p)$ is conversely hereditary

implies A-Heredity and Converse A-Heredity. Here, we shall investigate the analogous maximal class of frames such that for every pseudo-valuation v the conjunction of the conditions of p-Heredity and:

(p-Ancestrality) for every propositional variable p, the set $v(p)$ is ancestral

implies A-Heredity and A-Ancestrality. These frames are interesting because on them we can define ancestral valuations inductively. It is enough to define v for propositional variables so that p-Heredity and p-Ancestrality hold, and use $(v\rightarrow)$, $(v\wedge)$, $(v\vee)$, and $(v\bot)$ as clauses in an inductive definition, in order to obtain an ancestral valuation. Ancestral Kripke models on such frames will be called *inductive ancestral Kripke models*. To describe these models, we shall first review some notions introduced in [2] and introduce some analogous new notions.

For a frame $\langle W, R\rangle$, a nonempty $X \subseteq W$ is called an *ω-chain from x* iff there is a mapping f from the ordinal ω *onto* X such that $f(0) = x$ and $(\forall n \in \omega) f(n) R f(n+1)$. Let $\omega(x) = \{X \subseteq W$: *$X$ is an ω-chain from x*$\}$. A nonempty $X \subseteq W$ is called an *ω^{-}-chain from x* iff there is a mapping f from ω *onto* X such that $f(0) = x$ and $(\forall n \in \omega) f(n) R^{-1} f(n+1)$. Let $\omega^{-}(x) = \{X \subseteq W$: *$X$ is an ω^{-}-chain from x*$\}$.

Next, on PW, we define the operations Cl_ω and Int_ω by:

$$\mathrm{Cl}_\omega X = \{y : (\forall Y \in \omega(y)) Y \cap X \neq \varnothing\},$$

$$\mathrm{Int}_\omega X = \{y : (\exists Y \in \omega^{-}(y)) Y \subseteq X\}.$$

These operations resemble a closure and interior operation respectively, since they satisfy:

$$X \subseteq \mathrm{Cl}_\omega X, \qquad \mathrm{Int}_\omega X \subseteq X,$$

$$\mathrm{Cl}_\omega \mathrm{Cl}_\omega X = \mathrm{Cl}_\omega X, \qquad \mathrm{Int}_\omega X = \mathrm{Int}_\omega \mathrm{Int}_\omega X,$$

$$\mathrm{Cl}_\omega X \cup \mathrm{Cl}_\omega Y \subseteq \mathrm{Cl}_\omega (X \cup Y), \qquad \mathrm{Int}_\omega (X \cap Y) \subseteq \mathrm{Int}_\omega X \cap \mathrm{Int}_\omega Y.$$

If R is serial, then $\mathrm{Cl}_\omega \varnothing = \varnothing$, and, if R^{-1} is serial, then $W = \mathrm{Int}_\omega W$. However, we need not have:

$$\mathrm{Cl}_\omega (X \cup Y) \subseteq \mathrm{Cl}_\omega X \cup \mathrm{Cl}_\omega Y, \qquad \mathrm{Int}_\omega X \cap \mathrm{Int}_\omega Y \subseteq \mathrm{Int}_\omega (X \cap Y).$$

It is easy to show that:

Cone X is the least hereditary superset of X;

$\mathrm{Cl}_\omega X$ is the least conversely hereditary superset of X

(the second assertion is proved in Proposition 16 of [2]). So, X is hereditary iff $X = \mathrm{Cone}\, X$, and X is conversely hereditary iff $X = \mathrm{Cl}_\omega X$. For Int_ω, we can prove the following:

Proposition 4 *The set* $\mathrm{Int}_\omega X$ *is the greatest ancestral subset of* X.

Proof: To show that $\mathrm{Int}_\omega X$ is ancestral, suppose $y \in \mathrm{Int}_\omega X$. Then there is a $Y \in \omega^{-}(y)$ such that $Y \subseteq X$ and a $z \in Y$ such that zRy. The set obtained from Y by rejecting y is an ω^{-}-chain from z included in X; i.e., $z \in \mathrm{Int}_\omega X$.

To show that $\mathrm{Int}_\omega X$ is the greatest ancestral subset of X, suppose Y is ancestral, $Y \subseteq X$, and $y \in Y$. Then, by repeatedly using the ancestrality of Y, we obtain a z_1 such that $z_1 R y$ and $z_1 \in Y$, a z_2 such that $z_2 R z_1$ and $z_2 \in Y$, etc. The set of all these z_n's makes an ω^{-}-chain from y included in X. So, $y \in \mathrm{Int}_\omega X$.

As a corollary of this proposition, we obtain that X is ancestral iff $X = \mathrm{Int}_\omega X$.

It is easy to check that $-\mathrm{Cone}_{inv} - X = \{x : \forall y (xR^* y \Rightarrow y \in X)\}$, and that $-\mathrm{Cone}_{inv}-$, which is an interior operation, satisfies:

$-\mathrm{Cone}_{inv} - X$ is the greatest hereditary subset of X.

So, Cone is analogous to Cl_ω and $-\mathrm{Cone}_{inv}-$ to Int_ω. Hereditary sets may be characterized either as sets X such that $X = \mathrm{Cone}\, X$ or, alternatively, as sets X

such that $X = -\mathrm{Cone}_{inv} -X$. However, an operation that applied to X would give the greatest conversely hereditary subset of X need not exist, as the following counterexample shows. Let $W = \{a,b,c,d\}$ and $R = \{\langle a,a\rangle,\langle a,b\rangle,\langle b,c\rangle,\langle b,d\rangle,\langle c,c\rangle,\langle d,d\rangle\}$. Then *the* greatest conversely hereditary subset of $\{c,d\}$ does not exist ($\{c\}$ and $\{d\}$ are conversely hereditary, but $\{c,d\}$ is not, since $\mathrm{Cl}_\omega\{c,d\} = \{b,c,d\}$). Analogously, an operation that applied to X would give the least ancestral superset of X need not exist. A counterexample is obtained by taking the frame of the previous counterexample with R inverted, i.e., $W = \{a,b,c,d\}$ and $R = \{\langle a,a\rangle,\langle b,a\rangle,\langle c,b\rangle,\langle d,b\rangle,\langle c,c\rangle,\langle d,d\rangle\}$. Then *the* least ancestral superset of $\{a,b\}$ does not exist ($\{a,b,c\}$ and $\{a,b,d\}$ are ancestral, but $\{a,b\}$ is not, since $\mathrm{Int}_\omega\{a,b\} = \{a\}$). Note that, in the frames of these counterexamples, both R and R^{-1} are serial. Hence, these frames are appropriate for ancestral Kripke models.

It is easy to check that, for every frame $\langle W,R\rangle$, the relation R is reflexive iff $(\forall X \subseteq W)\mathrm{Cl}_\omega X = X$ (see Proposition 17 of [2]), or, alternatively, iff $(\forall X \subseteq W)X = \mathrm{Int}_\omega X$. Proposition 18 of [2] asserts that, if X is hereditary, then $\mathrm{Cl}_\omega X$ is hereditary. Analogously, we have the following:

Proposition 5 *If X is hereditary, then* $\mathrm{Int}_\omega X$ *is hereditary.*

Proof: Suppose $x \in \mathrm{Int}_\omega X$ and xRy. Then there is a $Y \in \omega^-(x)$ such that $Y \subseteq X$, and $Y \cup \{y\} \in \omega^-(y)$. Since X is hereditary, $Y \cup \{y\} \subseteq X$, and, hence, $y \in \mathrm{Int}_\omega X$.

So, for every X, the set $\mathrm{Cl}_\omega \mathrm{Cone}\, X$ is hereditary and conversely hereditary, and, for every hereditary and conversely hereditary X, we have $X = \mathrm{Cl}_\omega \mathrm{Cone}\, X$. Analogously, for every X, the set $\mathrm{Int}_\omega \mathrm{Cone}\, X$ is hereditary and ancestral, and, for every hereditary and ancestral X, we have $X = \mathrm{Int}_\omega \mathrm{Cone}\, X$.

We can prove the following proposition, related to Proposition 19 of [2]:

Proposition 6 *The condition*:
(a) *for every pseudo-valuation v on $\langle W,R\rangle$, if v satisfies p-*Heredity *and* Converse p-Heredity, *then v satisfies A-*Heredity *and* Converse A-Heredity
is equivalent with the conjunction of the following conditions:
(a1) *for every $X, Y \subseteq W$,* $\mathrm{Cl}_\omega \mathrm{Cone}\, X \to_R \mathrm{Cl}_\omega \mathrm{Cone}\, Y$ *is hereditary*;
(a2) *for every $X, Y \subseteq W$,* $\mathrm{Cl}_\omega \mathrm{Cone}\, X \cup \mathrm{Cl}_\omega \mathrm{Cone}\, Y$ *is conversely hereditary*;
(a3) *$\varnothing$ is conversely hereditary.*

Proof: We show first that (a) implies (a1), (a2), and (a3). If (a1) fails for some X and Y, then there is a pseudo-valuation v that satisfies p-Heredity and Converse p-Heredity such that $v(p_1) = \mathrm{Cl}_\omega \mathrm{Cone}\, X$ and $v(p_2) = \mathrm{Cl}_\omega \mathrm{Cone}\, Y$, but $v(p_1 \to p_2)$ is not hereditary. If (a2) fails, we proceed similarly to obtain that $v(p_1 \vee p_2)$ is not conversely hereditary. If (a3) fails, then $v(\bot)$ is not conversely hereditary.

That the conjunction of (a1), (a2), and (a3) implies (a) is shown by induction on the complexity of A. Note that the following conditions are satisfied for every frame $\langle W,R\rangle$ and every $X, Y \subseteq W$:

$\mathrm{Cl}_\omega \mathrm{Cone}\, X \to_R \mathrm{Cl}_\omega \mathrm{Cone}\, Y$ is conversely hereditary;
$\mathrm{Cl}_\omega \mathrm{Cone}\, X \cap \mathrm{Cl}_\omega \mathrm{Cone}\, Y$ is hereditary and conversely hereditary;

$\mathrm{Cl}_\omega\,\mathrm{Cone}\,X \cup \mathrm{Cl}_\omega\,\mathrm{Cone}\,Y$ is hereditary;
$\varnothing$ is hereditary.

Let $\mathrm{Cone}^-X = -\mathrm{Cone}_{inv}X$ (as in [2], §3). It follows from the proof of Proposition 19 of [2] that the conditions (a1), (a2), and (a3) can be expressed equivalently as follows:

(a1) $\forall x, z(xR^2z \Rightarrow (\forall Z \in \omega(z))\exists t(xRt \;\&\; t \in \mathrm{Cl}_\omega\,\mathit{Cone}\{z\} \;\&\; t \notin \mathrm{Cl}_\omega\,\mathrm{Cone}^-Z))$;
(a2) $\forall x(\forall X_1, X_2 \in \omega(z))\exists y(xRy \;\&\; y \notin \mathrm{Cl}_\omega\,\mathrm{Cone}^-X_1 \;\&\; y \notin \mathrm{Cl}_\omega\,\mathrm{Cone}^-X_2)$;
(a3) $\forall x \exists y(xRy)$, i.e., R is serial.

Condition (a1), called *prototransitivity* in [2], follows from transitivity (let t be z), but not the other way around. Condition (a2), called *protoreflexivity* in [2], follows from reflexivity (let y be x), but not the other way around (the intuitive meaning of these conditions is explained in more detail in [2]).

Now, we can prove the following analog of Proposition 6:

Proposition 7 *The condition*:
(b) *for every pseudo-valuation v on $\langle W, R\rangle$, if v satisfies p-*Heredity *and p-*Ancestrality, *then v satisfies A-*Heredity *and A-*Ancestrality
is equivalent with the conjunction of the following conditions:
(b1) *for every $X, Y \subseteq W$,* $\mathrm{Int}_\omega\,\mathrm{Cone}\,X \to_R \mathrm{Int}_\omega\,\mathrm{Cone}\,\mathrm{Y}$ *is hereditary*;
(b2) *for every $X, Y \subseteq W$,* $\mathrm{Int}_\omega\,\mathrm{Cone}\,X \cap \mathrm{Int}_\omega\,\mathrm{Cone}\,\mathrm{Y}$ *is ancestral*;
(b3) *for every $X, Y \subseteq W$,* $\mathrm{Int}_\omega\,\mathrm{Cone}\,X \to_R \mathrm{Int}_\omega\,\mathrm{Cone}\,\mathrm{Y}$ *is ancestral.*

Proof: We show first that (b) implies (b1), (b2), and (b3). If (b1) fails for some X and Y, then there is a pseudo-valuation v that satisfies p-Heredity and p-Ancestrality such that $v(p_1) = \mathrm{Int}_\omega\,\mathrm{Cone}\,X$ and $v(p_2) = \mathrm{Int}_\omega\,\mathrm{Cone}\,Y$, but $v(p_1 \to p_2)$ is not hereditary. If (b2) or (b3) fails, we proceed similarly to obtain that $v(p_1 \vee p_2)$, or respectively $v(p_1 \to p_2)$, is not ancestral.

That the conjunction of (b1), (b2), and (b3) implies (b) is shown by induction on the complexity of A. Note that the following conditions are satisfied for every frame $\langle W, R\rangle$ and every $X, Y \subseteq X$:

$\mathrm{Int}_\omega\,\mathrm{Cone}\,X \cap \mathrm{Int}_\omega\,\mathrm{Cone}\,Y$ is hereditary;
$\mathrm{Int}_\omega\,\mathrm{Cone}\,X \cup \mathrm{Int}_\omega\,\mathrm{Cone}\,Y$ is hereditary and ancestral;
$\varnothing$ is hereditary and ancestral.

It is a lengthy, but not very difficult, exercise to show that the conditions (b1) and (b2) can be expressed equivalently as follows:

(b1) $\forall x, z(xR^2z \Rightarrow (\forall Z \in \omega^-(z))\exists t(xRt \;\&\; t \in \mathrm{Int}_\omega\,\mathrm{Cone}\,Z \;\&\; t \notin \mathrm{Int}_\omega\,\mathrm{Cone}^-\{z\}))$;
(b2) $\forall x(\forall X_1, X_2 \in \omega^-(z))\exists y(yRx \;\&\; y \in \mathrm{Int}_\omega\,\mathrm{Cone}\,X_1 \;\&\; y \in \mathrm{Int}_\omega\,\mathrm{Cone}\,X_2)$.

The analogy of (b1) with (a1) is obvious; (b1) also follows from transitivity (let t be z), but not the other way around. Similarly, (b2) is analogous to (a2); (b2) also follows from reflexivity (let y be x), but not the other way around. By definition, (b3) may be written as:

(b3) $(\forall X, Y \subseteq W)\forall x(\forall y(xRy \Rightarrow (y \in \mathrm{Int}_\omega\,\mathrm{Cone}\,X \Rightarrow y \in \mathrm{Int}_\omega\,\mathrm{Cone}\,Y)) \Rightarrow$
$\exists z(zRx \;\&\; \forall t(xRt \Rightarrow (t \in \mathrm{Int}_\omega\,\mathrm{Cone}\,X \Rightarrow t \in \mathrm{Int}_\omega\,\mathrm{Cone}\,Y))))$.

When we instantiate X and Y by the same set, it is clear that (b3) implies that $\forall x \exists z(zRx)$, i.e. that R^{-1} is serial. When we instantiate X by W and Y by $\emptyset$, then (b3) implies that $\forall x \exists y(xRy)$, i.e. that R is serial.

These considerations show that the assumptions of reflexivity and transitivity for ordinary Kripke models do not function in exactly the same manner. Transitivity secures (a1) and (b1), and is tied to implication. Reflexivity secures (a2) and (b2), and is hence tied to disjunction and conjunction, but it also secures (a3) and (b3), and is hence also tied to $\bot$ and implication. Reflexivity also secures at one stroke Converse A-Heredity and A-Ancestrality. With reflexivity, we have reduced assumptions about valuations to an assumption about frames, which does not mention valuations.

The strong soundness and completeness of H with respect to inductive ancestral Kripke models is an easy consequence of Proposition 3 and the fact that ordinary quasi-ordered Kripke models are inductive ancestral Kripke models.

3 Nonhereditary Kripke models In [2] and [3], and the previous sections of this paper, we have Kripke-style models for H that differ from ordinary quasi-ordered Kripke models only in the conditions concerning frames. The conditions of Converse A-Heredity and A-Ancestrality, which we assumed for valuations in addition to A-Heredity, are satisfied in ordinary Kripke models, because the frames of these models are reflexive. Since the conditions for valuations concerning connectives, which we have stated in the definition of pseudo-valuation, do not differ from the standard conditions, we may say that we have not altered the conditions for valuations. In this section, we shall consider Kripke-style models for H where conditions for frames need not be changed, i.e., these frames may be quasi-ordered, but conditions for valuations will be changed. We want to show that there are such Kripke-style models where none of the heredity conditions, A-Heredity, Converse A-Heredity, and A-Ancestrality, need be satisfied. This will be achieved at the cost of changing also the conditions for valuations concerning connectives. We shall find these Kripke-style models by considering some particular embeddings of H into the modal logic $S4$ (the ordinary Kripke models for H are inspired by another such embedding; see [9], p. 92). So, we shall first introduce these embeddings.

Let us enlarge our propositional language with the unary connective $\Box$, and let $L\Box$ be the set of all formulas of this enlarged language. The modal logic $S4$ in $L\Box$ is axiomatized by adding to our axiomatization of H the following axiom-schemata:

$$((A \to B) \to A) \to A,$$

$$\Box(A \to B) \to (\Box A \to \Box B), \ \Box A \to A, \ \Box A \to \Box\Box A,$$

and the primitive rule of necessitation (i.e., from A, infer $\Box A$).

Consider now the translations (i.e. one-one mappings) t and s from L into $L\Box$ defined by the following inductive clauses:

$$t(p) = p, \qquad s(p) = p,$$

$$t(A \to B) = \Box t(A) \to \Box t(B), \qquad s(A \to B) = \Box s(A) \to s(B),$$

$$t(A \wedge B) = \Box t(A) \wedge \Box t(B), \qquad s(A \wedge B) = s(A) \wedge s(B),$$

$$t(A \vee B) = \Box t(A) \vee \Box t(B), \qquad s(A \vee B) = \Box s(A) \vee \Box s(B),$$

$$t(\bot) = \bot, \qquad s(\bot) = \bot.$$

The translation t prefixes $\Box$ to every proper subformula of a formula of L, and s resembles a translation considered in ([7], IV, §5.1) for embedding intuitionistic logic into linear logic (the translation related to ordinary Kripke models for H prefixes $\Box$ to propositional variables and implications; cf., for example, [6], Chap. 3, §7). In order to connect the translations t and s, we will use the following theorems of $S4$:

(1) $$\Box(\Box A \to \Box B) \leftrightarrow \Box(\Box A \to B),$$

(2) $$\Box(\Box A \wedge \Box B) \leftrightarrow \Box(A \wedge B).$$

Note that (1) and the rule converse to necessitation (i.e., from $\Box A$, infer A) may replace $\Box A \to A$ and $\Box A \to \Box\Box A$ in the axiomatization of $S4$. We can prove by induction on the complexity of A that, for every A:

(3) $$\Box t(A) \leftrightarrow \Box s(A)$$

is a theorem of $S4$. In the induction step, we use (1) when A is of the form $B \to C$, and (2) when A is of the form $B \wedge C$. Then we can easily establish the following proposition:

Proposition 8 *For every $A \in L$:*

A is provable in $H \Leftrightarrow t(A)$ is provable in $S4$,

$\Leftrightarrow s(A)$ is provable in $S4$.

Proof: Note that $\Box t(A)$ is the result of prefixing $\Box$ to every subformula of A. It is well-known that A is provable in H iff $\Box t(A)$ is provable in $S4$. By (3), we have that A is provable in H iff $\Box s(A)$ is provable in $S4$. Then we use the fact that $S4$ is closed under necessitation and the rule converse to necessitation.

Next, to fix notation and terminology, we introduce as follows the standard Kripke modelling of $S4$. A *modal Kripke model* for $S4$ is a triple $\langle W, R, v\rangle$ where $\langle W, R\rangle$ is a quasi-ordered frame and $v : L\Box \to PW$ satisfies:

$$\begin{aligned} v(A \to B) &= -v(A) \cup v(B), \\ v(A \wedge B) &= v(A) \cap v(B), \\ v(A \vee B) &= v(A) \cup v(B), \\ v(\bot) &= \varnothing, \\ v(\Box A) &= -\mathrm{Cone}^{-1} -v(A) \\ &= \{x : \forall y(xRy \Rightarrow y \in v(A))\}. \end{aligned}$$

As before, A holds in $\langle W, R, v\rangle$ iff $v(A) = W$.

Let $\Gamma \vdash_{S4} A$ mean that there is a proof of A in $S4$ from hypotheses in Γ *without necessitation*, i.e., there is a sequence of formulas terminating with A each of which is either in Γ, or a theorem of $S4$, or obtained by modus ponens from formulas preceding it in the sequence; and let $\Gamma \Vdash_{S4} A$ mean that there is a proof of A in $S4$ from hypotheses in Γ *with necessitation*, i.e., there is a sequence of formulas terminating with A each of which is either in Γ, or a theorem of $S4$, or obtained by modus ponens or necessitation from formulas preceding it in the sequence. Next, let $\Box\Gamma = \{\Box C : C \in \Gamma\}$. It is easy to establish the following equivalence:

$$(4) \qquad \Gamma \Vdash_{S4} A \Leftrightarrow \Box\Gamma \vdash_{S4} A.$$

We need this equivalence to prove the second of the following two strong soundness and completeness propositions (for the proof of these propositions, we rely on standard notions of modal logic, like the notions of canonical model and generated submodel; see, for example, [8]):

Proposition 9 *For every* $\Gamma \subseteq L\Box$ *and every* $A \in L\Box$:

$$\Gamma \vdash_{S4} A \Leftrightarrow (**) \textit{ for every } \langle W, R, v\rangle,\ \textstyle\bigcap_{C\in\Gamma} v(C) \subseteq v(A),$$

where $\langle W, R, v\rangle$ *ranges over modal Kripke models for* $S4$.

Proof: (Soundness) We use the fact that, for $\vdash_{S4}$, we have the deduction theorem; i.e., $\Gamma \cup \{C\} \vdash_{S4} B$ implies $\Gamma \vdash_{S4} C \rightarrow B$.

(Completeness) We can infer from *not* $\Gamma \vdash_{S4} A$ that $\Gamma \cup \{\neg A\}$ can be extended to a maximal consistent set Γ'. In the canonical model, $\Gamma' \in \bigcap_{C\in\Gamma} v(C)$ and $\Gamma' \notin v(A)$.

Proposition 10 *For every* $\Gamma \subseteq L\Box$ *and every* $A \in L\Box$:

$$\Gamma \Vdash_{S4} A \Leftrightarrow (*) \textit{ for every } \langle W, R, v\rangle,\ \textstyle\bigcap_{C\in\Gamma} v(C) = W \Rightarrow v(A) = W,$$

where $\langle W, R, v\rangle$ *ranges over modal Kripke models for* $S4$.

Proof: (Soundness) From the left-to-right direction of (4), we have that $\Gamma \Vdash_{S4} A$ implies $\Box\Gamma \vdash_{S4} A$. Since $v(C) = W$ implies $v(\Box C) = W$, we have that $\bigcap_{C\in\Gamma} v(C) = W$ implies $\bigcap_{C\in\Gamma} v(\Box C) = W$. Then it is enough to apply the soundness part of Proposition 9.

(Completeness) From the right-to-left direction of (4), we have that *not* $\Gamma \Vdash_{S4} A$ implies *not* $\Box\Gamma \vdash_{S4} A$. So, by the completeness part of Proposition 9, there is a $\langle W, R, v\rangle$ and an $x \in W$ such that $x \in \bigcap_{C\in\Gamma} v(\Box C)$ and $x \notin v(A)$. Then take the submodel $\langle W_x, R_x, v_x\rangle$ of $\langle W, R, v\rangle$ generated by x. For this model, we have $\bigcap_{C\in\Gamma} v(C) = W_x$ and $v(A) \neq W_x$.

It is clear that $\Gamma \vdash_{S4} A$ implies $\Gamma \Vdash_{S4} A$ and that $(**)$ implies $(*)$, but the converse implications fail, since we have $\{p\} \Vdash_{S4} \Box p$ and *not* $\{p\} \vdash_{S4} \Box p$. It is also clear that the deduction theorem of the soundness part of the proof of Proposition 9 fails for $\Vdash_{S4}$, since $p \rightarrow \Box p$ is not provable in $S4$.

We can now strengthen Proposition 8. Namely, if $\Gamma \vdash A$ means, as in the first section, that there is a proof of A in H from hypotheses in Γ, and if $t(\Gamma) = \{t(C) : C \in \Gamma\}$ and $s(\Gamma) = \{s(C) : C \in \Gamma\}$, then we can prove:

Proposition 11 *For every* $\Gamma \subseteq L$ *and every* $A \in L$:

$$\Gamma \vdash A \Leftrightarrow \Box t(\Gamma) \vdash_{S4} t(A),$$

$$\Leftrightarrow t(\Gamma) \Vdash_{S4} t(A),$$

$$\Leftrightarrow \Box s(\Gamma) \vdash_{S4} s(A),$$

$$\Leftrightarrow s(\Gamma) \Vdash_{S4} s(A).$$

Proof: To establish the third equivalence, we use Proposition 8 and the fact that, for both $\vdash$ and $\vdash_{S4}$, we have the deduction theorem. Then we obtain the remaining equivalences by applying (3) and (4).

It follows easily that, for every $\Gamma \subseteq L$ and every $A \in L$:

$$t(\Gamma) \vdash_{S4} t(A) \Rightarrow \Gamma \vdash A,$$

$$s(\Gamma) \vdash_{S4} s(A) \Rightarrow \Gamma \vdash A,$$

but the converse implications may fail, since we have neither $\{p, \Box p \to \Box q\} \vdash_{S4} q$ nor $\{p, \Box p \to q\} \vdash_{S4} q$.

The embedding of H into $S4$ by the translation s, from Proposition 8, suggests the following Kripke-style models for H, which we shall call *nonhereditary Kripke models*. A nonhereditary Kripke model is a triple $\langle W, R, v\rangle$ where $\langle W, R\rangle$ is a quasi-ordered frame and $v: L \to PW$, called a *nonhereditary valuation*, satisfies the following conditions for every $A, B \in L$:

$(\boldsymbol{v_s}\boldsymbol{\to})$ $\quad v(A \to B) = \text{Cone}^{-1} - v(A) \cup v(B) = \{x : \forall y(xRy \Rightarrow y \in v(A)) \Rightarrow x \in v(B)\},$

$(\boldsymbol{v}\boldsymbol{\wedge})$ $\quad v(A \wedge B) = v(A) \cap v(B),$

$(\boldsymbol{v_s}\boldsymbol{\vee})$ $\quad v(A \vee B) = -\text{Cone}^{-1} - v(A) \cup -\text{Cone}^{-1} - v(B) = \{x : \forall y(xRy \Rightarrow y \in v(A)) \textit{ or } \forall y(xRy \Rightarrow y \in v(B))\},$

$(\boldsymbol{v}\boldsymbol{\bot})$ $\quad v(\bot) = \varnothing.$

This differs from the conditions for pseudo-valuations in replacing $(v\to)$ and $(v\vee)$ by $(v_s\to)$ and $(v_s\vee)$. Note that now we have:

$$x \in v(\neg A) \Leftrightarrow \exists y(xRy \;\&\; y \notin v(A)),$$

which is quite different from the usual semantic clause for $\neg$, induced by conditions for pseudo-valuations:

$$x \in v(\neg A) \Leftrightarrow \forall y(xRy \Rightarrow y \notin v(A)).$$

As before, A holds in $\langle W, R, v\rangle$ iff $v(A) = W$.

From Proposition 8 and the standard Kripke modelling of $S4$, it is easy to infer the ordinary soundness and completeness of H with respect to nonhereditary Kripke models; i.e., for every $A \in L$:

(0) $\;A$ is provable in $H \Leftrightarrow A$ holds in every nonhereditary Kripke model

(cf., for example, [6], Chap. 3, §7). Actually, we can also prove the following strong soundness and completeness:

Proposition 12 *For every* $\Gamma \subseteq L$ *and every* $A \in L$:

$$\Gamma \vdash A \Leftrightarrow (*) \textit{ for every } \langle W, R, v\rangle,\ \textstyle\bigcap_{C \in \Gamma} v(C) = W \Rightarrow v(A) = W,$$

where $\langle W, R, v\rangle$ *ranges over nonhereditary Kripke models.*

Proof: (Soundness) If Γ is empty, then we use the soundness direction of (0). If Γ is not empty and $\Gamma \vdash A$, then, for some $\{C_1, \ldots, C_n\} \subseteq \Gamma$, where $n \geq 1$,

$(C_1 \wedge \ldots \wedge C_n) \rightarrow A$ is provable in H. By the soundness direction of (0), for every nonhereditary Kripke model $\langle W, R, v\rangle$, $v((C_1 \wedge \ldots \wedge C_n) \rightarrow A) = W$. We easily infer that, if $v(C_1 \wedge \ldots \wedge C_n) = W$, then $v(A) = W$, from which (*) follows.

(Completeness) Suppose *not* $\Gamma \vdash A$. Then, by Proposition 11, *not* $s(\Gamma) \Vdash_{S4} s(A)$, and, by the completeness part of Proposition 10, there is a modal Kripke model for $S4$ in which all the formulas in $s(\Gamma)$ hold, but $s(A)$ does not hold. This model yields a nonhereditary Kripke model that falsifies (*).

However, we cannot prove the strong soundness and completeness of H with respect to (**) of Proposition 3, because, for nonhereditary Kripke models $\langle W, R, v\rangle$, we need not have:

$$\Gamma \vdash A \Rightarrow (**) \text{ for every } \langle W, R, V\rangle,\ \textstyle\bigcap_{C\in\Gamma} v(C) \subseteq v(A).$$

For example, we have $\{p, p \rightarrow q\} \vdash q$, but we need not have $v(p) \cap v(p \rightarrow q) \subseteq v(q)$. We falsify this last inclusion in the nonhereditary Kripke model where $W = \{a, b\}$, $R = \{\langle a,a\rangle, \langle a,b\rangle, \langle b,b\rangle\}$, $v(p) = \{a\}$, and $v(q)$ is either $\varnothing$ or $\{b\}$; in this model, $a \in v(p) \cap v(p \rightarrow q)$ and $a \notin v(q)$ (cf. the remark after the proof of Proposition 11, and Proposition 9).

The following algebraic fact stands behind this failure of strong soundness and completeness with respect to (**). In an arbitrary frame, let $X \rightarrow_s Y$ be an abbreviation for Cone^{-1} $-X \cup Y$; i.e., $\rightarrow_s$ is the binary operation involved in $(v_s\rightarrow)$. Then we can prove:

$$X \cap Y \subseteq Z \Rightarrow X \subseteq Y \rightarrow_s Z,$$

but the converse implication may fail. In the frame of our counterexample above, we have $\{a, b\} \subseteq \{a\} \rightarrow_s \varnothing$, but we don't have $\{a, b\} \cap \{a\} \subseteq \varnothing$ (we may put $\{b\}$ instead of $\varnothing$ as well). This means that $\rightarrow_s$ is not the residual of $\cap$, and that we cannot use it to get the relative pseudo-complement of a Heyting algebra where $\cap$ is the meet operation. On the other hand, in [2], it is explained how the binary operation $\rightarrow_R$ involved in $(v\rightarrow)$ makes the relative pseudo-complement of a Heyting algebra of hereditary and conversely hereditary sets where $\cap$ is the meet operation. In the context of ancestral Kripke models, we have the following analogous fact. For every frame $\langle W, R\rangle$ and every nonempty set $\mathbf{A}$ of hereditary and ancestral subsets of W closed under $\rightarrow_R$ and $\cap$, for every $X, Y, Z \in \mathbf{A}$, we have:

$$X \cap Y \subseteq Z \Leftrightarrow X \subseteq Y \rightarrow_R Z.$$

When we verify this equivalence from left to right, we use the assumption that X is hereditary. For the other direction, we first establish that $W \in \mathbf{A}$ (there is a V in $\mathbf{A}$ and $V \rightarrow_R V = W$; since W is ancestral, R^{-1} must be serial). Then we use the assumption that Y is hereditary and $W \rightarrow_R Z$ ancestral (cf. the proof of Proposition 1).

The frames of nonhereditary Kripke models must be quasi-ordered. This can be inferred from the fact that $S4$ is the weakest normal modal logic in which H can be embedded by s. Namely, $s(p \rightarrow p)$ is $\Box p \rightarrow p$, and, for every normal modal logic, $s(p \rightarrow ((p \vee p) \vee (p \vee p)))$ is equivalent with $\Box p \rightarrow \Box\Box p$.

An alternative notion of nonhereditary Kripke model may be obtained by relying on the translation t instead of s. In the conditions for nonhereditary valuations, we can replace $(v_s\!\rightarrow)$ and $(v\wedge)$ by:

$$(\boldsymbol{v_t\rightarrow})\quad v(A\rightarrow B) = \mathrm{Cone}^{-1} - v(A) \cup -\mathrm{Cone}^{-1} - v(B)$$
$$= \{x : \forall y(xRy \Rightarrow y \in v(A)) \Rightarrow \forall y(xRy \Rightarrow y \in v(B))\},$$
$$(\boldsymbol{v_t\wedge})\quad v(A\wedge B) = -\mathrm{Cone}^{-1} - v(A) \cap -\mathrm{Cone}^{-1} - v(B)$$
$$= \{x : \forall y(xRy \Rightarrow y \in v(A)) \,\&\, \forall y(xRy \Rightarrow y \in v(B))\}.$$

With this alternative notion of nonhereditary Kripke model based on quasi-ordered frames, we could rewrite most of this section with essentially the same results.

If we replace $(v_s\!\rightarrow)$ and $(v\wedge)$ by $(v_t\!\rightarrow)$ and $(v_t\wedge)$, we can introduce a further change in our notion of nonhereditary Kripke model by rejecting the assumption that the frames of these models are quasi-ordered. Instead, for these frames, we assume only $R^2 = R$, i.e.:

$$\forall x, y(xR^2y \Leftrightarrow xRy),$$

which is transitivity and its converse, called *weak density.* Then we can still prove (0), i.e. the ordinary soundness and completeness of H with respect to these new nonhereditary Kripke models. This can be inferred from the fact that the weakest normal modal logic in which H can be embedded by t is obtained from our axiomatization of $S4$ by replacing $\Box A \rightarrow A$ by $\Box\Box A \rightarrow \Box A$ (this system is called $Kt'\circ$ in [1], and the translation t is there called t'; note that, here, $\perp$ is primitive and $\neg$ defined). The necessity of $\Box\Box A \leftrightarrow \Box A$ is inferred from the presence of $((B \rightarrow B) \rightarrow A) \leftrightarrow A$ in H.

Actually, for proving the necessity of $R^2 = R$, it does not matter whether we take $(v\wedge)$ or $(v_t\wedge)$ as our condition for conjunction. (Note, by the way, that, in a normal modal logic with $\Box\Box A \leftrightarrow \Box A$, we can prove $\Box(\Box A \wedge \Box B) \leftrightarrow \Box(A \wedge B)$.) It equally does not matter whether, with $(v_s\!\rightarrow)$ instead of $(v_t\!\rightarrow)$, we assume $(v\wedge)$ or $(v_t\wedge)$. Reflexivity and transitivity are necessary in either case. So, the change hinges only on implication.

However, we cannot prove the strong soundness and completeness of H with respect to nonhereditary Kripke models with $(v_t\!\rightarrow)$ and $(v_t\wedge)$ based on frames that must satisfy only $R^2 = R$. A counterexample is obtained by taking $W = \{a, b\}$, $R = \{\langle a, b\rangle, \langle b, b\rangle\}$, $v(p) = W$, and $v(q) = \{b\}$. It is clear that $R^2 = R$ and, by $(v_t\!\rightarrow)$, we obtain $v(p \rightarrow q) = W$. Since, of course, we have $\{p, p \rightarrow q\} \vdash q$, strong soundness with respect to (*), and a fortiori (**), must fail.

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