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A Generalization of the Adequacy Theorem for the Quasi-Senses

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Abstract In the present paper, based on Bressan's sense language SL_α^ν, a version of the adequacy theorem for quasi-senses is proved that is applicable in every case, even when SL_α^ν collapses into an extensional language. Thus this version affords a new result also for Bressan's modal language ML^ν, which is substantially identical to SL_1^ν. Furthermore, some conditions of the adequacy theorem are weakened: the basic well-formed expressions (wfes) can contain primitive constants. Then we consider a theory T based on SL_α^ν, a definition system D, and strong (weak) extensions of T in connection with a semantics for which the senses of the wfes are (are not) preserved by the principles of λ-conversion. The designation rules for quasi-senses are given in a complete form, also for strong theories. In fact, by means of the notion of a T-correspondent of a wfe, every defined constant has a quasi-sense. Synonymy relations are extended to strong and weak extensions of T. Finally, the previous version of the adequacy theorem is further generalized by making the wfes contain primitive and defined constants, and making the valuations be noninjective on their free variables. By means of this result it is possible to construct quasi-senses for any choice of a synonymy notion.

1 Introduction Many papers have been devoted to sense logic, starting with Church [15] and Carnap [13] and [14]. In [13] Carnap deals with some special modal languages and, at the end, he makes some substantial hints about synonymy and a sense language capable of treating simple (noniterated) belief sentences. Various attempts to construct a rather general and systematic theory of belief sentences were proposed later, e.g. by means of λ-categorial or quotational languages. Among the published papers on this subject we should mention Lewis [19], Cresswell [17] and [18], and Bigelow [2]. In particular, in the aforementioned papers of Cresswell, where the literature and the actual situation connected with the problem are described, several deficiences and limitations of past approaches are clearly presented.

Recently, the results of Church's paper [15] have been generalized (see, e.g.,

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Parsons [20]). Furthermore, a first-order theory capable of dealing with belief sentences of any finite order and universal and existential quantifiers is presented in Bealer [1].

The approach to sense in the present paper (see Bonotto–Bressan [9] and Bressan [11]) is based on a very different point of view in which uniformity and generality features are taken into account, and it is, so to speak, purely modal, which does not invalidate the extensionality thesis. Furthermore, we approach sense with a view to dealing explicitly with Church's λ operator, the ι operator for descriptions, general operators forms, synonymy, and, e.g., belief sentences of transfinite orders.

Senses are tightly connected with the notion of synonymy. This notion has been studied in itself, independently of its relation to senses, in Bonotto–Bressan [6]–[8] and in Bonotto [3] and [4], in connection with an extensional language and a modal one, respectively. The thesis that several natural notions of synonymy can be considered is emphasized in connection with an interpreted theory endowed with a definition system D (see Bonotto [3]). The one studied there substantially affords a positive answer to the question raised in Cresswell (cf. [17], p. 37, fn. 16). Roughly speaking, the principles of λ-conversion preserve the meaning (or the sense) relevant to a synonymy notion presented in Bonotto [3]. However, other answers are also possible here, as was shown in [9].

Bonotto and Bressan in [9] refer to a general interpreted modal calculus, $\mathcal{MC}^\nu$, and any interpreted theory $(\mathfrak{T}, D, I)$ based on it and endowed with a definition system D. The interpretation I is supposed to be admissible, i.e. a model for D. In connection with such a theory, four particular synonymy notions $\asymp_0$, $\asymp_1$, $\asymp_2$, and $\asymp_3$ are introduced first. They are regarded as binary relations among well-formed expressions (wfes) of $(\mathfrak{T}, D)$. Let us stress that they are characterized only by means of conditions on the forms of the wfes among which they hold. Among them $\asymp_0$ and $\asymp_1$ are defined, first, only for empty D, because the principles of λ-conversion are not meaning-preserving in connection with them. Therefore they may appear too weak (not extended enough) or too rich in content. On the one hand, $\asymp_0$ also has a basic role in treating quasi-senses connected with any other synonymy notion. On the other hand, the definitions of $\asymp_0$ and $\asymp_1$ can be extended to a certain theory $\mathfrak{T}^*$ endowed with the definition system D of $\mathfrak{T}$, provided D is of a suitable kind. In order to obtain a unified theory for the various (interesting) synonymy notions, a general rigorous definition of synonymy is introduced in [9]. For any synonymy notion $\asymp$ we have $\asymp_0 \subset \asymp$; if $\asymp_2 \subset \asymp$, then $\asymp$ is said to be *weak*.

In [9] we introduced suitable quasi-senses to represent the senses connected with any choice of $\asymp$, and assigned them to the wfes of $(\mathfrak{T}, D, I)$. These quasi-senses, to be denoted by ${}^{\asymp}QSs$, are constructed (for $\asymp\, = \,\asymp_0$) as suitable equivalence classes of ${}^{\asymp_0}QSs$. The ${}^{\asymp}QSs$ (and the corresponding senses) are (fail to be) preserved by the principles of λ-conversion when the defining conditions of $\asymp_2$ hold (when $\asymp$ is $\asymp_0$ or $\asymp_1$).

For every choice of $\asymp$ the quasi-senses have to fulfill certain natural adequacy requirements. Among them are the following:

Theorem 1.1. *If Δ and Φ are constant free wfes, while V and W are ostensive v-valuations that are injective on the variables free in Δ and Φ, respectively,* *then* Δ *has (with respect to* V*) the same quasi-sense as* Φ *(with respect to* W*) iff* Φ (Δ) *can (briefly) be obtained from* Δ (Φ) *by replacing the variables free in* Δ (Φ) *with those free in* Φ (Δ) *suitably rearranged.*

In [9], Theorem 1.1 is proved only for an effectively modal language. In fact, in its proof, the following assumption was used:

(a) The class Γ of the elementary possible cases is infinite.

On the basis of [9], $\mathcal{MC}^{\nu}$ has been extended into the interpreted sense calculus $\mathcal{SC}_{\alpha}^{\nu}$ (where α is a possibly transfinite ordinal) capable of dealing with belief sentences whose iteration orders may be transfinite.

The logical symbols of the language $\mathcal{SL}_{\alpha}^{\nu}$, on which $\mathcal{SC}_{\alpha}^{\nu}$ is based, include $\sim$, $\supset$, $\Box$, $\forall$, $=$, $\imath$ (for descriptions), and Church's primitive lambda λ^{p}; the other symbols are the variables v_{tn}^{β} and constants $c_{t\mu}^{\beta}$, where the (sense) order β can take any value $<\alpha$ (where α is a possibly uncountable ordinal, $t \in \tau^{\nu}$ and the index μ may be transfinite, unlike n).

Any semantics to be considered for $\mathcal{SL}_{\alpha}^{\nu}$ on the basis of [9] must involve senses, hence it must be based on a synonymy relation $\asymp$. The corresponding interpreted language can be denoted by ${}^{\asymp}\mathcal{SL}_{\alpha}^{\nu}$. In [15], only $\asymp_0$ is discussed; therefore, the index $\asymp_0$ was dropped, and we shall also drop it here. After presenting the formation rules for $\mathcal{SL}_{\alpha}^{\nu}$ in Section 2 and some useful definitions and conventions in Section 3, we present the main features of the semantical structure for $\mathcal{SL}_{\alpha}^{\nu}$ in Section 4.

Every wfe Δ of order β has a hyper-quasi-intension (hyper-quasi-extension) of order $\leq\beta$ which represents its hyper-intension (hyper-extension). In addition, Δ has a quasi-sense of order $\leq\beta$, which represents its sense.

Intuitively, every hyper-quasi-intension is a function from Γ into a set of hyper-quasi-extensions. Hyper-quasi-extensions are constructed in the usual type-theoretical fashion except that, in case a hyper-quasi-extension is a function, its domain is formed with hyper-quasi-intensions and quasi-senses. A relevant feature of this construction is that the quasi-senses must have an order lower than that of the function involved.

The entities assignable to variables and constants of order β are quasi-intensions of order β or quasi-senses of order $<\beta$.

Since expressions may contain both constants and variables, quasi-senses are relative to a valuation of the constants and variables. Roughly speaking, the senses of variables and constants are their valuations, whereas the quasi-sense of a compound expression Δ is a sequence $\langle\chi, x_1, \ldots, x_n\rangle$ where χ is a marker depending on the form of Δ and $x_1, \ldots, x_n$ are senses (of the components of Δ) or functions (depending on the senses of the components of Δ).

The quasi-senses—to be defined by conditions $(s_{1\text{-}10})$ in Section 4—have to fulfill certain natural adequacy requirements. In particular, an analogue of Theorem 1.1 must hold.

In the present paper, a version of Theorem 1.1 for $\mathcal{SL}_{\alpha}^{\nu}$ is presented which is applicable in every case, and hence also when $\mathcal{SL}_{\alpha}^{\nu}$ collapses into an extensional language, since assumption (a) is not used. Thus it affords a new result also in connection with $\mathcal{ML}^{\nu}$, which is substantially identical to $\mathcal{SL}_{1}^{\nu}$.

Furthermore, some conditions of the theorem are weakened. In fact, Theorem 1.1 is extended to cases in which the wfes Δ and Φ contain primitive constants, which involves some obvious changes in the proof (see Section 5).

Then in the present paper we consider a theory $\mathfrak{T}$ based on $\mathcal{SL}_\alpha^\nu$ and a definition system D (see Section 6). It is useful to consider strong (weak) extensions of $\mathfrak{T}$ in connection with a semantics for which the senses of wfes are [are not] preserved by the principles of λ-conversion.

In Section 7 the designation rules for quasi-senses, given in [11] for weak theories, are given in a complete form for strong theories. In fact, by means of the notion of T-correspondent of a well-formed expression (see Section 7) every defined constant also has a quasi-sense.

Furthermore, the relations from $\asymp_0$ to $\asymp_3$, introduced in [9] for theories based on $\mathcal{MC}^\nu$, are rigorously extended to strong and weak extensions of $\mathfrak{T}$ in Section 6.

Then by means of some notions introduced in Section 7, Theorem 1.1 can be further generalized. In Section 8, Theorem 1.1 is shown to hold even when Δ and Φ contain some primitive and some defined constants, and in case V and W can be noninjective on their free variables.

Now the adequacy requirement is proved and it is possible to construct quasi-senses in connection with any choice of synonymy notion. They are introduced as suitable equivalence classes of quasi-senses ${}^{\asymp_0}QS$. This treatment and further results are left for future papers.

2 *The sense language $\mathcal{SL}_\alpha^\nu$ of order α: Formation rules* Let α be any ordinal number, possibly uncountable. The sense language $\mathcal{SL}_\alpha^\nu$ of (sense) order α is based on the type system τ^ν, which is the smallest set τ^ν such that

(i) $\{0,1,\ldots,\nu\} \subset \tau^\nu$ and
(ii) if $n \in N_*$ ($=_D N - \{0\}$) and $t_0,\ldots,t_n \in \tau^\nu$, then the $n+1$ tuple $\langle t_1,\ldots,t_n,t_0\rangle \in \tau^\nu$.

We say that 0 is the *sentence type* (because of the use made of it), 1 to ν are *individual types*, and $\langle t_1,\ldots,t_n,t_0\rangle$, with $t_0,\ldots,t_n \in \tau^\nu$ and $t_0 = 0$ [$t_0 \neq 0$], is a *relation* (*function*) *type*. We also set

$$(2.1)\qquad \begin{cases} (t_1,\ldots,t_n) =_D \langle t_1,\ldots,t_n,0\rangle, \\ (t_1,\ldots,t_n : t_0) =_D \langle t_1,\ldots,t_n,t_0\rangle \textit{ if } t_0,\ldots,t_n \in \tau^\nu \textit{ and } t_0 \neq 0. \end{cases}$$

For $t_0,\ldots,t_n$, θ, $\varphi \in \tau^\nu$ and $n \in N_*$, we define the *operator type*

$$(2.2)\qquad (t_1,\ldots,t_n;\theta,\varphi) =_D \langle\langle t_1,\ldots,t_n,\theta\rangle,\varphi\rangle.$$

The symbols of $\mathcal{SL}_\alpha^\nu$ are the following logical symbols: comma, left and right parentheses, the connectives $\sim$ and $\supset$, $\Box$, $\forall$, $=$ (for contingent identity), ι (for descriptions), and λ^P (primitive Church's lambda); plus the variables v_{tn}^β and constants $c_{t\mu}^\beta$, of *order* β, *type* t, and *index* n or μ ($\beta < \alpha$, $t \in \tau^\nu$, $n \in N_*$, and $0 < \mu < \alpha + \omega_0$ where ω_0 is the first infinite ordinal).

If A is an *expression* of $\mathcal{SL}_\alpha^\nu$, i.e. a finite sequence of symbols of $\mathcal{SL}_\alpha^\nu$, then the largest among the orders of the constants and variables occurring in A will be called its (*sense*) *order* and will be briefly denoted by A^{ord}.

The class E_t of the wfes of $\mathcal{SL}^\nu_\alpha$ of type t ($\in \tau^\nu$) is defined recursively by conditions (f_1) to (f_{10}) below, regarded as holding for $n \in N_*$ and t, $t_0, \dots, t_n, \theta, \varphi \in \tau^\nu$:

(f_1) $c^\beta_{t\mu}, v^\beta_{tn} \in E_t$ for $\beta < \alpha$ and $0 < \mu < \alpha + \omega_0$.
(f_2) If $\Delta_i \in E_{t_i}$ ($i = 1, \dots, n$) and $\Delta \in E_{\langle t_1, \dots, t_n, t\rangle}$, then $\Delta(\Delta_1, \dots, \Delta_n) \in E_t$.
(f_3) If $\Omega \in E_t$ with $t = (t_1, \dots, t_n; \theta, \varphi)$, x_1 to x_n are n (distinct) variables, $x_i \in E_{t_i}$ ($i = 1, \dots, n$), and $\Delta \in E_\theta$, then $(\Omega x_1, \dots, x_n)\Delta \in E_\varphi$.
(f_{4-8}) If p, $q \in E_0$, then $\sim p$, $p \supset q$, $\Box p$, $(\forall v^\beta_{tn})p \in E_0$ and $(\imath v^\beta_{tn})p \in E_t$.
(f_9) If Δ_1, $\Delta_2 \in E_t$, then $\Delta_1 = \Delta_2 \in E_0$.
(f_{10}) If $\Delta \in E_t$ and x_1 to x_n are n variables with $x_i \in E_{t_i}$ ($i = 1, \dots, n$), then $(\lambda^p x_1, \dots, x_n)\ \Delta \in E_{\langle t_1, \dots, t_n, t\rangle}$.

(2.3) If $A^{<\beta} =_D \bigcup_{\delta<\beta} A^\delta$, $A^{\leq\beta} =_D \bigcup_{\delta\leq\beta} A^\delta$, $A^{\not<\beta} =_D A^\beta - A^{<\beta}$ ($A^{<0} = \emptyset$).

For $t \in \tau^\nu$ we also set

$$E^\beta_t =_D \{\Delta \in E_t | \Delta^{ord} \leq \beta\}; \quad \text{wfe}^\beta =_D \bigcup_{t\in\tau^\nu} E^{\beta\not<}_t, \tag{2.4}$$

so that the wfe$^\beta$s are the wfes of order β.

By identifying the variables v_{tn} and the constants c_{tn} of the modal language $\mathcal{ML}^\nu$ considered in [9] with v^0_{tn} and c^0_{tn}, respectively, ($n \in N$) the wfes of $\mathcal{ML}^\nu$ turn out to be those of $\mathcal{SL}^\nu_\alpha$ in which only symbols of $\mathcal{ML}^\nu$ occur.

$\wedge$, $\vee$, $\equiv$, $(\exists x)$, and $\Diamond$ would be introduced in the usual (metalinguistic) ways.

3 Some conventions and metalinguistic definitions

Convention 3.1 By $x, y, z, x_1, \dots, p, q, r, p_1, \dots$, and $\Delta, \Delta_1, \dots$, will be denoted arbitrary variables, wffs, and wfes (of $\mathcal{SL}^\nu_\alpha$), respectively. By $x^\beta, \dots$, $\Delta^\beta, \Delta^\beta_1, \dots$ we will denote wfe$^\beta$s of the respective kinds above.

Definition 3.1 We say that Δ is an equivalent of Φ if Δ and Φ are wfes and Δ can be obtained from Φ by a series of steps which consist of alphabetic changes of bound variables.

Convention 3.2 If (i) Δ is a wfe, (ii) u_1 to u_a are constants or variables, and u_i, $\Delta_i \in E_{t_i}$ with $t_i \in \tau^\nu$ ($i = 1, \dots, b$), then $\Delta(u_i/\Delta_i)_b$, as well as $\Delta[u_1, \dots, u_b/\Delta_1, \dots, \Delta_b]$, denotes the result of substituting Δ_1 to Δ_b simultaneously for u_1 to u_b respectively (at the free occurrences) in a certain equivalent Δ' of Δ such that Δ_i is free for u_i in Δ' for $i = 1, \dots, b$ (the precise description of this equivalent would be of no interest for what follows).

Convention 3.3 If x_1 to x_b are b variables and a wfe Δ is denoted by $\Phi(x_1, \dots, x_b)$, then $\Phi(\Delta_1, \dots, \Delta_b)$ denotes $\Delta(x_i/\Delta_i)_b$.

The synonymy relation $\asymp$ and the nonexisting object of type t can be defined within $\mathcal{SL}^\nu_\alpha$ itself metalinguistically:

(3.1) $\Delta_1 \asymp \Delta_2 \equiv_D (F)F(\Delta_1) = F(\Delta_2)$, with $F^{ord} = 1 + \max\{\Delta^{ord}_1, \Delta^{ord}_2\}$, where $\Delta_1, \Delta_2 \in E_t^\beta$ and F is the first variable of type (t) that satisfies (3.1) and is nonfree in Δ_1 and Δ_2.

$$(3.2) \qquad a^* =_D a_t^* =_D (\imath v_{t1}) v_{t1} \neq v_{t1}.$$

By rule (f_9) in Section 2, = can be applied also to wffs, as a substitute for equivalence (and a_0^* will turn out to be equivalent to $(x)x \neq x$). Hence definition (3.1) applies also to wffs; and definitions of the relational and functional Church's (nonprimitive) λ-operators become the following:

$$(3.3) \qquad (\lambda x_1, \ldots, x_n)\, \Delta \equiv_D (\imath f).(\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = \Delta \bigwedge (\forall y_1, \ldots, y_n). \sim (\exists x_1, \ldots, x_n) \bigwedge_{i=1}^{n} x_i \asymp y_i \supset f(y_1, \ldots, y_n) = a^*,$$

where (i) $\Delta \in E_t^\beta$, (ii) x_1 to x_n are n variables of the respective types t_1 to t_n and arbitrary orders, (iii) f is the first variable of type $\langle t_1, \ldots, t_n, t_0 \rangle$ and nonfree in Δ, such that

$$(3.4) \qquad f^{ord} = \max\{\Delta^{ord}, x_1^{ord}, \ldots, x_n^{ord}\},$$

and (iv) y_1 to y_n are the first n variables different from x_1 to x_n, of the same order as f, and of the respective types t_1 to t_n.

4 The semantical structure for $\mathcal{SL}_\alpha^\nu$ The structure for the semantics for $\mathcal{SL}_\alpha^\nu$ is based on $\nu + 2$ sets $D_0, D_1, \ldots, D_\nu, \Gamma$. For them we require that $D_0 = \{T, F\}$ and D_1 to D_ν contain at least two elements, one of which is F.

Intuitively, every hyper-quasi-intension is a function from Γ into a set of hyper quasi-extensions. Hyper-quasi-extensions are constructed in the usual type-theoretical way except that, in case a hyper-quasi-extension is a function, its domain is formed with hyper-quasi-intensions and quasi-senses. A relevant feature of the construction is that the quasi-senses must have orders lower than that of the function.

For every $t \in \tau^\nu$ and $\beta < \alpha$, in the semantical structure we have a set HQE_t^β of hyper-quasi-extensions, a set HQI_t^β of hyper-quasi-intensions and a set A_t^β of entities assignable to variables v_{tn}^β and constants $c_{t\mu}^\beta$.

These sets are defined by induction on the order β and, for any given β, by induction on the complexity of t.

The general construction rules are R_1 to R_4 below, where, for any pair of sets X and Y, $X \to Y$ denotes the set of all functions from X into Y and $X \mapsto Y$ denotes the set of all functions from a subset of X into Y.

(R_1) $HQE_r^\beta = D_r$ for $r \in \{0, 1, \ldots, \nu\}$
(R_2) $HQI_t^\beta = (\Gamma \to HQE_t^\beta)$, for $t \in \tau^\nu$
(R_3) $A_t^\beta = HQI_t^\beta \cup QS_t^{<\beta}$ $\mathcal{QE}_t^\beta = HQE_t^\beta - \{F\}$ for $t \in \tau^\nu$
(R_4) $HQE_{\langle t_1, \ldots, t_0, t_n \rangle}^\beta = (A_{t_1}^\beta \times \cdots \times A_{t_n}^\beta \mapsto \mathcal{QE}_{t_0}^\beta) \cup \{F\}$ for $t_0, t_1, \ldots, t_n \in \tau^\nu$.

Of course, as they stand, these rules provide only the initial step of the construction; they also require the definition of QS_t^β, given A_t^β for $t \in \tau^\nu$.

Note that $A_t^0 = HQI_t^0$, which can be substantially identified with QI_t as defined in [9].

The set QS_t^β is defined as the set of the quasi-senses of expressions of type t and order $\leq\beta$. Since expressions may contain free variables and constants, quasi-senses are relative to a valuation of the variables and constants. Let V^β be the set of the v-valuations of order β, that is, $V \in V^\beta$ iff V is a function defined on all variables of order $\delta \leq \beta$ and

$$(4.1) \qquad V(v_{tn}^\delta) \in A_t^\delta \qquad (\text{where } t \in \tau^\nu,\ n \in N_*).$$

Similarly, the set of the c-valuations of order β will be denoted by I^β. The elements of I^β are defined in the obvious way; in particular, for every $I \in I^\beta$ and every $\delta \leq \beta$,

$$(4.2) \qquad I(c_{t\mu}^\delta) \in A_t^\delta \qquad (\text{where } t \in \tau^\nu,\ 0 < \mu < \alpha + \omega_0).$$

Roughly speaking, the quasi-senses of variables and constants are their valuations, whereas the quasi-sense of a compound expression Δ is a sequence $\langle \chi, x_1, \ldots, x_n \rangle$ where χ is a marker depending on the form of Δ and $x_1, \ldots, x_n$ are senses (of the components of Δ) or functions (depending on the senses of the components of Δ).

The quasi-sense of the expression Δ, under the v-valuation V and c-valuation I, will be denoted by $sens_{IV}\Delta$. It is defined by ($s_{1\text{-}10}$) below, where the following convention will be used.

Convention 4.1 For $X = \{v_{t_1}^\delta, \ldots, v_{t_n}^\delta\}$ we shall denote by $g(\Delta, X, V, I)$ the function $\left\{\langle \xi_1, \ldots, \xi_n, \sigma\rangle : \xi_i \in A_{t_i}^{\delta_i},\ \sigma \neq I_F \text{ and } \sigma = sens_{IV'}\Delta, \text{ where } V' = V\begin{pmatrix} v_{t_1}^\delta, \ldots, v_{t_n}^\delta \\ \xi_1, \ldots, \xi_n \end{pmatrix}\right\}$.

Rule	If Δ is	then $\breve{\Delta} = sens_{IV}\Delta$ is
(s_1)	v_{tn}^δ or $c_{t\mu}^\delta$	$V(v_{tn}^\delta)$ or $I(c_{t\mu}^\delta)$, respectively.
(s_2)	$\Delta_0(\Delta_1, \ldots, \Delta_n)$	$\langle \breve{\Delta}_0^{ord}, \breve{\Delta}_0, \breve{\Delta}_1, \ldots, \breve{\Delta}_n\rangle$.
(s_3)	$(\Omega x_1, \ldots, x_n)\Delta'$	$\langle \Omega^{ord}, \breve{\Omega}, g(\Delta, \{x_1, \ldots, x_n\}, V, I)\rangle$.
($s_{4\text{-}6}$)	$\sim\Delta_1, \Delta_1 \supset \Delta_2, \Box\Delta_1$ ($t_1 = t_2 = 0$)	$\langle \sim, \breve{\Delta}\rangle$, $\langle \supset, \breve{\Delta}_1, \breve{\Delta}_2\rangle$, $\langle \Box, \breve{\Delta}_1\rangle$.
($s_{7\text{-}8}$)	$(\forall x)\Delta'$, $(\iota x)\Delta'$ ($t' = 0$)	$\langle \forall, g(\Delta', \{x\}, V, I)\rangle$, $\langle \iota, g(\Delta', \{x\}, V, I)\rangle$.
(s_9)	$\Delta_1 = \Delta_2$ ($t_1 = t_2$)	$\langle =, \breve{\Delta}_1, \breve{\Delta}_2\rangle$.
(s_{10})	$(\lambda^p x_1, \ldots, x_n)\Delta'$	$\langle \lambda^p, g(\Delta', \{x_1, \ldots, x_n\}, V, I)\rangle$.

Now let us define the class QS_t^β for $t \in \tau^\nu$ and $\beta < \alpha$ by

$$(4.3) \qquad QS_t^\beta =_D \{sens_{IV}\Delta \mid V \in V^\beta, \Delta \in E_t^\beta\}.$$

The function V (I) defined on the variables (constants) of $\mathcal{SL}_\alpha^\nu$ will be said to be a v-valuation (c-valuation) relative to Γ and D_1 to D_ν if it satisfies the first (second) of the relations

$$(4.4) \qquad V(v_{tn}^\beta) \in A_t^\beta, \quad I(c_{t\mu}^\beta) \in A_t^\beta$$

$$(\text{where } t \in \tau^\nu,\ n \in N_*,\ 0 < \mu < \alpha + \omega_0 \text{ and } \beta < \alpha).$$

The v-valuations (c-valuations) assigning a hyper-quasi-intension to every variable (constant) will be called *ostensive v-valuations* (*c-valuations*).

The designation rules, which assign hyper-quasi-intensions to wfes of $\mathcal{SL}_{\alpha}^{\nu}$, are not relevant to this paper. A detailed presentation of these rules can be found in [11].

5 An adequacy theorem A theory $\mathcal{T}$ is said to be *based on* $\mathcal{SL}_{\alpha}^{\nu}$ if its symbols are those of $\mathcal{SL}_{\alpha}^{\nu}$, except for some (perhaps all) constants. The constants of $\mathcal{T}$ are regarded as primitive. Furthermore, if Δ is a wfe of $\mathcal{T}$, the primitive constants and free variables occurring in Δ will be referred to as *elementary expressions of* Δ.

Now we can prove the following:

Theorem 5.1 *Assume that*: (i) Δ *and* Φ *are wfes of* $\mathcal{T}$ *defined constants free and of type t*; (ii) *I is an ostensive c-valuation, V and W are ostensive c-valuations, and the set-theoretical unions* $I \cup V$ *and* $I \cup W$ *are injective functions*[1] *on the elementary expressions of* Δ *and* Φ *respectively*; (iii) $sens_{IV}\Delta = sens_{IW}\Phi$; *and* (iv) u_1 *to* u_a *is a bijective list formed with the elementary expressions of* Δ $(a \geq 0)$. *Then* (a) Δ *and* Φ *have the same length, and* (b) *we can arrange the elementary expressions of* Φ *in the list* $w_1, \ldots, w_a$ *and choose equivalents* (see Convention 3.1) Δ', Φ' *of* Δ *and* Φ, *respectively, for which* (see Convention 3.2)

$$(5.1) \qquad \Delta' = \Phi(w_i/u_i)_a \text{ (or } \Phi' = \Delta(u_i/w_i)_a\text{)}, \ (I \cup V)(u_i) = (I \cup W)(w_i) \quad (i = 1, \ldots, a).$$

Proof: Note that the existence of Φ' satisfying the second part of (5.1) is a straightforward consequence of the existence of Δ' satisfying the first part of (5.1).

By conditions (s_{1-10}) in Section 4 and assumption (iii), Δ and Φ have the same length, say ℓ.

We use induction on ℓ: Assume $\ell = 1$; then Δ and Φ are elementary expressions. By (s_1) in Section 4 and assumption (iii), $I \cup V(\Delta) = I \cup W(\Phi)$. By (iv), $u_1 = \Delta$, and hence (5.1) holds for $w_1 = \Phi$ and $\Delta' = \Delta$ ($\Phi' = \Phi$). This concludes the initial step.

Now assume $\ell > 1$ and let the thesis hold for $\bar{\ell} < \ell$. We consider only the cases $\Delta = \Delta_0(\Delta_1, \ldots, \Delta_n)$ and $\Delta = (\Omega_{\Delta} x_1, \ldots, x_n)\Delta_0$. The other cases can be proved in a similar way.

Let

$$(5.2) \qquad \Delta = \Delta_0(\Delta_1, \ldots, \Delta_n).$$

A trivial consequence of (iii) is that

$$(5.3) \qquad \Phi = \Phi_0(\Phi_1, \ldots, \Phi_n),$$

and by (s_2) in Section 4

$$(5.4) \qquad \langle \Delta_0^{ord}, sens_{IV}\Delta_0, \ldots, sens_{IV}\Delta_n \rangle = \langle \Phi_0^{ord}, sens_{IW}\Phi_0, \ldots, sens_{IW}\Phi_n \rangle.$$

Then

(5.5) $\Delta_0^{ord} = \Phi_0^{ord}$ and $sens_{IV}\Delta_\kappa = sens_{IW}\Phi_\kappa$ (where $\kappa = 0, \ldots, n$).

For $\kappa = 0, \ldots, n$, the lengths of Δ_κ and Φ_κ are less than ℓ and conditions (i) to (iv) hold for Δ_κ and Φ_κ (since, in this case, the elementary expressions of Δ_κ are also elementary expressions of Δ, and similarly for Φ_κ and Φ).

Hence by the inductive hypothesis we can choose an equivalent Δ'_κ of Δ_κ and arrange the elementary expressions in Φ_κ into the list w_1^κ to $w_{m_\kappa}^\kappa$ in such a way that

(5.6) $$\Delta'_\kappa = \Phi_\kappa(w_i^\kappa/u_i^\kappa)_{m_\kappa}, \quad I \cup V(u_i^\kappa) = I \cup W(w_i^\kappa)$$
$$\text{(where } i = 1, \ldots, m_k\text{)},$$

where u_1^κ to $u_{m_\kappa}^\kappa$ are the elementary expressions of Δ_κ.

The conclusion above holds for $\kappa = 0, \ldots, n$. Furthermore, the elementary expressions u_i^κ and w_i^κ (where $i = 1, \ldots, m_k$, $\kappa = 0, \ldots, n$) are the elementary expressions in Δ_κ and Φ_κ, respectively. Hence, by (iv), the former are u_1 to u_a. Furthermore, by the injectivity property of $I \cup W$, for $i = 1, \ldots, a$ there is exactly one elementary expression w_j^κ that satisfies the second part of condition (5.6) for $u_j^\kappa = u_i$. We identify w_i with this w_j^κ. The correspondence $u_i \to w_i$ $(i = 1, \ldots, a)$ thus obtained between the elementary expressions of Δ and those of Φ is one-to-one and surjective. Hence, $(5.6)_2$ implies $(5.1)_3$.

Now we set $\Delta' = \Delta'_0(\Delta_1, \ldots, \Delta'_n)$. By (5.2), Δ' is an equivalent of Δ. Hence $(5.6)_1$, true for $\kappa = 0, \ldots, n$, implies $(5.1)_1$. We can conclude that the thesis holds in this case.

Now let

(5.7) $$\Delta = (\Omega_\Delta x_1, \ldots, x_n)\Delta_0.$$

By (iii),

(5.8) $$\Phi = (\Omega_\Phi y_1, \ldots, y_n)\Phi_0,$$

and the lengths of Δ_0 and Φ_0 are less than ℓ.

By (s_3) in Section 4,

(5.9) $$\langle \Omega_\Delta^{ord}, sens_{IV}\Omega_\Delta, g(\Delta_0, \{x_1, \ldots, x_n\}, V, I)\rangle =$$
$$\langle \Omega_\Phi^{ord}, sens_{IW}\Omega_\Phi, g(\Phi_0, \{y_1, \ldots, y_n\}, W, I)\rangle$$

(see Convention 4.1). Then we have

(5.10) $$g(\Delta_0, \{x_1, \ldots, x_n\}, V, I) = g(\Phi_0, \{y_1, \ldots, y_n\}, W, I).$$ [2]

By Convention 4.1 and rules ($s_{1\text{-}10}$) we have for $\xi_i \in A_{t_i}^{\delta_i}$ (where $i = 1, \ldots, n$),

(5.11) $$sens_{IV'}\Delta_0 = sens_{IW'}\Phi_0,$$

where

$$V' = V\begin{pmatrix} x_1 \ldots x_n \\ \xi_1 \ldots \xi_n \end{pmatrix} \quad \text{and} \quad W' = W\begin{pmatrix} y_1 \ldots y_n \\ \xi_1 \ldots \xi_n \end{pmatrix}.$$

Conditions (i), (iii), and (iv) hold for Δ_0 and Φ_0. With a view to dealing with (ii) we choose the n-tuple $\xi_1, \ldots, \xi_1$ in such a way that, if s_1 to s_p are the elementary expressions of Φ_0, different from y_1 to y_n, then

$$(5.12) \qquad \xi_1 = (I \cup W)(s_1).$$

By (5.11) we have

$$(5.13) \qquad sens_{IV'}\Delta_0 = sens_{IW'}\Phi_0,$$

where

$$V' = V\begin{pmatrix} x_1 \ldots x_n \\ \xi_1 \ldots \xi_1 \end{pmatrix} \quad \text{and} \quad W' = W\begin{pmatrix} y_1 \ldots y_n \\ \xi_1 \ldots \xi_1 \end{pmatrix}.$$

We note that V' and W' are ostensive c-valuations. Furthermore, we prove easily from (5.11) that the variables among x_1 to x_n, which are free in Δ_0, are as many as those, among y_1 to y_n, which are free in Φ_0. By (5.13) there exists one d_k among $d_1, \ldots, d_q$ (which are the elementary expressions of Δ_0 different from x_1 to x_n) such that $\xi_1 = IUV(d_k)$. This can be proved by induction on the length of Δ_0 (by using reductio ad absurdum also). We consider now the list $d_1, \ldots, d_q, d_{q+1}, \ldots, d_{q+m}$ $(s_1, \ldots, s_p, s_{p+1}, \ldots, s_{p+m})$ where the m variables $d_{q+1}, \ldots, d_{q+m}$ $(s_{p+1}, \ldots, s_{p+m})$ are those, among x_1 to x_n (y_1 to y_n), which are free in Δ_0 (Φ_0). We set

$$(5.14) \qquad \bar{\Delta}_0 = \Delta_0 \ (d_{q+1}, \ldots, d_{q+m}/d_k, \ldots, d_k),$$
$$\bar{\Phi}_0 = \Phi_0 \ (s_{p+1}, \ldots, s_{p+m}/s_1, \ldots, s_1).$$

We can easily prove the following:

Lemma 5.1 *Assume that u_1 to u_a (t_1 to t_a) is a bijective (possibly nonbijective) list of the elementary expressions occurring in the wfe Ψ and that V and W are c-valuations; then*

$$(5.15) \qquad \Psi' = \Psi(u_i/t_i)_a, \quad (I \cup V)(u_i) = (I \cup W)(t_i)$$
$$(\textit{where } i = 1, \ldots, a) \Rightarrow sens_{IV}(\Psi) = sens_{IW}(\Psi').$$

By Lemma 5.1 we have

$$(5.16) \qquad sens_{IV'}\bar{\Delta}_0 = sens_{IV'}\Delta_0, \quad sens_{IW'}\bar{\Phi}_0 = sens_{IW'}\Phi_0;$$

hence by (5.13) and (5.16) we have

$$(5.17) \qquad sens_{IV'}\bar{\Delta}_0 = sens_{IW'}\bar{\Phi}_0.$$

We conclude that conditions (i) to (iv) hold for $\bar{\Delta}_0$, $\bar{\Phi}_0$, V', and W'. Then by the inductive hypothesis (the length to $\bar{\Delta}_0$ is obviously less than ℓ) the thesis also holds for these entities.

Hence we can arrange the elementary expressions of $\bar{\Phi}_0$ into a list φ_1 to φ_q and can choose an equivalent $\bar{\Delta}'_0$ of $\bar{\Delta}_0$ for which

$$(5.18) \qquad \bar{\Delta}'_0 = \bar{\Phi}_0(\varphi_i/d_i), \quad I \cup V(d_i) = I \cup W(\varphi_i) \qquad (\text{where } i = 1, \ldots, q).$$

Then, for $i = 1, \ldots, q$, d_i and φ_i are the elementary expressions of Δ_0 and Φ_0, respectively, different from x_i and y_i (where $i = 1, \ldots, n$).

By (5.9), we also have

(5.19) $$sens_{IV}\Omega_\Delta = sens_{IW}\Omega_\Phi.$$

The lengths of Ω_Δ and Ω_Φ are less than ℓ and conditions (i) to (iv) hold for Ω_Δ and Ω_Φ (since, in this case, the elementary expressions of Ω_Δ are also elementary expressions of Δ, and similarly for Ω_Φ and Φ).

Hence we can arrange the elementary expressions of Ω_Φ into a list ω_1 to ω_r and can choose an equivalent Ω'_Δ of Ω_Δ for which

(5.20) $$\Omega'_\Delta = \Omega_\Phi(\omega_i/\delta_i), \quad I \cup V(\delta_i) = I \cup W(\omega_i) \qquad (\text{where } i = 1,\dots,r),$$

where δ_1 to δ_r are the elementary expressions of Ω_Δ.

Hence the elementary expressions δ_1 to δ_r and d_1 to d_q are those of Δ. Hence, by (iv), they are u_1 to u_a. Furthermore, by the injectivity properties of $I \cup W$, for $i = 1,\dots,a$, there is exactly one elementary expression of Φ that satisfies condition $(5.18)_2$ or $(5.20)_2$ for u_i in δ_1 to δ_r or in d_1 to d_q. We denote this elementary expression by w_i. The correspondence $u_i \to w_i$ (where $i = 1,\dots,a$) thus obtained between the elementary expressions in Δ and those in Φ is a bijection. Hence $(5.18)_2$ and $(5.20)_2$ imply $(5.1)_3$.

Now it is clear that by (5.7), (5.8), $(5.18)_1$, $(5.20)_1$, and the metalinguistic definition

(5.21) $$\Delta' = ([\Omega_\Phi y_1 \dots y_n]\Phi_0)(w_i/u_i)_a,$$

Δ' is an equivalent of Δ.

We conclude that the theorem holds also in this case.

6 Admissible definitions; strong and weak extensions of a theory; the synonymies from $\asymp_0$ to $\asymp_3$ We define recursively the class AD_t^n of *admissible definienda* of type t and degree n ($t \in \tau^\nu$, $n \in N_*$) by conditions (a) to (c) below (see [3]).

(a) $c_{t\mu} \in AD_t^0$.
(b) If $\Delta \in AD^n_{\langle t_1,\dots,t_m,t\rangle}$ and x_1 to x_m are m variables (of suitable types) distinct from those occurring in Δ, then $\Delta(x_1,\dots,x_m) \in AD_t^{n+1}$.
(c) If $\Omega \in AD^n_{(t_1,\dots,t_m;\theta,\varphi)}$ and x_0 to x_m *are* $m+1$ variables (of suitable types) not occurring in Ω, then $(\Omega x_1,\dots,x_m)x_0(x_1,\dots,x_m) \in AD_\varphi^{n+1}$.

By induction one can easily prove the assertions

(d) If $\Delta \in AD_t^n$, then only one constant occurs in Δ and only once, and
(e) If $\Delta \in AD_t^n \cap AD_t^m$, then $n = m$.

Then n can be called the *degree* of the admissible definiendum Δ. The class of admissible definienda of type t is defined by

(6.1) $$AD_t = \bigcup_{n\in N} AD_t^n \qquad (\text{where } t \in \tau^\nu).$$

Now assume that: (i) $\Delta \in AD_t^n$ for some $t \in \tau^\nu$ and $n \in N$; (ii) K is a class of constants different from the one, c_r, occurring in Δ; (iii) $\Delta' \in E_t$; (iv) the constants occurring in Δ' are in K; and (v) the free variables in Δ' are free also in Δ. Then we say that (α) the wff

$$(6.2) \qquad \Delta = \Delta' \quad (\textit{equivalent to } \Delta \equiv \Delta' \textit{ for } t = 0)$$

is an *admissible definition of c_r in terms of the constants in K*, (β) its *degree* is n, (γ) Δ is its *definiendum*, and (δ) Δ' is its *definiens*.

By use of Church's lambda operator, the degree of many admissible definienda can be lowered in the following sense. If the relations

$$(6.3) \qquad \Delta \in AD^n_{\langle t_1,\ldots,t_m,t\rangle},\ \Omega \in AD^n_{(t_1,\ldots,t_m;\theta,\varphi)}$$

hold, then by means of a suitable choice of Δ and x_0 to x_m, the wffs

$$(6.4) \qquad \Delta(x_1,\ldots,x_m) = \Delta',\ (\Omega x_1,\ldots,x_m)x_0(x_1,\ldots,x_m) = \Delta',$$

are admissible definitions of degree $n + 1$. As a consequence, the equalities

$$(6.5) \qquad \Delta = (x_1,\ldots,x_m)\Delta',\ \Omega = (\lambda x_0)\Delta',$$

are admissible definitions of degree n, to be called *directly associated* with $(6.4)_1$ and $(6.4)_2$ respectively. This relation generates an equivalence relation R. If two definitions are related by R, we say that they are *associated*.

Following [3], we give the following definition:

Definition 6.1 The wfes $\Delta(x_1,\ldots,x_n)$ and $\Phi(y_1,\ldots,y_n)$, briefly Δ and Φ, will be said to be $(x_1,y_1,\ldots,x_n,y_n)$-similar if for $i = 1$ to n, x_i and y_i are n variables of the same order and $\Delta(v_1,\ldots,v_n)$ is equivalent to $\Phi(v_1,\ldots,v_n)$ whenever $v_1,\ldots,v_n$ are variables which do not occur in Δ or Φ.

Let $\mathfrak{T}$ be any theory based on $\mathcal{SL}^{\nu}_{\alpha}$. In connection with $\mathfrak{T}$ the synonymy $\asymp_0$ [$\asymp_1$] can be defined recursively as the smallest equivalence relation among wfes of $\mathfrak{T}$ that satisfies conditions ($C_{1\text{-}2}$) (($C_{1\text{-}4}$)) below in the (binary) relation $\asymp$.

(C_1) If $\Delta \asymp \Delta'$ and $\Delta_i \asymp \Delta_i'$ (where $i = 1,\ldots,n$), then $\Delta(\Delta_1,\ldots,\Delta_n) \asymp \Delta'(\Delta_1',\ldots,\Delta_n')$ where $\Delta,\Delta' \in E_{\langle t_1,\ldots,t_n,t\rangle}$ and Δ_i, $\Delta_i' \in E_{t_i}$ (where $i = 1,\ldots,n$).
(C_2) If $\Delta \asymp \Delta'$, $\Omega \asymp \Omega'$ and Δ' and Δ'' are $(x_1,y_1,\ldots,x_n,y_n)$-similar (see Definition 6.1), then $(\Omega x_1,\ldots,x_n)\Delta \asymp (\Omega y_1,\ldots,y_n)\Delta''$, where $\Delta,\Delta',\Delta'' \in E_\theta$ and $\Omega,\Omega' \in E_{(t_1,\ldots,t_n;\theta,\varphi)}$.[3]
(C_3) $(\Omega x_1,\ldots,x_n)\Delta \asymp \Omega[(\lambda x_1,\ldots,x_n)\Delta]$, where $\Delta \in E_\theta$ and $\Omega \in E_{(t_1,\ldots,t_n;\theta,\varphi)}$.
(C_4) $(\lambda x_1,\ldots,x_n)\Delta \asymp (\lambda^p x_1,\ldots,x_n)\Delta$, where $\Delta \in E_\theta$.

Now let χ be a countable (possible tranfinite) ordinal and let $\{c_\varphi\}_{\varphi<\chi}$ be an injective sequence of constants that do not belong to $\mathfrak{T}$. For every $\varphi < \chi$, let D_φ be an admissible definition of c_φ in terms of the constants c_ψ with $\psi < \varphi$ and the constants belonging to $\mathfrak{T}$—i.e., the primitive constants of $\mathfrak{T}$.

The weak and strong extensions of

$$(6.6) \quad \mathfrak{T}^w = (\mathfrak{T},D)_w,\ \mathfrak{T}^s = (\mathfrak{T},D)_s \qquad (\text{where } D = \{D_\varphi\}_{\varphi<\chi},\ D_\varphi \equiv_D \Delta_\varphi = \Delta_\varphi')$$

have the symbols of $\mathfrak{T}$ added (only) with the constants c_φ ($\varphi < \chi$). The wfes of $\mathfrak{T}^s$ are those of $\mathfrak{T}$ formed with symbols of $\mathfrak{T}^s$, while the wfes of $\mathfrak{T}^w$ are obtained (roughly speaking) from those of $\mathfrak{T}$ and the definienda Δ_φ ($\varphi < \chi$) by substitution of some among these wfes, or some among already constructed wfes, for some variables free in a wfe of the same kind.[4]

Considering $\mathfrak{T}^s$ ($\mathfrak{T}^w$) is useful in connection with a semantics for which the senses of wfes are (are not) preserved by the principles of λ-conversion. Note that theory $\mathfrak{T}^w$, unlike $\mathfrak{T}^s$, generally fails to be based on $\mathcal{SL}_\alpha^\nu$, but it is a proper part of such a theory.

Now in connection with $\mathfrak{T}^s$ we define the synonymy relation $\asymp_2$ ($\asymp_3$) as the smallest equivalence relation between wfes of $\mathfrak{T}^s$ that satisfies both conditions ($C_{1\text{-}4}$) above and ($C_{5\text{-}7}$) (($C_{5\text{-}8}$)) below in the relation $\asymp$.

(C_5) $\Delta_\varphi \asymp \Delta'_\varphi \ (\varphi < \chi)$.
(C_6) $\Delta \asymp (\lambda^p x_1, \ldots, x_n)(\Delta[x_1, \ldots, x_n])$, where $\Delta \in E_{\langle t_1, \ldots, t_n, t\rangle}$.
(C_7) $([\lambda^p x_1, \ldots, x_n]\Phi)(\Delta_1, \ldots, \Delta_n) \asymp \Phi(x_i/\Delta_i)_n$ where $\Phi \in E_t$.
(C_8) $p \wedge q \asymp q \wedge p$ for all wffs p and q.

In connection with $\mathfrak{T}^w$ the synonymy relation $\asymp_0$ ($\asymp_1$) can be defined by means of conditions ($C_{1\text{-}2}$) (($C_{1\text{-}4}$)) above and the following:

($C_{5'}$) If $\Delta_i \asymp \Delta'_i$ (where $i = 1, \ldots, n$), then $\Delta_\varphi(x_i/\Delta_i)_n \asymp \Delta'_\varphi(x_i/\Delta_i)_n$.

7 *Semiotic and semantic preliminaries*

One could say that the $(\nu + 2)$-tuple $\mathbf{I} = \langle D_1, \ldots, D_\nu, \Gamma, I\rangle$ where I is a c-valuation relative to Γ and D_1 to D_ν (see (4.4)) is an *interpretation for* $\mathcal{SL}_\alpha^\nu$, and the v-valuations relative to Γ and D_1 to D_ν are **I**-*valuations*.

We consider v-valuated wfes of $\mathfrak{T}^s$ (or, more precisely, **I**-valuated wfes) defined as couples $\langle \Delta, V\rangle$ where Δ is a wfe of $\mathfrak{T}^s$ and V is an **I**-valuation.

We now introduce some notations that will be useful in what follows.

Definition 7.1 (a) Let $\zeta = \langle \Delta, V\rangle$ be an **I**-valuated wfe of $\mathfrak{T}^s$, whose elementary expressions can be arranged in the (bijective) list u_1 *to* u_a. Furthermore, let v_i be u_i if, for no variable v_{tn} free in Δ, we have

$$V(v_{tn}) = (V \cup I)(u_i) \tag{7.1}$$

(so that u_i is a constant); otherwise, let v_i be the v_{tn} that satisfies (7.1), with the least n. Then we denote v_i by $u_i(\Delta, V)$ and we say that the (possibly nonbijective) sequence $\langle v_1, \ldots, v_a\rangle$ is the Δ-V-reduction of $\langle u_1, \ldots, u_a\rangle$. Furthermore, let $\zeta^V = \langle \Delta^V, V\rangle$, where

$$\Delta^V = \Delta(u_i/v_i)_a = \Delta[u_1, \ldots, u_a/v_1, \ldots, v_a]. \tag{7.2}$$

Then we say that (b) Δ^V is the V-reduction of Δ, and (c) ζ^V is the V-reduction of ζ.

Now: for any wfe $\mathfrak{T}^s$ of Δ and **I**-valuation V, (i) the function $I \cup V$ is injective on the elementary expressions of Δ^V, (ii) $(\Delta^V)^V = \Delta^V$, (iii) V is injective on the variables free in Δ iff $\Delta = \Delta^V$, in case no primitive constants occur in Δ.

For every definition D_φ in D of $\mathfrak{T}^s$, let $c_\varphi = \bar{\Delta}_\varphi$ be the associate definition of degree zero (see Section 6). By transfinite induction we now define $\bar{\Delta}_\varphi^T$ for $\varphi < \chi$.

$$\bar{\Delta}_0^T = \bar{\Delta}_0, \ \bar{\Delta}_\varphi^T = \bar{\Delta}_\varphi(c_{\psi_i}/\bar{\Delta}_{\psi_i}^T)_\mu, \tag{7.3}$$

where ψ_1 to ψ_μ are the μ values of ψ ($<\chi$) with which c_ψ occurs in Δ_φ, and hence in $\bar{\Delta}_\varphi$.

Furthermore, if Δ is a wfe of $\mathfrak{T}^s$, we set

$$\Delta^T = \Delta(c_{\varphi_i}/\bar{\Delta}^T_{\varphi_i})_\mu, \tag{7.4}$$

φ_1 to φ_μ being the μ values of φ ($<\chi$) with which c_φ occurs in Δ, and we say that Δ is the *T-correspondent* of Δ.

In connection with any admissible interpretation **I** for $\mathfrak{T}^s$, i.e. an interpretation for $\mathcal{SL}^\nu_\alpha$ that satisfies the definition D_φ ($\varphi < \chi$) of $\mathfrak{T}^s$, and any **I**-valuation V, the quasi-sense of any wfe Δ of theory $\mathfrak{T}^s$ (briefly: $sens_{IV}\Delta$) is defined by simultaneous recursion on the type t of Δ, by means of rules ($s_{2\text{-}10}$) and rules (s_1) and s_1') below:

(s_1) if Δ has the form v_{tn} or $c_{t\mu}$, where $c_{t\mu}$ is a primitive constant, then $sens_{IV}(\Delta)$ is $V(v_{tn})$ or $I(c_{t\mu})$, respectively,

(s_1') if Δ is a defined constant, then $sens_{IV}(\Delta)$ is $\text{sens}_{\text{IV}}(\Delta^T)$.

8 A strong version of the adequacy theorem Theorem 8.1 below is an adequacy theorem stronger than Theorem 5.1, since it does not involve the assumptions that $I \cup V$ and $I \cup W$ are injective and that no defined constant occurs in Δ and Φ.

Theorem 8.1 *Assume that* (i) Δ *and* Φ *are wfes of* $\mathfrak{T}^s$ *of type* t, (ii) $\mathbf{I} = \langle D_1, \ldots, D_\nu, \Gamma, I\rangle$ *is an admissible interpretation for* $\mathfrak{T}^s$, *where* **I** *is an ostensive c-valuation, and* V *and* W *are ostensive* **I**-*valuations*, (iii) $sens_{IV}\Delta = sens_{IW}\Phi$, (iv) u_1 *to* u_a *is a bijective list formed with the elementary expressions of* $\Delta^{TV}(a \geq 0)$. *Then* (a) Δ^{TV} *and* Φ^{TW} *have the same length, and* (b) *we can arrange the elementary expressions of* Φ^{TW} *in the list* w_1 *to* w_a *and can choose equivalents* Δ', Φ' *of* Δ^{TV} *and* Φ^{TW} *for which* (*see Convention* 3.2)

$$\Delta' = \Phi^{TV}(w_i/u_i) \quad (or\ \Phi' = \Delta^{TW}(u_i/w_i)_a) \quad I \cup V(u_i) = I \cup V(w_i) \tag{8.1}$$

$$(where\ i = 1, \ldots, a).$$

Proof: By induction on the length of Δ, we can prove that

$$sens_{IV}\Delta = sens_{IV}\Delta^T \quad \text{and} \quad sens_{IW}\Phi = sens_{IW}\Phi^T \tag{8.2}$$

and by Lemma 5.1 in Section 5, we have

$$sens_{IV}\Delta^T = sens_{IV}\Delta^{TV} \quad \text{and} \quad sens_{IW}\Phi^T = sens_{IW}\Phi^{TW}. \tag{8.3}$$

Then by (8.2), (8.3), and assumption (iii), we have

$$sens_{IV}\Delta^{TV} = sens_{IW}\Phi^{TW}. \tag{8.4}$$

Furthermore, all hypotheses of Theorem 5.1 hold for Δ^{TV} and Φ^{TW}; hence, so does the thesis.

NOTES

1. By $I \cup V$ we denote the function obtained as a set-theoretical union of the function I and V which have disjoint domains.

2. Obviously, for $i = 1, \ldots, n$, x_i has the same order and type as y_i.

3. In particular, y_i and x_i (where $i = 1, \ldots, n$) can coincide as well as Δ' and Δ''.

4. For a more precise description, see [9], Section 20.

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