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The Dual Cantor-Bernstein Theorem and the Partition Principle

BERNHARD BANASCHEWSKI and GREGORY H. MOORE

Abstract This paper examines two propositions, the Dual Cantor-Bernstein Theorem and the Partition Principle, with respect to their logical interrelationship and their history. It is shown that the Refined Dual Cantor-Bernstein Theorem is equivalent to the Axiom of Choice.

1 Introduction We first recall some standard notation. If $x \leq y$ means that there is an injection $f: x \to y$, the dual relation $x \leq^* y$ is taken to mean that, if x is nonempty, there is a surjection $g: y \to x$. Analogously, $x < y$ means that $x \leq y$ and not $y \leq x$, while $x <^* y$ means that $x \leq^* y$ and not $y \leq^* x$. Letting $x \approx y$ mean that there is a bijection $f: x \to y$, we can express the Cantor-Bernstein Theorem as the proposition that if $x \leq y$ and $y \leq x$, then $x \approx y$. Likewise, the Dual Cantor-Bernstein Theorem (CB*) states that if $x \leq^* y$ and $y \leq^* x$, then $x \approx y$. The Partition Principle (PP) connects $\leq$ and $\leq^*$ by stating that $x \leq^* y$ implies $x \leq y$.

Neither the Dual Cantor-Bernstein Theorem nor the Partition Principle can be proved in Zermelo-Fraenkel set theory (ZF), but both are theorems if the Axiom of Choice (AC) is permitted. It is easily seen that CB* follows from PP in ZF, by means of the Cantor-Bernstein Theorem, and also that the converse of PP is a theorem of ZF. Now the Trichotomy of Cardinals (TC) states that, for all x and y, $x < y$ or $y < x$ or $x \approx y$. Likewise, the Dual Trichotomy of Cardinals (TC*) states that $x <^* y$ or $y <^* x$ or $x \approx y$. It turns out that TC is equivalent to AC (Hartogs [5]), and so is TC* (Lindenbaum and Tarski [9]; Sierpiński [18]).

It will be useful to introduce a certain refinement for each of CB*, PP, AC, and TC. The $\aleph_\alpha$-Dual Cantor-Bernstein Theorem ($\aleph_\alpha$-CB*) states that, for every x, if $x \leq^* \aleph_\alpha$ and $\aleph_\alpha \leq^* x$, then $x \approx \aleph_\alpha$. Analogously, the $\aleph_\alpha$-Partition Principle ($\aleph_\alpha$-PP) states that, for every x, if $\aleph_\alpha \leq^* x$, then $\aleph_\alpha \leq x$. It is clear that, for each α, $\aleph_\alpha$-PP implies $\aleph_\alpha$-CB*. In a similar vein, $\aleph_\alpha$-AC states that if $x \approx \aleph_\alpha$, then there is a function f such that $f(y) \in y$ for every nonempty member y of

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x. Finally, for every α, $\aleph_\alpha$-TC states that for every x, $x < \aleph_\alpha$ or $\aleph_\alpha < x$ or $x \approx \aleph_\alpha$.

2 Some historical remarks The Partition Principle has a long history.[1] In 1883 Cantor employed $\aleph_1$-PP in proving that if a point set P in $\mathbb{R}^n$ has a countable set of limit points, then the αth derived set of P is empty for some countable α ([4], p. 413). In 1896 Burali-Forti introduced what is (almost) the Partition Principle as an axiom for set theory in the following form: If S is a family of nonempty classes, then $S \leq \bigcup S$. Unfortunately, Burali-Forti's axiom is false, as Russell noted in 1906 ([12], p. 49), unless S is assumed to be a disjoint family. With this assumption, Burali-Forti's axiom is a form of the Partition Principle.

The first clear and correct statement of the Partition Principle was made in 1902 by Beppo Levi [7], who expressed it in a way that makes its name apparent: if a set A is partitioned into a family S of disjoint nonempty sets, then $S \leq A$. Yet Levi explicitly introduced PP precisely in order to reject it in general and to object, in particular, to Bernstein's use of PP in 1901 to prove that the family of closed subsets of the real line has the power of the continuum. Bernstein rejected Levi's criticism in turn and affirmed PP thus: "I regard this principle as one of the most important in set theory, and I see no objection to using it" ([2], p. 558). All of this occurred before Zermelo [20] introduced the Axiom of Choice late in 1904. When Zermelo proposed this axiom, he mentioned, as an argument for accepting it, the fact that PP cannot be proved without AC.

In 1906, in an unpublished manuscript entitled "Multiplicative Axiom" [13], Russell asserted without proof that PP is equivalent to the Multiplicative Axiom, which in 1908 he proved equivalent to AC [14]. More precisely, in the 1906 manuscript he claimed that the proposition FR (that every function includes a one–one function with the same range) is equivalent to the Multiplicative Axiom (as is, in fact, the case), showed that FR implies PP, and claimed without proof that PP implies FR.[2] To this day, however, it remains uncertain whether PP is equivalent to AC.

Further research on PP and CB* was largely due to the Warsaw school of set theorists. Thus in 1918 Sierpiński stated PP in the following form: For every set A and function f, $f''A \leq A$ ([15], p. 109). In 1926 Lindenbaum and Tarski [9] first introduced the relation $\leq *$ and formulated the Weak Partition Principle (WPP): If $x \leq * \, y$, then not $y < x$. They asserted that WPP, which clearly follows from CB*, implies the existence of a nonmeasurable subset of $\mathbb{R}$ as well as of an uncountable subset of $\mathbb{R}$ lacking a perfect subset.[3] Tarski established that WPP also implies $\aleph_1 \leq 2^{\aleph_0}$ ([16], p. 227). In 1926 Lindenbaum and Tarski [9] also pointed out that $\aleph_0$-PP implies $\aleph_0$-TC.

In 1965 Levy [8] stated that it was not known whether, in ZF, WPP implies CB*, or CB* implies PP, or PP implies AC, or whether none of these is the case. On the other hand, Tarski observed that the Weak Power Hypothesis (WPH) implies CB*, where WPH states that $\mathcal{P}(x) \approx \mathcal{P}(y)$ implies $x \approx y$ ([8], p. 225).

Pincus [11] established in 1978 that PP implies the proposition: For all α, $\aleph_\alpha$-AC. This proposition is known, however, not to imply even $\aleph_1$-TC ([6], p. 127). Actually, Pincus's proof shows even more, namely that if for all α we have $\aleph_\alpha$-PP, then for all α we have $\aleph_\alpha$-AC. After remarking that $\aleph_\alpha$-TC implies $\aleph_\beta$-PP for all $\beta \leq \alpha$, Pelc [11] gave an equivalence between CB* and PP under certain conditions: If NDS, then PP if and only if CB* and DC and IP.[4] Tarski [19] had already established that NDS implies $\aleph_0$-AC restricted to finite sets (cf. [6], p. 161).

3 An equivalence result The remainder of this paper is devoted to showing that a natural strengthening of CB* is equivalent to AC. Let us now rename the Cantor-Bernstein Theorem as the "Raw Cantor-Bernstein Theorem", and analogously for the Dual Cantor-Bernstein Theorem. Then the Refined Cantor-Bernstein Theorem will be the following proposition: If $f:x \to y$ and $g:y \to x$ are injections, then there is a bijection $h:x \to y$ such that $h \subseteq f \cup g^{-1}$. Likewise, the Refined Dual Cantor-Bernstein Theorem states that if $f:x \to y$ and $g:y \to x$ are surjections, then there is a bijection $h:x \to y$ such that $h \subseteq f \cup g^{-1}$. Now the Refined Cantor-Bernstein Theorem, which clearly implies the Raw Cantor-Bernstein Theorem, is provable in ZF. Indeed, all known proofs in ZF of the Raw Cantor-Bernstein Theorem actually establish the Refined Cantor-Bernstein Theorem (see [1]). On the other hand, the Refined Dual Cantor-Bernstein Theorem, which clearly implies CB*, is quite strong:

Theorem *The Refined Dual Cantor-Bernstein Theorem is equivalent to* AC.

Proof: ($\Leftarrow$): Given any surjections $f:x \to y$ and $g:y \to x$, AC supplies right inverses $u:x \to y$ and $v:y \to x$ for g and f, respectively, and by the Refined Cantor-Bernstein Theorem there exists a bijection $h:x \to y$ such that $h \subseteq u \cup v^{-1}$. Now $u \subseteq g^{-1}$ and $v \subseteq f^{-1}$ since gu and fv are identity maps. Therefore $h:x \to y$ is a bijection included in $f \cup g^{-1}$.

($\Rightarrow$): We establish AC by showing that any surjection $f:x \to y$ has a right inverse. For this, let

$$z = \{0\} \cup y \cup (x \times \omega),$$

assuming this union to be disjoint, and define a map $k:z \to z$ by

$k(0) = 0,$

$k(s) = 0$ for all $s \in y$,

$k(t,0) = f(t)$ for all $t \in x$,

$k(t,n+1) = (t,n)$ for all $t \in x$ and all $n \in \omega$.

Obviously k is surjective since f is, and by applying the Refined Dual Cantor-Bernstein Theorem to the pair $k:z \to z$, $k:z \to z$ we obtain a bijection $h:z \to z$ such that $h \subseteq k \cup k^{-1}$. In particular, we have that $h|y = \{(s,h(s))|s \in y\} \subseteq k \cup k^{-1}$.

If $h|y \subseteq k^{-1}$, then $kh(s) = s$ for each s in y, and since k maps only $x \times \{0\}$ into y this implies $h(s) \in x \times \{0\}$ for every $s \in y$. Then there is a map $g:y \to x$ for which $h(s) = (g(s),0)$. It follows that

$$s = kh(s) = k(g(s),0) = fg(s)$$

for every $s \in y$, showing that g is a right inverse of f.

If, on the other hand, $h|y$ is not a subset of k^{-1}, then there is exactly one

$s_0 \in y$ such that $h(s_0) = 0$, and the same argument as before applies to all the s in $y_0 = y \setminus \{s_0\}$. Hence we have a map $g_0 : y_0 \to x$ such that $fg_0(s) = s$ for all $s \in y_0$, and the desired right inverse $f: y \to x$ of f is now obtained by picking any $t_0 \in x$ for which $f(t_0) = s_0$ is the missing value of g at s_0.

It should be pointed out that the ($\Rightarrow$) part of the above proof actually uses only the special case of the Refined Dual Cantor–Bernstein Theorem in which $y = x$ and $g = f$. If we refer to this case as the Special Refined Dual Cantor–Bernstein Theorem, we obtain the following result:

Corollary AC *implies the Refined Dual Cantor–Bernstein Theorem and follows from the Special Refined Dual Cantor–Bernstein Theorem.*

In closing, we note that the above equivalence theorem may alternatively be obtained from the topos-theoretical result concerning the Refined Dual Cantor–Bernstein Theorem given in [1]. There the argument for ($\Leftarrow$) is expressed in such a way that it becomes valid in any topos; for the counterpart of ($\Rightarrow$), however, one requires the existence of a natural number object (the topos-theoretical expression of the Axiom of Infinity) as well as a certain technical condition on the subobjects of 1. The present theorem then follows from the observation that the topos of sets and maps determined by ZF is of the required kind. The direct proof for ZF given here is a good deal more transparent.

4 Open problems PP and CB* are a rich source of problems. Several open problems concern the equivalence of AC with various of the propositions considered in this paper. Problem (1) below is another way of asking the natural question raised by the theorem proved above; namely, does the Raw Dual Cantor–Bernstein Theorem imply the Refined Dual Cantor–Bernstein Theorem?

(1) Is CB* equivalent to AC?
(2) Is PP equivalent to AC?
(3) is WPP equivalent to AC?
(4) Is NDS equivalent to AC?

The answer to (4) is almost certainly negative, but thus far there is no proof. Yet it should be mentioned that in 1905 Schoenflies asserted that NDS is equivalent to the Well-Ordering Theorem (so that the answer to (4) is affirmative), while Zermelo in 1908 rejected Schoenflies's claim ([10], p. 149).

Certain problems arise from asking whether known implications are reversible:

(5) Does CB* imply PP?
(6) Does WPP imply CB*?

Of these, problems (1), (4), and (5) were posed in Pelc [11], but no progress seems to have been made concerning them. Pelc's fourth problem (Is CB* provable in ZF?) can be answered in the negative, since CB* implies WPP, which in turn implies $\aleph_1 \leq 2^{\aleph_0}$, a proposition known not to be a theorem of ZF.

A similar group of problems concerns NDS, which is known to imply $\aleph_0$–TC:

(7) Does NDS imply WPP (or CB* or even PP)?
(8) Does PP imply NDS?
(9) Does NDS imply $\aleph_1$-TC?

The last group of problems is somewhat miscellaneous:

(10) Does PP imply the Boolean Prime Ideal Theorem (or even that every set can be ordered)?
(11) Find a proposition P, where P does not imply AC, such that PP and P jointly imply AC. (One candidate for P is the Boolean Prime Ideal Theorem.)
(12) Does WPH imply PP (or even AC)?
(13) Does CB* (or even WPP) imply DC?
(14) Does PP follow from the proposition that for all α, $\aleph_\alpha$-PP?

Apropos of (14), recall that TC follows from the proposition that for all α, $\aleph_\alpha$-TC, but that AC does not follow from the proposition that for all α, $\aleph_\alpha$-AC.

The corresponding question for CB* (Does CB* follow from the proposition that for all α, $\aleph_\alpha$-CB*?) can be answered negatively since, for each α, $\aleph_\alpha$-CB* is provable in ZF. This proof is an easy consequence of the lemma that if $x \leq^* y$ and y is well-orderable, then $x \leq y$. The first form in which $\aleph_\alpha$-WPP could be stated (if $x \leq^* \aleph_\alpha$, then not $\aleph_\alpha < x$) is also provable in ZF, as is the second form (if $\aleph_\alpha \leq^* y$, then not $y < \aleph_\alpha$), by means of $\aleph_\alpha$-CB*. Consequently, with either form of $\aleph_\alpha$-WPP, we have that WPP does not follow from the proposition that for all α, $\aleph_\alpha$-WPP.

NOTES

1. These remarks are limited to the history of PP and of CB*. For a discussion of the history of the Cantor-Bernstein Theorem, see [1] and [10], pp. 42–50. On the history of the Axiom of Choice, see [10].

2. Russell's proposition FR, that every function $f: x \to y$ includes an injection F with the same range, is clearly the same (except for the algebraic terminology) as the later form of AC stating that every surjection $f: x \to y$ has a right inverse $g: x \to y$ (namely, take g as F^{-1}).

3. Proofs were only published in 1947 by Sierpiński [17], and they reveal that these consequences (and $\aleph_1 \leq 2^{\aleph_0}$) already follow from $2^{\aleph_0}$-WPP, i.e., from the proposition that if $\mathbb{R} \leq^* y$, then not $y < \mathbb{R}$.

4. Here NDS, for "no decreasing sequence of cardinals", states that there is no function f on the natural numbers such that for every n, $f(n+1) < f(n)$; DC is the Principle of Dependent Choices; and IP, for "intermediate power", states that if $x <^* y$ then, for some z, $x \leq^* z$ and $z < y$.

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Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario L8S 4K1
Canada