Notre Dame Journal of Formal Logic
Volume 32, Number 4, Fall 1991

Full Satisfaction Classes: A Survey

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Abstract We give a survey (with no proofs) of the theory of full satisfaction classes for models of Peano Arithmetic.

Arithmetize the language L_{PA} of Peano Arithmetic (PA). Let $M \models \text{PA}$. By a *full satisfaction class for M* we mean a subset $S \subseteq Sent^M$ (the set of sentences in the sense of M) which satisfies the usual conditions on truth given by Tarski, i.e.

(i) If φ is of the form $S^m0 + S^k0 = S^r0$ then $\varphi \in S$ iff $m + k = r$.
(ii) The same for other atomic formulas.
(iii) $(\neg\varphi) \in S$ iff $\varphi \notin S$ for each sentence φ.
(iv) $(\varphi \,\&\, \psi) \in S$ iff both φ, ψ are in S.
(v) $(\exists v_k \varphi) \in S$ iff $\varphi(S^m0) \in S$ for some m.

Think of this as follows: S is just the notion of truth for all sentences in the sense of M, including nonstandard ones. Robinson [16] was the first to treat seriously the nonabsoluteness of the finiteness in the very definitions of the language. The notion of a full satisfaction class was defined explicitly by Krajewski [11], who proved the following nonuniqueness result.

Theorem 1 (Krajewski [11]) *There exists M, a model of* PA, *which has many different satisfaction classes. To be more specific, if S_0 is a full satisfaction class for a countable M so that (M, S_0) is recursively saturated then S_0 has $2^{\aleph_0}$ automorphic images, i.e.*

$$\{S \subseteq M : \exists g \in \text{Aut}(M)\ S = g * S_0\}$$

is of power continuum.

The idea of the proof of Krajewski's result is to apply the countable version of the Chang-Makkai Theorem (cf. Schlipf [17]). In order to verify the assumption, use Tarski's Theorem on Undefinability of Truth.

After having shown that the nonstandard language determined by an $M \models$ PA does not have uniquely determined semantics, the question arises: for which models M of PA does there exist such a semantics? That is, which models admit a full satisfaction class?

Received July 20, 1990, revised April 10, 1991

Theorem 2 (Kotlarski, Krajewski, Lachlan [8]) *Every countable and recursively saturated model M of* PA *admits a full satisfaction class.*

Theorem 3 (Lachlan [13]) *If $M \models$ PA, M is nonstandard and admits a full satisfaction class, then M is recursively saturated.*

We comment on these results. We say that a full satisfaction class S for M is Σ_n-inductive if (M, S) satisfies induction for Σ_n formulas in $L_{\mathrm{PA}} \cup \{S\}$. Lachlan's result (i.e., Theorem 3) is obvious under the stronger assumption that M admits a Σ_1-inductive satisfaction class. To see why this is so, note that if $p(v, b)$ is a recursive type in the variable v and parameter $b \in M$, we represent it in PA and hence are able to argue as follows:

$$\forall k \in \mathcal{N}(M, S) \models \exists v \forall \varphi < k[\varphi \in p \Rightarrow S(\varphi; v, b)].$$

By Σ_1 induction, this holds for some $k > \mathcal{N}$. Then the v granted by this formula realizes p because its elements are standard. Elimination of Σ_1 induction from the above argument requires an ingenious trick (see [13]).

For Theorem 2, define the following system of ω-logic. Let Γ_0 be the set of all true atomic and negated false atomic sentences, and all sentences of the form $\neg\varphi \vee \varphi$. We let $\Gamma_{n+1} =$ the set of all sentences in M which may be derived from sentences of Γ_n by a single application of one of the rules of the predicate calculus or by a single application of the ω-rule:

$$\text{infer } \eta \vee \forall z \psi(z) \text{ from all sentences } \eta \vee \psi(S^z O).$$

Then one can express the notion of Γ_n in PA (indeed, the function $n \mapsto \Gamma_n$ is primitive recursive).

Let A_i be the disjunction of copies of the sentence $0 = 0$ with the following distribution of parentheses: A_0 is $(0 = 0)$, A_{i+1} is $(A_i \vee A_i)$. The main lemma for Theorem 2 is:

Lemma 4 *For each k there exists m so that $\neg\Gamma_k(A_m)$.*

(This is intuitively obvious for clearly there is no other way to derive A_i's than to first check that $\Gamma_0(A_0)$ holds, then derive A_1 from A_0, etc., so the greater m is the greater k must be. However, a precise proof of this requires some work. It is nontrivial to show that there is no essentially simpler proof of all the A_i's, say in five steps.)

Granted this, we see that for $m \in M \setminus \mathcal{N}$, $M \models \neg\Gamma_k(A_m)$ for all k. Thus, it suffices to prove the following lemma.

Lemma 5 *If M is a countable recursively saturated model of* PA *and $\varphi \in M$ is such that*

$$\forall k \in \mathcal{N}\ M \models \neg\Gamma_k(\neg\varphi),$$

then there exists a full satisfaction class for M making φ true.

(This is proved much like the Completeness Theorem. The ω-rule is essential to choose constants.)

It should be noticed that this method allows us to construct an S for M making all the axioms of PA true (all in the sense of M, so including nonstandard instances of the induction scheme), but, as we shall see later, the existence of an

S making all the theorems of PA true requires more than mere recursive saturation of M.

As we have seen, the heart of the matter in Theorem 2 was to prove that derivations of A_i's needed many steps. Questions of this sort were studied also by Czech mathematicians; see Krajíček [12] and references in his paper.

The natural question whether the countability assumption is essential in Theorem 2 and Lemma 5 was settled by Smith ([18] and [20]). His result is as follows:

Theorem 6 *Kaufmann's model has no full satisfaction class.*

Kaufmann [1] constructed ω_1-like recursively saturated models M for PA with the property of being "rather classless", i.e.

> if $X \subseteq M$ is such that $\forall b \in M\, X \cap < b$ is (coded) in M then X is definable with parameters in M.

See also Schmerl [21] for more in this direction. Once again, it is quite easy to see that Kaufmann's model has no full Σ_1-inductive satisfaction class. In Smith's result, eliminating induction requires a lot of work to ensure that only the standard Tarski conditions on truth are used.

Ratajczyk [15] studied satisfaction classes from the point of view of partition properties, in the Paris-Harrington style. Let us state his result.

Definition (in PA). c is a sequence of $n + \max$ indiscernibles for φ iff

(i) c is an increasing sequence

(ii) $\forall b_1, c_1, b_2, c_2$ subsequences of c if $lh(b_1) = n = lh(b_2)$ and $lh(c_1) = \max(b_1)$ & $lh(c_2) = \max(b_2)$ then $[\varphi(b_1 \cap c_1) \Leftrightarrow \varphi(b_2 \cap c_2)]$.

Here φ is a formula in one free variable; $b_1 \cap c_1$ and $b_2 \cap c_2$ are parameters. Consider the sentences

$$\forall b_1 \exists c_1 \cdots \forall b_m \exists c_m \{ \&_{i \le m} [c_i \text{ is an increasing sequence and } b_i < \min(c_i) \text{ and } b_i < lh(c_i)]$$

$$\& \; [\&_{i<m-1} \max(c_i) < \min(c_{i+1})]$$

$$\& \; [c_1 \cap \cdots \cap c_m \text{ is a sequence } n + \max - \text{indiscernible for } \varphi]\}.$$

Denote these sentences

$$\omega \xrightarrow{\varphi} (m^* < \omega)^{n+\max}.$$

Denote

$$\omega \xrightarrow{\Delta_0} (m^* < \omega)^{n+\max}$$

the scheme

$$\omega \xrightarrow{\varphi} (m^* < \omega)^{n+\max}, \quad \varphi \in \Delta_0.$$

Theorem 7 *If M is a countable recursively saturated model of* PA *then M admits a full inductive satisfaction class iff*

$$\forall m, n \in \mathcal{N}\; M \models \omega \xrightarrow{\Delta_0} (m^* < \omega)^{n+\max}.$$

Ratajczyk (in the same paper [14]) gave also a combinatorial sentence φ, in the Paris-Harrington style, such that φ is independent from the theory PA(S), i.e. PA + S is a full inductive satisfaction class. Let me state his result. Define (in PA) a finite set X to be *k-large* if there exist sets $X_1 \cdots X_k$ such that $X = X_1 \cup \cdots \cup X_k$ and $\max(X_i) \leq \min(X_{i+1})$ and $\min(X_i) \leq \mathrm{card}(X_i)$, $i < k$. Write $M \underset{*}{\longrightarrow} (k)_r^{e+\max}$ if for every partition $P: [M]^{\leq M} \to r$ there exists a k-large $H \subseteq M$ and $c \leq r$ so that for each increasing sequence $a \cap b \subseteq H$ with $lh(a) = e$, $lh(b) = \max(a)$ we have $P(a \cap b) = c$.

Theorem 8 (again [14])

(i) $\forall k, r, e \in \mathcal{N}$ $\mathrm{PA}(S) \vdash \exists M\, M \underset{*}{\longrightarrow} (k)_r^{e+\max}$

(ii) *the combinatorial principle* $\forall k, r, e \, \exists M\, M \underset{*}{\longrightarrow} (k)_r^{e+\max}$ *is independent from* $\mathrm{PA}(S)$.

Krajewski's Nonuniqueness Theorem (i.e. Theorem 1) was strengthened by Kossak [2]. Let us recall that Krajewski obtained continuum many distinct automorphic images of a single S, provided that (M, S) is countable and recursively saturated. Call two satisfaction classes S_1, S_2 for M *isomorphic* (*elementarily equivalent*) if S_1 is an automorphic image of S_2 ((M, S_1) and (M, S_2) satisfy the same theory in $L_{\mathrm{PA}} \cup \{S\}$). Kossak's results are stated below.

Theorem 9 *If a countable nonstandard M admits an inductive full satisfaction class then*

(i) *it admits continuum many nonelementarily equivalent inductive full satisfaction classes*

(ii) *it admits continuum many elementarily equivalent but nonisomorphic satisfaction classes.*

Kossak's proofs use the fact that some results obtained originally for models of PA work for models for $\mathrm{PA}(S)$ as well.

In Kotlarski [6] I considered models for the theory $\Delta_0 - \mathrm{PA}(S) = \mathrm{PA} + S$ is a full satisfaction class + induction for bounded formulas in $L_{\mathrm{PA}} \cup \{S\}$. The first result is as follows.

Theorem 10 [6] $\Delta_0 - \mathrm{PA}(S)$ *is finitely axiomatizable. One of the axiomatizations is* $\mathrm{PA}^- + S$ *is a full satisfaction class* $+ \forall \varphi[(\mathrm{PA} + \varphi) \Rightarrow S(\varphi)]$.

Let us recall that in Theorem 2 we were able to obtain a full satisfaction class making all the axioms of PA true whenever M was countable and recursively saturated. Theorem 10 shows in particular that the existence of S making all the theorems of PA true requires not just that M be recursively saturated. In addition M must satisfy some theory stronger than PA. Indeed, $S(0 = 1)$ cannot hold, so $\mathrm{PA} \vdash 0 = 1$ cannot hold in M, i.e. M must think that PA is consistent.

In order to specify the theory we are speaking about we change the system of ω-logic described in the comments to Theorem 2. In that system, the use of each rule increased the complexity of the proof. Now we work with the system in which only the use of the ω-rule increases the complexity of the proof. The definition is as follows. Define formulas $\Gamma_n \in L_{\mathrm{PA}}$ by the following induction.

$$\Gamma_0(\varphi) \text{ is } \mathrm{PA} \vdash \varphi$$

$$\Gamma_{n+1/2}(\varphi) \text{ is "}\varphi \text{ is of the form } \eta \vee \forall z \psi(z) \text{ and } \forall z \Gamma_n(\eta \vee \psi(S^z 0))\text{"}.$$

$$\Gamma_{n+1}(\varphi) \text{ is "there exists a proof of } \varphi \text{ from } \mathrm{PA} \cup \{\psi : \Gamma_{n+1/2}(\psi)\}\text{"}.$$

Theorem 11 [6] *If M is countable and recursively saturated then M admits a full Δ_0-inductive satisfaction class iff* $\forall n \in \mathcal{N}\ M \models \neg\Gamma_n(0 = 1)$.

On the other hand, we have the following result.

Theorem 12 *The theory* $\Delta_0 - \mathrm{PA}(S) + \exists j S(\Gamma_j(0 = 1))$ *is consistent. Indeed, if a countable nonstandard model M of* PA *admits an S so that* $(M, S) \models \Delta_0 - \mathrm{PA}(S)$, *then M admits also an S so that* $(M, S) \models \Delta_0 - \mathrm{PA}(S) + \exists j S(\Gamma_j(0 = 1))$.

Theorem 12 was obtained in [6] with the use of the usual method of Gödel using diagonalization. Later, I realized that one of the model-theoretic constructions of Section 4 of that paper allows one to obtain Theorem 12 with no diagonalization.

Thus, there arises the following problem: what conditions should be imposed on the theory of M in order to ensure that (if M is countable and recursively saturated) M admits a Σ_n-inductive full satisfaction class? This question was solved by Ratajczyk and myself ([9] and [10]). In order to state the results, we define the transfinite iterations of ω-logic. Fix a "natural" system of notations for ordinals $< \epsilon_0$ in PA; this is given by the Cantor Normal Form Theorem for ordinals $< \epsilon_0$. We define by transfinite induction on $\alpha < \epsilon_0$ theories T^α and Γ_n^α. We put $T^0 = \mathrm{PA}$ and $\Gamma_0^0 = \mathrm{PA}$. Γ_n^α are defined as previously, i.e. $\Gamma_0^\alpha(\varphi)$ is $T^\alpha \vdash \varphi$, $\Gamma_{n+1/2}^\alpha(\varphi)$ is

$$\text{“}\varphi \text{ is of the form } \eta \vee \forall z \psi(z) \text{ and } \forall z \Gamma_n^\alpha(\eta \vee \psi(z))\text{”},$$

$\Gamma_{n+1}^\alpha(\varphi)$ is $T^\alpha \cup \Gamma_{n+1/2}^\alpha \vdash \varphi$. Finally

$$T^{\alpha+1} = T^\alpha \cup \{\neg\Gamma_n^\alpha(0 = 1) : n\}$$

and $T^\lambda = \bigcup_{\alpha<\lambda} T^\alpha$ for λ limit.

There is a well-known trick (using the Recursion Theorem) which allows us to formalize this definition in PA.

Define, for an ordinal α, the sequence $\omega_m(\alpha)$ by putting $\omega_0(\alpha) = \alpha$, $\omega_{m+1}(\alpha) = \omega^{\omega_m(\alpha)}$ (ordinal exponentiation).

Theorem 13 *Fix a natural number m. Then for any countable and recursively saturated model M of* PA, *M admits a full Σ_m-inductive satisfaction class iff*

$$\forall k \in \mathcal{N}\ M \models \neg\Gamma_k^{\omega_m(k)}(0 = 1).$$

Corollary 14 *If $M \models$* PA *is countable and recursively saturated, then M admits a full inductive satisfaction class iff*

$$\forall n \in \mathcal{N}\ M \models \neg\Gamma_n^{\omega_n}(0 = 1).$$

Here $\omega_n = \omega_n(\omega)$ is the natural sequence convergent to ϵ_0.

Corollary 14 follows from Theorem 13 immediately. The proof of Theorem 13 is in two steps. In the first step, we prove that if M is a countable recursively saturated model of PA + consistency of sufficiently much of this ω-logic, then M has an S so that $(M, S) \models \Delta_0 - \mathrm{PA}(S)$ and some rapidly growing functions are total. In the second step, we prove that if $(M, S) \models \Delta_0 - \mathrm{PA}(S)$ + totality of these functions, then M admits also a Σ_m-inductive full satisfaction class. We stress that the ideas come much more from the theory of models for PA rather than Proof Theory.

In [10] we extended the methods to obtain a proof theoretic description of the strength of the existence of a full inductive satisfaction class.

Theorem 15 *If M is a countable recursively saturated model of* PA, *then M admits an S so that* $(M,S) \models \Sigma_m - \mathrm{PA}(S)$ *iff*

$$\forall k \in \mathcal{N}\ M \models \textit{transfinite induction over } \epsilon_{\omega_m(k)}.$$

Similarly, we have the following result.

Corollary 16 *If M is a countable and recursively saturated model of* PA, *then M has a full inductive satisfaction class iff*

$$\forall k \in \mathcal{N}\ M \models \textit{transfinite induction over } \epsilon_{\omega_k}.$$

Once again, the methods used to prove Theorem 15 and Corollary 16 are model-theoretic. We did not use the Cut Elimination Theorem.

There were some applications of a bit weaker notion, that of a *partial* satisfaction class, to study recursively saturated models for PA. Thus in [3] and Kossak and Schmerl [5] an ω_1-like recursively saturated rigid model for PA was constructed using partial satisfaction classes. But, the most interesting applications of partial satisfaction classes are due to Schmerl [21]. Also, Smith ([18] and [19]) gives interesting applications of full satisfaction classes to recursively saturated and resplendent models.

Another survey of satisfaction classes is due to Murawski [14], who mentions other applications.

Acknowledgment – The author would like to thank Julia F. Knight and her husband Bill for their hospitality at the University of Notre Dame where this paper was written.

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