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Modal Definability in Enriched Languages

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Abstract The paper deals with polymodal languages combined with standard semantics defined by means of some conditions on the frames. So a notion of "polymodal base" arises which provides various enrichments of the classical modal language. One of these enrichments, viz. the base $\mathcal{L}(R,-R)$, with modalities over a relation and over its complement, is the paper's main paradigm. The modal definability (in the spirit of van Benthem's correspondence theory) of arbitrary and Δ-elementary classes of frames in this base and in some of its extensions, e.g., $\mathcal{L}(R,-R,R^{-1},-R^{-1})$, $\mathcal{L}(R,-R,\neq)$ etc., is described, and numerous examples of conditions definable there, as well as undefinable ones, are adduced.

Introduction Undoubtedly, first-order languages are reliable and universal tools for formalization. However, in some cases the cost of this universality is not fully acceptable: on the one hand we have the undecidability results, and on the other the fact that the expressive power of first-order languages does not allow any possibility for a categorical characterization of a given infinite model since it is elementarily equivalent to any of its ultrapower. So it is desirable, sometimes even necessary, to seek alternative languages for particular types of

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structures. One solution can provide the propositional modal languages. Let L_2 be, for binary relational structures (frames) $\langle W,R\rangle$, a second-order language which contains a countable set of unary predicate variables $P_1,P_2,\dots$, and let $\mathcal{L}(\Box)$ be a propositional modal language. Define a translation ST of the formulas of $\mathcal{L}(\Box)$ into L_2 (see [6]) as follows:

(1) $\mathrm{ST}(p_i) \rightleftharpoons P_i x$
(2) $\mathrm{ST}(\neg\varphi) \rightleftharpoons \neg\mathrm{ST}(\varphi)$
(3) $\mathrm{ST}(\varphi \wedge \psi) \rightleftharpoons \mathrm{ST}(\varphi) \wedge \mathrm{ST}(\psi)$
(4) $\mathrm{ST}(\Box\varphi) \rightleftharpoons \forall y(Rxy \to \mathrm{ST}(\varphi)\,[y/x])$

where x is a fixed individual variable and y is an individual variable, different from x and not occurring in $\mathrm{ST}(\varphi)$. This translation reflects exactly the relational semantics for the modal language: a modal formula φ is valid in a frame F iff the L_2-formula $\forall Q_1\dots\forall Q_m\forall x\mathrm{ST}(\varphi)$ is valid in F (considered as an L_2-model), where $Q_1,\dots,Q_m$ are the predicate variables corresponding to the propositional variables $q_1,\dots,q_m$ occurring in φ. So the validity of a modal formula in a frame is expressed by a second-order universal L_2-formula, i.e., the modal language appears as a (fragment of) a nonelementary language alternative to the usual first-order one for binary relational structures. This nontraditional role of the modal language takes shape in works of Goldblatt, Fine, Sahlqvist, Thomason (see, e.g., [11], [9], [12], [20], [23]), and especially van Benthem (see [1], [2], [5], and [6]) where it forms the so-called correspondence theory. The main problem of this theory can be formulated in two directions: which (first-order) properties of the relational structures are expressible in the modal language (*modal definability*), and which modal formulas have interpretations that can be expressed by first-order conditions (*first-order definability*). The entire ideology of this theory as well as a detailed systematization of the achievements in the field are discussed in [5] and [6].

The correspondence theory can be naturally generalized and the basic results are directly transferrable into polymodal languages with relational semantics over frames $\langle W,R_1,\dots,R_n\rangle$. However, in the concretely arising polymodal languages (e.g., languages for tense and dynamic logics) some conditions over the frames are imposed; the frames satisfying these conditions are the "standard" ones for the language. It is convenient to introduce the notion of a "*polymodal base*" $\mathcal{L}_T(R_1,\dots,R_n)$, which consists of a polymodal language $\mathcal{L}(\Box_1,\dots,\Box_n)$ and a (first-order) theory T for structures $\langle W,R_1,\dots,R_n\rangle$. The models of T are just the *standard frames*; the standard semantics for this base is the relational semantics, however only in standard frames. Now, in the class of standard frames a relativized variant of the correspondence theory arises. It should be noticed that the classical (poly)modal language is not powerful enough to really compete with the first-order language. However, the polymodal bases provide opportunities to construct enriched modal languages—when the theory T explicitly defines some of the relations $R_1,\dots,R_n$ by means of the rest. Here is a typical example: the language for tense logics is a bimodal base with standard frames $\langle W,R,R^{-1}\rangle$ and can be considered as a modal language, enriched with an additional modality $\Box_{-1}$ with a nonstandard relational semantics $x \models \Box_{-1}\varphi$ iff $\forall y(Ryx \Rightarrow y \models \varphi)$. In such a way appropriate bases can enrich the modal language in order to obtain desirable expressive capacities. The main purpose of this paper is to suggest a general approach to the problem of modal definability in polymodal bases in the spirit of the correspondence theory and to investigate in that direction some concrete bases providing enriched modal languages. The main paradigm is the bimodal base $\mathcal{L}(R,-R)$ ($-R$ is the complement of R) with standard frames $\langle W,R,-R\rangle$. This base considerably extends the expressive power of the classical modal language, e.g., every universal first-order formula for R and $=$ is definable in it. It is investigated in detail in this paper, and the modal definability of arbitrary (Section 3) and Δ-elementary (Section 4) classes of standard frames are described and some consequences are obtained. Also, various particular examples of definable and indefinable properties are adduced (Section 5). Some other bases, viz., $\mathcal{L}(R,-R,\neq)$, $\mathcal{L}(R,-R,R^{-1},-R^{-1})$, $\mathcal{L}(R_1,-R_1,R_2,-R_2,R_1 \cap R_2)$, etc., are introduced and briefly investigated in a similar manner in Sections 6 and 7.

One more note. It is clear that the main advantage of the modal languages is their two-faced nature: propositional languages with a second-order interpretation. This advantage can be realized only when the deductive reliability, concerning the standard semantics, of a given axiomatic is ensured, i.e., when completeness with respect to the standard frames is proved. In the completeness-proving procedure in polymodal bases additional difficulties appear in comparison with usual modal languages. These difficulties are connected with the "standardizing" of the frames refuting nontheorems. A general technique and various completeness results in the bimodal base $\mathcal{L}(R,-R)$ can be found in [13].

1 Preliminaries Basic notions of modal logic (within the bounds of the initial sections of [16] and [11] or [6]) will be assumed to be familiar, viz.: valuation and model over a given frame, general (first-order) frame; modal algebra; forcing ($\models$) and validity in a model/frame, general frame, modal algebra; the basic frame constructions: generated subframe, p-morphism, disjoint union, ultrafilter extension (ue), and Stone representation (SR); also the algebraic notions: subalgebra, congruence, homomorphism, direct product. We specify that the notion of a "generated subframe" will be reserved for the ones generated from one point, and the others will simply be called subframes. If F and G are frames and $F \cong \mathrm{ue}(G)$ then G will be called an *ultrafilter contraction of* F. All these notions and the basic facts connected with them are naturally generalized in polymodal languages. Now let us set forth some definitions of ultraproducts (see [11] and [6]).

Definitions

(1) The *ultraproduct of a family of sets* $\{W_i\}_{i\in I}$ *over an ultrafilter* D *in* I is the quotient-set $\Pi_D\{W_i\}_{i\in I}$ of the direct product $\Pi\{W_i\}_{i\in I}$, over the equivalence relation $\equiv_D$ defined by $f \equiv_D g$ iff $\{i : f(i) = g(i)\} \in D$.

(2) The *ultraproduct of a family of frames* $\{F_i = \langle W_i,R^i_1,\ldots,R^i_n\rangle\}_{i\in I}$ *over an ultrafilter* D *in* I is the frame $\Pi_D\{F_i\}_{i\in I} = \langle \Pi_D\{W_i\}_{i\in I},R_1,\ldots, R_n\rangle$, where $R_k = \{\langle f/{\equiv_D},g/{\equiv_D}\rangle : f/{\equiv_D},g/{\equiv_D} \in \Pi_D\{W_i\}_{i\in I}$ & $\{i | R^i_k f(i)g(i)\} \in D\}$.

(3) The *ultraproduct of a family of general frames* $\{\mathfrak{F}_i = \langle F_i,\mathbb{W}_i\rangle\}_{i\in I}$ *over*

an ultrafilter D in I is a general frame $\Pi_D\{\mathfrak{F}_i\}_{i\in I} = \langle F, \mathbb{W}\rangle$, where $F = \Pi_D\{F_i\}_{i\in I}$ and $\mathbb{W} = \Pi\{\mathbb{W}_i\}_{i\in I}/\equiv_D = \{X/\equiv_D : X \in \Pi\{\mathbb{W}_i\}_{i\in I}\}$.

(4) The *ultraproduct of a family of polymodal algebras* $\{\mathfrak{A}_i\}_{i\in I}$ *over an ultrafilter D in I* is the quotient-algebra $\Pi_D\{\mathfrak{A}_i\}_{i\in I} = \Pi\{\mathfrak{A}_i\}_{i\in I}/\equiv_D$ (the equivalence $\equiv_D$, defined above, is a congruence).

Note 1 (see Chapter 4 of [6]) Let $\{F_i\}_{i\in I}$ be a family of frames and D an ultrafilter in I. If the frames of the family are considered as full general frames $\langle W_i, 2^{W_i}\rangle$ then Definition (3) above provides an ultraproduct which is not (in general) a full frame, hence is distinct from the ultraproduct obtained by (2) above, so two different notions of ultraproduct of frames exist. The ultraproduct obtained by (3) will be called a *weak ultraproduct.* Unlike the usual ultraproduct (by (2)), it preserves the validity of modal formulas. Denote it by $\Pi_D^w\{F_i\}_{i\in I}$.

Fact 1.1 (Łos's Theorem, 4.19 in [7]) *Let α be a formula of a first-order language L and $\{\mathfrak{A}_i\}_{i\in I}$ a family of L-models. Then $\Pi_D\{\mathfrak{A}_i\}_{i\in I} \vDash \alpha$ iff $\{i \mid \mathfrak{A}_i \vDash \alpha\} \in D$.*

We now define some operators over classes of algebras. Let A be a class of algebras of some signature Ω. Then:

I(A) is the class of all isomorphic copies of algebras from A
S(A) is the class of all subalgebras of algebras from A
H(A) is the class of all homomorphic images of algebras from A
P(A) is the class of all direct products of algebras from A
U(A) is the class of all ultraproducts of algebras from A.

Note 2 When a sequence of operators is applied, the unneeded brackets will be omitted, e.g., IS(A) will be written instead of I(S(A)). Also we shall write, e.g., $S(\mathfrak{A})$ instead of S$(\{\mathfrak{A}\})$. Equality and inclusion of operators are naturally defined.

Fact 1.2 (Section 23 in [15]) (i) $X^2 = X$ *where* $X \in \{\mathrm{I,S,H,P}\}$; (ii) SH$(A) \subseteq$ HS(A); (iii) PH$(A) \subseteq$ HP(A); (iv) PS$(A) \subseteq$ SP(A).

Fact 1.3 ([11]) *All of the operators introduced above preserve the validity of modal formulas.*

Let us note that the class of $\mathfrak{L}$-algebras is defined by means of identities, hence it forms a variety. Denote the variety, generated by a class of algebras A, by VAR(A). Then:

Fact 1.4 (Birkhoff's Theorem, see [15]) VAR(A) = HSP(A).

Let us now define some operators over classes of frames. Let C be such a class. Then:

$\mathrm{I_f}(C)$ is the class of all isomorphic copies of frames from C
$\mathrm{S_f}(C)$ is the class of all subframes of frames from C
$\mathrm{H_f}(C)$ is the class of all p-morphic images of frames from C
$\mathrm{D_f}(C)$ is the class of all disjoint unions of frames from C
$\mathrm{U_f}(C)$ is the class of all weak ultraproducts of frames from C.

Note 3 The last operator sends C to a class of general frames.

Fact 1.5 ([11]) *All of the operators described above preserve the validity of modal formulas.*

Fact 1.6 (6.5 and 7.8 in [11]) *If $\{\mathfrak{F}_i\}_{i\in I}$ is a family of general frames and D is an ultrafilter in I then $(\Sigma\{\mathfrak{F}_i\}_{i\in I})^+ \cong \Pi\{\mathfrak{F}_i^+\}_{i\in I}$ and $(\Pi_D\{\mathfrak{F}_i\}_{i\in I})^+ \cong \Pi_D\{\mathfrak{F}_i^+\}_{i\in I}$.*

Some more notation: Let $\mathcal{L}(R_1,\ldots,R_n)$ be a polymodal language (base) with a set of formulas $\Phi_{\mathcal{L}}$. Then $L_0(R_1,\ldots,R_n)$ is a first-order language with $=$ and binary predicates $R_1,\ldots,R_n$ and with a set of formulas $\mathrm{For}_0(R_1,\ldots,R_n)$. The modality corresponding to R_i will be denoted by $\Box_i$ (dual $\Diamond_i$), but more frequently by $[R_i]$ (dual $\langle R_i\rangle$).

If $F = \langle W,R_1,\ldots,R_n\rangle$ is a frame, $x \in W$, and $X \subseteq W$, then $R_i(x) = \{y \in W \mid R_ixy\}$, $R_i[X] = \bigcup_{x\in X} R_i(x)$, and $[R_i]X = \{x \in W \mid R_i(x) \subseteq X\}$.

If $\mathfrak{F} = \langle W,R_1,\ldots,R_n,\mathbb{W}\rangle$ is a general frame, then $\mathfrak{F}^+ = \langle \mathbb{W},-,\cap,[R_1],\ldots,[R_n],W\rangle$ is the polymodal algebra generated by $\mathfrak{F}$ (see [11]). If C is a set of (general) frames then set $C^+ \rightleftharpoons \{\mathfrak{A}/\exists F \in C \text{ such that } \mathfrak{A} \cong F\}$.

Finally, if β is a formula (modal or first-order) then $\mathrm{FR}(\beta)$ denotes the class of frames in which β is valid; $\mathrm{FR}(\Sigma)$ is defined analogously for a set of formulas Σ. If F is a frame (general frame, model, algebra) then $\mathrm{Th}_{\mathrm{mod}}(F) = \{\varphi \in \Phi_{\mathcal{L}} \mid F \models \varphi\}$. The definition of $\mathrm{Th}_{\mathrm{mod}}(C)$ for a class of frames (general frames, etc.) is analogous.

2 Absolute and relative modal definability

The problem of modal definability Let a polymodal language $\mathcal{L} = \mathcal{L}(\Box_1,\ldots,\Box_n)$ be fixed.

Definitions

(1) A class of $\mathcal{L}$-frames C is *modally definable* (MD) *in the language* $\mathcal{L}$ if $C = \mathrm{FR}(\Gamma)$ for some set of $\mathcal{L}$-formulas Γ.

(2) A formula α (set of formulas Σ) of the first-order language L_0 is *modally definable in* $\mathcal{L}$ if $\mathrm{FR}(\alpha)$ ($\mathrm{FR}(\Sigma)$) is such.

The problem arises of finding criteria for modal definability. Concretely, if a given class of frames is MD then a defining set of formulas Γ or an algorithm finding it has to be exhibited, or at least a nonconstructive proof for the existence of a defining set has to be given; if the class is not MD, this has to be proved.

Our purpose will be to find criteria for modal definability in the spirit of Birkhoff's Theorem: a class of frames is MD iff it is closed under certain operators.

The most general result for modal definability in the classical modal language (the generalization of which in polymodal languages is trivial) is the theorem of Goldblatt and Thomason (Theorem 3 of [12]) which translates Birkhoff's Theorem into frame notions. This introduces a rather complicated (*ad hoc*, as van Benthem notes) construction named by the authors the SA-construction. It is a translation of a composition of natural algebraic constructions—a subalgebra of a homomorphic image; however, this translation cannot be split in two natural analogues, so the SA-construction proves to be not quite elegant. Informally it is a contraction of a reduction of a frame; for brevity we shall call it a *collapse.*

Definitions

(1) Let $F = \langle W, R_1, \ldots, R_n \rangle$ and $F' = \langle W', R_1', \ldots, R_n' \rangle$. *$F'$ is obtained from F by collapse (F' is a collapse of F)* if there exists a general frame $\mathfrak{F} = \langle F, \mathbb{W} \rangle$ such that
  (i) W' is a set of ultrafilters in $\mathfrak{F}^+$, and for each $u, v \in W'$, $R_i' uv$ iff $[R_i]u \subseteq v$, i.e., $\forall X \in \mathbb{W}([R_i]X \in u \Rightarrow X \in v)$ for $i = 1, \ldots, n$ and the following conditions hold:
  (ii) $(\forall U \subseteq W')(\exists X \in \mathbb{W})(\forall u \in W')(u \in U \text{ iff } X \in u)$
  (iii) $(\forall u \in W')(\forall X \in \mathbb{W})((\forall v \in W')(R_i' uv \Rightarrow X \in v) \Rightarrow [R_i]X \in u)$ for $i = 1, \ldots, n$.

(2) The *collapse of a general frame* $\mathfrak{F}_1 = \langle F, \mathbb{W}_1 \rangle$ is defined analogously: F is a collapse of $\mathfrak{F}_1$ if there exists a general frame $\mathfrak{F} = \langle F, \mathbb{W} \rangle$ such that $\mathbb{W} \subseteq \mathbb{W}_1$ and conditions (i), (ii), and (iii) hold.

Denote the class of all collapses of a class of frames C by $\mathrm{C}(C)$.

Fact 2.1 (1 in [12]) $G \in \mathrm{I_f C}(F)$ *iff* $G^+ \in \mathrm{HS}(F^+)$.

Corollary 2.2 (2 in [12]) *If $G \in \mathrm{C}(F)$ and $\varphi \in \Phi_{\mathcal{L}}$ then $F \models \varphi \Rightarrow G \models \varphi$.*

A specification: if $G \in \mathrm{C}(F)$ then $G^+ \in \mathrm{H}(\mathfrak{F}^+)$, where $\mathfrak{F}$ is the general frame from the definition of collapse (see the proof of 2 in (12]).

Theorem 2.3 (3 in [12]) *The class of $\mathcal{L}$-frames C is modally definable in $\mathcal{L}$ iff C is closed under isomorphisms, disjoint unions, and collapses.*

Let C be a class of frames. Denote the class of all ultrafilter contractions of frames from C by $\mathrm{C_u}(C)$. As a matter of fact, the ultrafilter contraction is a particular case of a collapse, since $F^+ \leq (\mathrm{ue}(F))^+$, so the notation is coordinated with the terminology.

Note that when C is closed under elementary equivalence then modal definability obtains a more natural characterization (8 in [12]): C is modally definable in $\mathcal{L}$ iff it is closed under subframes, p-morphic images, disjoint unions, and ultrafilter contractions.

Relative modal definability The modal definability discussed so far is, in a sense, *absolute*; i.e., definability in the class of all $\mathcal{L}$-frames. When the polymodal language is replaced with a polymodal base a "relative" definability in the class of standard frames arises. Some definitions in this connection follow.

Definitions Let C and D be classes of frames of the same language $\mathcal{L}$ where $C \subseteq D$.

(i) *C is modally definable* (MD) *in D* if $C = \mathrm{FR}(\Gamma) \cap D$ for some $\Gamma \subseteq \Phi_{\mathcal{L}}$.
(ii) A *modally definable closure* (MDC) *of C in D*, denoted by $[C]_D$, is the least class containing C that is modally definable in D.

Note 4 $[C]_D$ always exists—it is an intersection of all classes modally definable in D and containing C (there are such classes, e.g. D).

Actually, $[C]_D$ is explicitly definable: $[C]_D \rightleftharpoons D \cap \mathrm{FR}(\mathrm{Th}_{\mathrm{mod}}(C))$. The modally definable closure of the class C in the class of all frames of the given language $\mathcal{L}$ will be denoted by $[C]_{\mathcal{L}}$ or simply by $[C]$.

Definitions

(1) Let $\mathcal{L} = \mathcal{L}_T(R_1, \ldots, R_n)$ be a fixed polymodal base. The class $\{F \mid F \models T\}$ of the standard frames will be denoted by $\mathbb{C}_{\mathrm{s}}$. The frames from $[\mathbb{C}_{\mathrm{s}}]$ will be called *basic frames* and their generated subframes will be called *total frames*. Denote the class of the basic (total) frames by $\mathbb{C}_{\mathrm{b}}$ ($\mathbb{C}_{\mathrm{t}}$).

(2) *A general $\mathcal{L}$-frame* $\mathfrak{F} = \langle F, \mathbb{W} \rangle$ is *standard* (*basic*, *total*) if $F \in \mathbb{C}_{\mathrm{s}}$ ($\mathbb{C}_{\mathrm{t}}$, $\mathbb{C}_{\mathrm{b}}$). Denote the class of standard (basic, total) general frames by $\mathbb{C}_{\mathrm{gs}}$ ($\mathbb{C}_{\mathrm{gb}}, \mathbb{C}_{\mathrm{gt}}$).

(3) *An $\mathcal{L}$-algebra $\mathfrak{A}$ is standard* (*basic*, *total*) if there exists an $\mathfrak{F} \in \mathbb{C}_{\mathrm{gb}}$ ($\mathbb{C}_{\mathrm{gt}}, \mathbb{C}_{\mathrm{gs}}$) such that $\mathfrak{A} \cong \mathfrak{F}^+$. Denote the class of standard (basic, total) $\mathcal{L}$-algebras by $\mathbb{M}_{\mathrm{b}}$ ($\mathbb{M}_{\mathrm{t}}, \mathbb{M}_{\mathrm{s}}$).

Theorem 2.4 (specifying of 2.3) $\quad [C] = \mathrm{I_f CD_f}(C)$.

Denote the family of classes modally definable in D by $\mathrm{MD}(D)$.

Theorem 2.5 *Let $C \subseteq D \subseteq E$ be classes of frames. Then:*
(i) *$C \in \mathrm{MD}(D)$ iff $C = [C]_D$*
(ii) $[C]_D = [C]_E \cap D$
(iii) *if $D \in \mathrm{MD}(E)$ then $[C]_D = [C]_E$; in particular, $C \in \mathrm{MD}(D)$ iff $C \in \mathrm{MD}(E)$.*

Proof: (i) Follows directly from the definitions.

(ii) Let the class $[C]_E$ be definable in E by a set of formulas $\Gamma \subseteq \Phi$. Then $[C]_E \cap D$ is defined in D by Γ and $C \subseteq [C]_E \cap D \Rightarrow [C]_D \subseteq [C]_E \cap D$. Conversely, let $[C]_D$ be defined in D by $\Delta \subseteq \Phi$ and let C_Δ be the class defined in E by Δ. Then $C \subseteq [C_D] \subseteq C_\Delta \Rightarrow [C]_E \subseteq [C_\Delta]_E = C_\Delta \Rightarrow [C]_E \cap D \subseteq C_\Delta \cap D = [C]_D$.

(iii) If $D \in \mathrm{MD}(E)$ then $[C]_E \subseteq D$ and, by (ii), $[C]_D = [C]_E$.

Some comments on the above results:

(i) The description of the MDC's provides, in particular, a description of the corresponding modal definability.
(ii) The MDC's will be described as closures with respect to some operators. If we describe the MDC's in a class containing the standard one, by means of operators preserving the standard class, this will provide a description of the MDC's in the standard class.
(iii) If the standard class is modally definable (e.g., in the bimodal base for tense logics) then the relative MDC's coincide with the absolute ones.

The problem of relative modal definability is to describe the modally definable subclasses of the class of the standard frames of a given polymodal base. The strategy for attacking this problem will be, in the spirit of the above comments, to describe the MDC's in the class of standard frames. Of course difficulties will arise since, in general, this class will not be closed under the basic constructions. So the problem of modal definability seems to be rather difficult to work out in the general case; here we shall investigate some interesting concrete modal bases, which are sufficiently representative to illustrate the general problem.

3 Modal definability in the base $\mathcal{L}(R,-R)$ The main purpose of this section is to describe the MDC's and MD classes in the modal base $\mathcal{L}(R,-R)$, consisting of a bimodal language $\mathcal{L}(\Box_1,\Box_2)$ and a first-order theory T_- with a single axiom, (–) $\forall xy(R_1xy \Leftrightarrow -R_2xy)$, hence with standard frames $\langle W,R,-R\rangle$. This base can be considered as a modal language with an additional modality $\boxbar$ $(=[-R]\neg)$ with the nonstandard relational semantics $x \models \boxbar\varphi$ iff $\forall y(y \models \varphi \Rightarrow Rxy)$. This modality has appeared in different authors and in different contexts: in Goldblatt [10] as a negation in a quantum logic; in Humberstone [17] as a modality over the "inaccessibility relation"; in van Benthem [3] as an "obligation"; and in Gargov, Passy, and Tinchev [14] as a "sufficiency".

Let us introduce some notation for the base $\mathcal{L}(R,-R)$ in the sense of Sections 3 and 4. In order to emphasize the standard semantics, and for convenience, we write $\boxplus \rightleftharpoons \Box_1$ and $\boxminus \rightleftharpoons \Box_2$ (duals $\diamond\!\!\!+$ and $\diamond\!\!\!-$). The modality $\blacksquare$ (dual $\blacklozenge$), corresponding to $R = R_1 \cup R_2$ in the frames $\langle W,R_1,R_2\rangle$ is explicitly defined by $\boxplus$ and $\boxminus$: $\blacksquare\varphi = \boxplus\varphi \wedge \boxminus\varphi$. In [14] it is proved that the minimal normal $\mathcal{L}(R,-R)$-logic, denoted there by $K^{\sim}$, is axiomatized by the S5-axioms for $\blacksquare$. This fact implies that the class of basic $\mathcal{L}(R,-R)$-frames $\mathbb{C}_b = [C_s]$ consists of exactly the frames $\langle W,R_1,R_2\rangle$ in which $R_1 \cup R_2$ is an equivalence relation; so the total $\mathcal{L}(R,-R)$-frames are those frames $\langle W,R_1,R_2\rangle$ in which $R_1 \cup R_2 = W^2$, hence in total frames $\blacksquare$ is the universal modality.

Now the strategy will be to describe subsequently the modal definability in $\mathbb{C}_b$, $\mathbb{C}_t$, and $\mathbb{C}_s$. Denote, for convenience, the corresponding MDC's by $[\ ]_b$, $[\ ]_t$, $[\ ]_s$. As a corollary of 2.4 and 2.5 we have:

Corollary 3.1 *If $C \subseteq \mathbb{C}_b$ then*
(i) $[C]_b = I_f CD_f(C)$
(ii) *C is modally definable in $\mathbb{C}_b$ iff it is closed under isomorphisms, disjoint unions, and collapses.*

Modal definability in $\mathbb{C}_t$ Indeed, the fact that $R_1 \cup R_2$ is a universal relation in the class $\mathbb{C}_t$ is not modally expressible, though there does exist a simple condition which characterizes the algebras from $\mathbb{M}_t$:

Lemma 3.2 *$\mathfrak{A} \in \mathbb{M}_t$ iff in $\mathfrak{A}$ the following condition holds: (τ) $\forall t \in A(\blacksquare t = 0$ or $t = 1)$ (equivalently, $\forall t \in A(\blacksquare t = 0$ or $\blacksquare t = 1)$).*

Proof: (1) Let $\mathfrak{A} \in \mathbb{M}_t$, $\mathfrak{A} \cong \mathfrak{F}^+$, $\mathfrak{F} = \langle W,R_1,R_2,\mathbb{W}\rangle \in \mathbb{C}_{gt}$, and $X \in \mathbb{W}$. Then $\blacksquare X = \boxplus X \wedge \boxminus X = \{x|R_1(x) \subseteq X \;\&\; R_2(x) \subseteq X\} = \{x|(R_1 \cup R_2)(x) \subseteq X\} = \{x|W \subseteq X\} = \begin{cases} W, \text{ if } X = W \\ \varnothing, \text{ otherwise} \end{cases}$; so condition (τ) holds in $\mathfrak{F}^+$ hence in $\mathfrak{A}$.

(2) Let (τ) hold in $\mathfrak{A}$. We shall prove that $\mathrm{SR}(\mathfrak{A}) \in \mathbb{C}_{\mathrm{gt}}$. Let $\mathrm{SR}(\mathfrak{A}) = \langle W_{\mathfrak{A}}, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, \mathbb{W}_{\mathfrak{A}}\rangle$. Assume that $R_1^{\mathfrak{A}} \cup R_2^{\mathfrak{A}} \neq W_{\mathfrak{A}}^2$; i.e., there exist $u, v \in W_{\mathfrak{A}}$ such that $\neg R_1^{\mathfrak{A}}uv$ & $\neg R_2^{\mathfrak{A}}uv$. Then there exists t_1 such that $\boxplus t_1 \in u$ & $t_1 \notin v$ and t_2 such that $\boxminus t_2 \in u$ & $t_2 \notin v$. Let $t = t_1 \cup t_2$. $\boxplus t_1 \leq \boxplus t \Rightarrow \boxplus t \in u$. Analogously, $\boxminus t \in u \Rightarrow \blacksquare t \in u \Rightarrow \blacksquare t \neq 0 \Rightarrow t = 1$, but $t \notin v$, which is a contradiction. So $\mathrm{SR}(\mathfrak{A}) \in \mathbb{C}_{\mathrm{gt}} \Rightarrow \mathfrak{A} \cong (\mathrm{SR}(\mathfrak{A}))^+ \in \mathbb{M}_{\mathrm{t}}$.

Lemma 3.3 *Each algebra from $\mathbb{M}_{\mathrm{t}}$ is simple, i.e., has no proper congruences.*

Proof: Let $\equiv$ be a congruence in $\mathfrak{A} \in \mathbb{M}_{\mathrm{t}}$, different from $=$; i.e., there exist $a, b \in \mathfrak{A}$ such that $a \equiv b$ and $a \neq b$. Then $(a \leftrightarrow b) \equiv 1 \Rightarrow \blacksquare(a \leftrightarrow b) \equiv 1$, but on the other hand $(a \leftrightarrow b) \neq 1 \Rightarrow \blacksquare(a \leftrightarrow b) = 0$, i.e. $0 \equiv 1$.

Note 5 Equivalent to Lemma 3.3 is this fact: The total frames do not have any proper subframes.

Corollary 3.4 *If $F \in \mathbb{C}_{\mathrm{t}}$ then $G \in \mathrm{I_f C}(F)$ iff $G^+ \in \mathrm{IS}(F^+)$.*

Theorem 3.5 *Let $A \subseteq \mathbb{M}_{\mathrm{t}}$. Then $\mathbb{M}_{\mathrm{t}} \cap \mathrm{HSP}(A) = \mathrm{ISU}(A)$.*

Proof: Let us note that the operators I, S, and U preserve the class $\mathbb{M}_{\mathrm{t}}$. (Condition (τ) *is a first-order formula in the signature of the $\mathcal{L}$-algebras, therefore,* by Łos's Theorem, it is preserved under ultraproducts.) All algebras from A are simple, hence subdirectly irreducible. Therefore, by the Jonsson result (3.2 in [18]) each subdirectly irreducible algebra from $\mathrm{HSP}(A)$ belongs to $\mathrm{HSU}(A)$. Moreover, $\mathrm{SU}(A) \subseteq \mathbb{M}_{\mathrm{t}} \Rightarrow \mathrm{HSU}(A) = \mathrm{ISU}(A)$ hence $\mathbb{M}_{\mathrm{t}} \cap \mathrm{HSP}(A) \subseteq \mathrm{ISU}(A)$. The converse inclusion goes as follows: $\mathrm{ISU}(A) \subseteq \mathrm{ISHP}(A) \subseteq \mathrm{HSP}(A) \Rightarrow \mathrm{ISU}(A) \subseteq \mathbb{M}_{\mathrm{t}}$.

Note 6 The quoted result of Jonsson's was pointed out to the author from the anonymous referee. This led to the replacement of the complicated direct proof of the theorem with the simple one above.

Lemma 3.6 *$\mathbb{C}_{\mathrm{t}}$ is closed under collapses.*

Proof: Let $F \in \mathbb{C}_{\mathrm{t}}$ and $G \in \mathrm{C}(F)$. Then $G^+ \in \mathrm{IS}(F^+)$. $F^+ \in \mathbb{M}_{\mathrm{t}}$ and condition (τ) is preserved under isomorphisms and subalgebras $\Rightarrow G^+ \in \mathbb{M}_{\mathrm{t}} \Rightarrow$ (see the proof of 3.2) $\mathrm{SR}(G^+) \in \mathbb{C}_{\mathrm{gt}} \Rightarrow \mathrm{ue}(G) \in \mathbb{C}_{\mathrm{t}}$. And G is embedded as a substructure in $\mathrm{ue}(G) \Rightarrow G \in \mathbb{C}_{\mathrm{t}}$.

Lemma 3.7 *Let $F = \langle W, R_1, R_2\rangle \in \mathbb{C}_{\mathrm{t}}$. Then $F' = \langle W', R_1', R_2'\rangle \in \mathrm{I_f C}(F)$ iff F' is isomorphic to a frame $F^\sim = \langle W^\sim, R_1^\sim, R_2^\sim\rangle$ for which there exists a general frame $\mathfrak{F} = \langle F, \mathbb{W}\rangle$ such that $\mathfrak{F}^+$ is a complete atomic (as a Boolean algebra) $\mathcal{L}$-algebra with a set of atoms $W^\sim$ and the following conditions hold:*
(1_i) For every $a, b \in W^\sim$, $R_i^\sim ab$ iff there exist $x \in a$ and $y \in b$ such that $R_i xy$
(2_i) $(\forall a \in W^\sim)(\forall X \in \mathbb{W})((\forall b \in W^\sim)(R_i^\sim ab \Rightarrow b \subseteq X) \Rightarrow R_i[a] \subseteq X)$, for $i = 1,2$.

Proof: (1): Let $F' = \langle W', R_1', R_2'\rangle \in \mathrm{C}(F)$. Then W' is a set of ultrafilters in an algebra $\mathfrak{F}^+$ for some general frame $\mathfrak{F} = \langle F, \mathbb{W}\rangle$, at that $\mathfrak{F}^+ \cong F'^+$ according to 3.4 and the specification after 2.2 and the conditions for $\mathfrak{F}$ from the definition of collapse hold. Let $h: \mathfrak{F}^+ \to F'^+$ be the corresponding isomorphism. $\mathfrak{F}^+$ is a complete atomic $\mathfrak{L}$-algebra (see [22]). Let $W^\sim$ be the set of atoms of $\mathfrak{F}^+$. Then the mapping $g: W' \to W^\sim$, defined by $g(u) = h^{-1}(\{u\})$ is a bijection, since:

- $g(u) \in W^\sim$ for each $u \in W'$: if $X \in \mathbb{W}$ and $X \subseteq g(u)$ then $h(X) \subseteq \{u\} \Rightarrow h(X) = \varnothing$ or $h(X) = \{u\} \Rightarrow X = \varnothing$ or $X = g(u)$
- g is an injection since h is a bijection
- g is onto: if $a \in W'$ then $h(a) \neq \varnothing$. Let us assume that there exist $u, v \in W'$ such that $u \neq v$ and $u, v \in h(a)$. Then $g(u), g(v) \subseteq a$ and $g(u) \neq g(v) \Rightarrow g(u) = \varnothing$ or $g(v) = \varnothing$, which is impossible. So $h(a)$ is $\{u\}$ for some $u \in W'$. Now let us define $R_i^\sim$ for $i = 1,2$: for every $a, b \in W^\sim$, $R_i^\sim ab$ iff $R_i' h(a)h(b)$ for $i = 1,2$. Then $\langle W^\sim, R_1^\sim, R_2^\sim \rangle \cong F'$ and:

  (1_i) $R_i^\sim ab$ iff $\forall X \in \mathbb{W}([R_i]X \in h(a) \Rightarrow X \in h(b))$ iff $\forall X \in \mathbb{W}(a \subseteq [R_i]X \Rightarrow b \subseteq X)$ iff $\forall X \in \mathbb{W}(R_i[a] \subseteq X \Rightarrow b \subseteq X)$ iff $R_i[a] \nsubseteq \bar{b}$ iff $b \nsubseteq \overline{R_i[a]}$ for $i = 1,2$;

  (2_i) is in fact condition (iii) from the definition of collapse.

(2): Conversely, let the conditions of the lemma hold, and let $\mathfrak{F}^+ = \langle F, \mathbb{W} \rangle$ be the complete atomic $\mathfrak{L}$-algebra with set of atoms $W^\sim$. Then the mapping $h: \mathfrak{A} \to F^{\sim +}$, defined by $h(X) = \{a \in W^\sim \mid a \subseteq X\}$, is an isomorphism, since:

- h is a bijection: $\mathfrak{F}^+$ is a complete algebra $\Rightarrow h$ is onto; if $h(X) = h(Y)$ then $h(X \setminus Y) = \varnothing$ and $h(Y \setminus X) = \varnothing \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$.
- h is a homomorphism: the only nontrivial checking is that $h([R_i]X) = [R_i^\sim]h(X)$: $h([R_i]X) = \{a \in W^\sim \mid R_i[a] \subseteq X\}$, $[R_i^\sim]h(X) = \{a \mid \forall b \in W^\sim(R_i^\sim ab \Rightarrow b \subseteq X)\}$. It follows from (2_i) that $[R_i^\sim]h(X) \subseteq h([R_i]X)$; conversely, let $a \in h([R_i]X)$, i.e., $R_i[a] \subseteq X$, and let $R_i^\sim ab$, i.e., there exist $x \in a$ and $y \in b$ such that $R_i xy$. Then $y \in X \Rightarrow b \cap X \neq \varnothing \Rightarrow b \subseteq X$. So $h([R_i]X) \subseteq [R_i^\sim]h(X)$. Therefore $\mathfrak{F}^+ \cong F^{\sim +} \cong F'^+ \Rightarrow F^{\sim +} \in \mathrm{IS}(F^+) \Rightarrow$ (by 2.4.5) $F^\sim \in \mathrm{I_f C}(F)$.

Note 7 Condition (1_i) is equivalent to

$(1_i')$ For any $a, b \in W^\sim$, $R_i^\sim ab$ iff for every $x \in a$ there exists a $y \in b$ such that $R_i xy$. Actually, if there exist $x \in a$ and $y \in b$ such that $R_i xy$ then $\varnothing \neq (a \cap \langle R_i \rangle b) \in \mathbb{W} \Rightarrow a \subseteq \langle R_i \rangle b$.

The conditions for the frame $F^\sim$ in the above lemma can be considered as a definition of a collapse in $\mathbb{C}_t$. Let us call it a *t-collapse*. Under the conditions of Lemma 3.7 $F^\sim$ will be called a *t-collapse of F, corresponding to* $\mathfrak{F}$. The respective operator will be denoted by '$\mathrm{C_t}$'. A t-collapse of a general total frame is defined in a similar manner.

Theorem 3.8 *Let* $C \subseteq \mathbb{C}_t$. *Then* $[C]_t = \mathrm{I_f C_t U_f}(C)$.

Proof: According to 1.5 and 2.2, $\mathrm{I_f C_t U_f}(C) \subseteq [C]_t$. Conversely, let $F \in [C]_t$. Then $F^+ \in \mathbb{M}_t \cap \mathrm{HSP}(C^+) = \mathrm{ISU}(C^+) \Rightarrow F \in \mathrm{I_f C_t U_f}(C)$.

Corollary 3.9 *A class* $C \subseteq \mathbb{C}_t$ *is modally definable in* $\mathbb{C}_t$ *iff* C *is closed under isomorphisms and* t*-collapses of weak ultraproducts.*

Modal definability in $\mathbb{C}_t$

Lemma 3.10 *Let $F = \langle W,R,-R\rangle \in \mathbb{C}_s$ and $F' = \langle W',R_1',R_2'\rangle \in C_t(F)$. Then $F' \in \mathbb{C}_s$ iff:*

(i) *For every $a \in W'$ and $x,y \in a$, $R(x) = R(y)$*

(ii) *For every $a,b \in W'$, $R_1'ab$* iff *for every $x \in a$ and $y \in b$, Rxy (which, for reasons similar to those of Note 7, is equivalent to $b \subseteq R[a]$)*

(iii) *The general frame $\mathfrak{F} = \langle F,\mathbb{W}\rangle$ corresponding to F' satisfies the condition: for every $a \in W'$, $R[a] \in \mathbb{W}$.*

Proof: (1) Let the conditions of the lemma hold. Then $R_2'ab$ iff there exist $x \in a$ and $y \in b$ such that $-Rxy$ iff $-R_2'ab$, hence $F' \in \mathbb{C}_s$.

(2) Let $F' \in \mathbb{C}_s$ and $\mathfrak{F} = \langle F,\mathbb{W}\rangle$ be the corresponding general frame. Then for every $a,b \in W'$, $-R_1'ab$ iff there exist $x \in a$ and $y \in b$ such that $-Rxy \Rightarrow R_1'ab$ iff for every $x \in a$ and $y \in b$ Rxy, hence (ii). Moreover, it follows from the definition of t-collapse that:

$$(*)\quad (\forall a \in W')(\forall X \in \mathbb{W})((\forall b \in W')(R_1'ab \Rightarrow b \subseteq X) \Rightarrow R[a] \subseteq X)\ \&\ (\forall b \in W')(-R_1'ab \Rightarrow b \subseteq X) \Rightarrow (-R)[a] \subseteq X).$$

Then $R_1'ab$ iff $-(-R_1')ab$ iff $b \subseteq -(-R)[a] = \widetilde{\bigcap_{x\in a}} R(x)$. Analogously $(-R_1')ab$ iff $b \subseteq \widetilde{\bigcap_{x\in a}} (-R)(x)$. Let $\widetilde{\bigcup}$ be the union in $\mathfrak{F}^+$ and for every $a \in W'$: $X_a = \widetilde{\bigcup} \{b \mid R_1'ab\}$, $Y_a = \widetilde{\bigcup} \{b \mid -R_1'ab\}$. $Y_a = W\backslash X_a$ since $\mathfrak{F}^+$ is a complete and atomic $\mathfrak{L}$-algebra.

Now, $\forall b \in W'(R_1'ab \Rightarrow b \subseteq X_a) \Rightarrow$ (by $(*)$) $R[a] \subseteq X_a$ and $\forall b \in W'(-R_1'ab \Rightarrow b \subseteq W\backslash X_a) \Rightarrow$ (by $(*)$) $(-R)[a] \subseteq W\backslash X_a$, i.e., $X_a \subseteq -(-R)[a] \Rightarrow R[a] \subseteq -(-R)[a]$, i.e., $\bigcup_{x\in a} R(x) \subseteq \bigcap_{x\in a} R(x) \Rightarrow \bigcap_{x\in a} R(x) = X_a = \bigcup_{x\in a} R(x) = R[a]$, whence (i) and (iii) follow.

The conditions of the above lemma define a collapse in the class $\mathbb{C}_s$; let us call it an *s-collapse* and denote the corresponding operator by 'C_s'. An s-collapse of a general frame is defined in a similar manner.

Theorem 3.11 *If $C \subseteq \mathbb{C}_s$ then* $[C]_s = I_f C_s U_f [C]$.

Proof: The ultraproducts preserve the standardness (a first-order condition), hence the assertion follows from 2.5 and 3.8.

Corollary 3.12 *$C \subseteq \mathbb{C}_s$ is modally definable in $\mathbb{C}_s$ iff C is closed under isomorphisms and* s-*collapses of weak ultraproducts.*

We should note that the obtained characterizations are rather nonconstructive (which is objectively conditioned) in order for them to serve as a practical criterion. That is why more natural characterizations of the modal definability of Δ-elementary classes of total and standard frames will be sought in the next section.

4 Modal definability of Δ-elementary classes and formulas in $\mathfrak{L}(R,-R)$

Modal definability of Δ-elementary classes in $\mathbb{C}_t$ Recall that a class of frames C is Δ-elementary if there exists a set $\Sigma \subseteq \mathrm{For}_0$ such that $C = \mathrm{FR}(\Sigma)$.

Theorem 4.1 *Let $C \subseteq \mathbb{C}_t$ be a Δ-elementary class. Then* $[C]_t = \mathrm{I_f C_t}(C)$.

Proof: $\mathrm{I_f C_t}(C) \subseteq [C]_t$. Conversely, let $F \in [C]_t = \mathrm{I_f C_t U_f}(C)$. Then there exist $\{F_i\}_{i \in I} \subseteq C$, an ultrafilter D in I, and $F_D = \Pi_D\{F_i\}_{i \in I}$ such that $F \in \mathrm{I_f C_t}(\Pi^w_D\{F_i\}_{i\in I}) \Rightarrow F^+ \in \mathrm{IS}(\Pi_D\{F_i^+\}_{i \in I}) \subseteq \mathrm{ISS}(F_D^+) = \mathrm{IS}(F_D^+) \Rightarrow F \in I_f C_t(F_D) \subseteq \mathrm{I_f C_t}(C)$ since the Δ-elementary class C is closed under ultraproducts.

Corollary 4.2
(i) *A Δ-elementary class $C \subseteq \mathbb{C}_t$ is modally definable in $\mathbb{C}_t$ iff C is closed under* t-collapses
(ii) *A set of elementary formulas $\Sigma \subseteq \mathrm{For}_0(R_1,R_2)$ is modally definable in $\mathbb{C}_t$ iff (the truth of Σ) is preserved under* t-*collapses.*

Another, more convenient characterization of the modally definable Δ-elementary classes in $\mathbb{C}_t$ can be obtained applying the results from Section 2 of [12].

Definition $\mathfrak{F} = \langle W, R_1, R_2, \mathbb{W}\rangle$ is a *replete* general frame if for each ultrafilter u in $\mathfrak{F}^+$:
(i) $\cap u \neq \varnothing$
(ii) $\cap\{\langle R_i\rangle X \mid X \in u\} \subseteq \langle R_i\rangle(\cap u)$, for $i = 1,2$.
If $\mathfrak{F}$ satisfies (ii) and
(i′) For every $w \in W$ such that $\cap u = \{w\}$,
then $\mathfrak{F}$ is *descriptive.*

Fact 4.3 (4 in [12]) *If $\mathfrak{F} = \langle F, \mathbb{W}\rangle$ then there exists a replete $\mathfrak{F}' = \langle F', \mathbb{W}'\rangle$ such that $\mathfrak{F}^+ \cong \mathfrak{F}'^+$ and $F \approx F'$ ($\approx$ denotes an elementary equivalence).*

Note 8 *If* $\mathfrak{F} \in \mathbb{C}_{gb}$ $(\mathbb{C}_{gt}, \mathbb{C}_{gs})$ then $\mathfrak{F}' \in \mathbb{C}_{gb}$ $(\mathbb{C}_{gt}, \mathbb{C}_{gs})$, since the classes $\mathbb{C}_{gb}$, $\mathbb{C}_{gt}, \mathbb{C}_{gs}$ are elementary.

Fact 4.4 (5 in [12]) *If $\mathfrak{F}' = \langle F', \mathbb{W}'\rangle$ is replete then there exists a descriptive $\mathfrak{F}'' = \langle F'', \mathbb{W}''\rangle$ such that $\mathfrak{F}'^+ \cong \mathfrak{F}''^+$ and F'' is a* p-*morphic image of F'.*

Note 9 The classes $\mathbb{C}_b$ and $\mathbb{C}_t$ are preserved under p-morphisms $\Rightarrow$ if $\mathfrak{F}' \in \mathbb{C}_{gb}$ $(\mathbb{C}_{gt})$ then $\mathfrak{F}'' \in \mathbb{C}_{gb}$ $(\mathbb{C}_{gt})$. However, the same does not hold for $\mathbb{C}_s$ (see 4.16).

Corollary 4.5 (6 in [12]) *If $\mathfrak{A} \leq F^+$ then there exists a descriptive $\mathfrak{F}'' = \langle F'', \mathbb{W}''\rangle$ such that $\mathfrak{F}''^+ \cong \mathfrak{A}$ and F'' is a* p-*morphic image of some $F' \approx F$.*

Fact 4.6 (7 in [12]) *If $\mathfrak{F} = \langle F, \mathbb{W}\rangle$ is descriptive and $F'^+ \in \mathrm{H}(\mathfrak{F}^+)$ then* $\mathrm{ue}(F') \in \mathrm{I_f C}(F)$.

Corollary 4.7 *If $\mathfrak{F} = \langle F, \mathbb{W}\rangle \in \mathbb{C}_{gt}$ is descriptive and $F'^+ \cong \mathfrak{F}^+$ then* $\mathrm{ue}(F') \cong F$.

Theorem 4.8 *Let $C \subseteq \mathbb{C}_t$ be a Δ-elementary class. Then* $[C]_t = \mathrm{C_u H_f}(C)$.

Proof: (1) $\mathrm{C_u H_f}(C) \subseteq [C]_t$, since $\mathrm{C_u}$ and $\mathrm{H_f}$ preserve the class $\mathbb{C}_t$ (it is defined by a universal formula that preserves the validity in substructures; the preserving under $\mathrm{H_f}$ follows from the definition of p-morphism) and the validity of modal formulas.

(2) Conversely, let $\mathrm{Th_{mod}}(C) = \Gamma$. We shall prove that Γ modally defines

$\mathrm{C_uH_f}(C)$ in $\mathbb{C}_t$. If $F \in \mathrm{C_uH_f}(C)$ then $F \models \Gamma$. Let $F \models \Gamma$. Then $F^+ \models \Gamma \Rightarrow F^+ \in \mathbb{M}_t \cap \mathrm{HSP}(C^+) = \mathrm{ISU}(C^+)$, i.e., there exist $\{F_i\}_{i \in I} \subseteq C$, an ultrafilter D in I, and $F_D = \Pi_D\{F_i\}_{i \in I}$ such that $F^+ \in \mathrm{IS}(F_D^+)$ (as in the proof of 4.1). C is Δ-elementary $\Rightarrow F_D \in C$. According to 4.5 there exists a descriptive $\mathfrak{F}'' = \langle F'', \mathbb{W}'' \rangle$ such that $\mathfrak{F}''^+ \cong F^+$ and F'' is a p-morphic image of some $F' \approx F_D$. Then $F' \in C \Rightarrow F'' \in \mathrm{H_f}(C)$ and, by 4.7, $F \in \mathrm{C_u}(F'') \Rightarrow F \in \mathrm{C_uH_f}(C)$.

Corollary 4.9
(1) *A Δ-elementary class $C \subseteq \mathbb{C}_t$ is modally definable in $\mathbb{C}_t$ iff C is closed under* p-*morphisms and ultrafilter contractions.*
(2) *A set of formulas $\Sigma \subseteq \mathrm{For}_0(R_1,R_2)$ is modally definable in $\mathbb{C}_t$ iff Σ is preserved under* p-*morphisms and ultrafilter contractions.*

Corollary 4.10 *A closed formula $\alpha \in \mathrm{For}_0(R_1,R_2)$ is modally definable in $\mathbb{C}_t$ iff α is preserved under* p-*morphisms and its negation is closed under ultrafilter extensions.*

Corollary 4.11 *A set of universal formulas $\Sigma \subseteq \mathrm{For}_0(R_1,R_2)$ is modally definable in $\mathbb{C}_t$ iff Σ is preserved under* p-*morphisms.*

Proof: The ultrafilter contraction is a substructure and all universal formulas are preserved in substructures.

Modal definability of Δ-elementary classes in $\mathbb{C}_s$

Theorem 4.12 *Let $C \subseteq \mathbb{C}_s$ be a Δ-elementary class. Then $[C]_s = \mathrm{I_fC_s}(C)$.*

Proof: The assertion follows from 3.11 and 4.1 since ultraproducts preserve "standardness" (a first-order condition).

Corollary 4.13
(1) *A Δ-elementary class of Kripke-frames $\langle W,R \rangle$ is modally definable in $\mathcal{L}(R,-R)$ iff it is closed under* s-*collapses.*
(2) *A set of formulas $\Sigma \subseteq \mathrm{For}_0(R)$ is modally definable in $\mathcal{L}(R,-R)$ iff Σ is preserved under* s-*collapses.*

The above characterization is more easily applicable for nonconstructive proofs of modal definability, but the characterizations following from 2.5 and 4.8 seem to be more convenient for negative results.

Corollary 4.14 *Let $C \subseteq \mathbb{C}_s$ be a Δ-elementary class. Then $[C]_s = \mathbb{C}_s \cap \mathrm{C_uH_f}(C)$.*

The description of the MDC's in a given modal base suggests the way to find a construction that proves for a given formula or a class of formulas that it cannot be modally defined, which will be illustrated by the following example.

Example 4.15 The formula $\exists x\, Rxx$ is not modally definable in $\mathcal{L}(R,-R)$.

Proof: Let $C = \mathbb{C}_s \cap \mathrm{FR}(\exists x\, Rxx)$. We shall prove that $C \neq [C]_s = \mathbb{C}_s \cap \mathrm{C_uH_f}(C)$. Let $F = \langle \mathbb{N},<,\geq \rangle$. $F \notin C$. Let $\mathrm{ue}(F) = \langle \mathbb{N}^*,<^*,\geq^* \rangle$ and $\mathbb{N}^* = \mathbb{N}_p \cup \mathbb{N}_f$, where $\mathbb{N}_p$ is the set of principal ultrafilters in $\mathbb{N}$ and $\mathbb{N}_f$ is the set of free ones. It is a matter of direct checking that if $X \subseteq \mathbb{N}$ then $[<]X =$ the biggest ray $[n,\infty)$ contained in X, and that $[\geq]X =$ the biggest segment $[0,n]$ contained in X. Then the structure $\mathrm{ue}(F)$ has the following properties:

(1) $\langle \mathbb{N}_p, <^*|_{\mathbb{N}_p}, \geq^*|_{\mathbb{N}_p}\rangle \cong \langle \mathbb{N},<,\geq\rangle$
(2) $<^*|_{\mathbb{N}_f} = \geq^*|_{\mathbb{N}_f} = \mathbb{N}_f^2$
(3) for each $u \in \mathbb{N}_p$, $v \in \mathbb{N}_f$: $u <^* v$, $\neg u \geq^* v$, $v \geq^* u$, and $\neg v <^* u$.

Now let us set $W \rightleftharpoons \mathbb{N}_p \times \{0\} \cup \mathbb{N}_f \times \{0,1\}$. Define the following relations in W:

$\langle u,i\rangle <' \langle v,j\rangle$ iff $u < v$ & $(u,v \in \mathbb{N}_f \Rightarrow i = j)$
$\langle u,i\rangle \geq' \langle v,j\rangle$ iff $u \geq v$ & $(u,v \in \mathbb{N}_f \Rightarrow i \neq j)$

Let $G = \langle W,<',\geq'\rangle$. $G \in \mathbb{C}_s$; if $u \in \mathbb{N}_f$ then $\langle u,0\rangle <' \langle u,0\rangle \Rightarrow G \in C$. It remains to observe that the mapping g: $G \to \mathrm{ue}(F)$, defined by $g(\langle u,i\rangle) = u$, is a p-morphism:

(1) $\langle u,i\rangle <' \langle v,j\rangle \Rightarrow u < v$
(2) $g(\langle u,i\rangle) <' v \Rightarrow \langle u,i\rangle <' \langle v,j\rangle$
(3), (4) analogously for $\geq'$.

Therefore $C \neq [C]_s \Rightarrow$ formula $\exists x\, Rxx$ is not modally definable.

The following lemma characterizes the p-morphisms preserving $\mathbb{C}_s$.

Lemma 4.16 *Let* $F = \langle W,R,-R\rangle \in \mathbb{C}_s$, $F' = \langle W',R_1',R_2'\rangle$, *and* g: $F \to F'$ *be a* p-*morphism. Then* $F' \in \mathbb{C}_s$ *iff the condition*

(@) *"for every* $x,y \in W$, Rxy *iff* $R_1'g(x)g(y)$*"*

holds.

Proof: If $F' \in \mathbb{C}_s$ then $R_2' = -R_1'$. So $Rxy \Rightarrow R_1'g(x)g(y)$ and $-Rxy \Rightarrow R_2'g(x)g(y)$, therefore $-R_2'g(x)g(y) \Rightarrow Rxy$, hence (@). Conversely condition (@) implies $-R_1'g(x)g(y) \Rightarrow -Rxy \Rightarrow R_2'g(x)g(y)$. If $R_2'g(x)g(y)$ then there exists $x' \in W$ such that $-Rxx'$ and $g(x') = g(y)$, i.e., $-R_1'g(x)g(y)$. So $R_2' = -R_1'$ and $F' \in \mathbb{C}_s$.

Definition A p-morphism satisfying (@) will be called a *bi-morphism.* (Thus, the morphisms in $\mathbb{C}_s$ are just the bi-morphisms.)

Let us note that the results obtained in this section are trivially generalized for polymodal bases $\mathcal{L}(R_1,-R_1,\ldots,R_n,-R_n)$.

5 Some demonstrations of the modal definability in $\mathcal{L}(R,-R)$

Definability of universal formulas Adding $[-R]$ to the classical modal language greatly strengthens its expressive possibilities. The following theorem is weighty, but it is hardly the only argument in support of this assertion.

Theorem 5.1 *Each universal formula* α *from* $\mathrm{For}_0(R)$ *is modally definable in* $\mathcal{L}(R,-R)$. (Cf. 14.5 of [6].)

Proof: It is not difficult to see that each s-collapse is a bi-morphic image of a substructure and that the validity of universal formulas is preserved in both substructures and bi-morphic images. However, we shall adduce an explicit algorithm, providing the modal equivalent to each universal formula. This equivalent will not be the shortest possible, but the improving of the algorithm or the reducing of the obtained formulas is connected with technicalities which will not be discussed here. We may assume that α is closed (otherwise the universal closure of α will be taken). Now we shall construct a modal formula φ such that $\mathrm{FR}(\alpha) = \mathrm{FR}(\varphi)$. So let $\alpha \equiv \forall y_1 \dots \forall y_n \beta$ where β is an open formula in conjunctive normal form, $\beta = \beta_1 \wedge \dots \wedge \beta_n$. Then $\alpha \equiv \bigwedge_{i=1}^{k} \forall y_1 \dots \forall y_n \beta_i$. For each member $\forall y_1 \dots \forall y_n \beta_i$ of this conjunction a modal equivalent φ_i will be constructed such that $\varphi \rightleftharpoons \varphi_1 \wedge \dots \wedge \varphi_k$. We may also assume that in each β_i there are no disjuncts of form $y_j = y_j$, otherwise $\beta_i \equiv \top$ and $\varphi_i \rightleftharpoons \top$. So, let us fix a member $\forall y_1 \dots \forall y_n \beta_i$. We then proceed as follows.

(1) If disjuncts of the form $y_j \neq y_k$ occur in β_i, they are subsequently removed in the following way: Let $\beta_i \equiv y_j \neq y_k \vee \gamma_i$ and, for definiteness, let $j \leq k$. If $j = k$ then $\beta_i \equiv \gamma_i$; if $j < k$ then $\forall y_1 \dots y_n \beta_i \equiv \forall y_1 \dots \forall y_{k-1} \forall y_{k+1} \dots \forall y_n \gamma_i [y_j / y_k] \rightleftharpoons \delta_i$. In the former case replace β_i by γ_i and in the latter replace $\forall y_1 \dots \forall y_n \beta_i$ by δ_i.

(2) Now let us suppose that all operations from (1) are performed. Let $\forall y_1 \dots \forall y_n \beta_i \equiv \neg \exists y_1 \dots \exists y_n \gamma_i$, where γ_i is a conjunction of formulas of type $y_j \neq y_k$, $R y_j y_k$, and $\neg R y_j y_k$. Put for each $j = 1, \dots, n$:

$r_j^+ \rightleftharpoons \{s \mid R y_s y_j \text{ occurs in } \gamma_i\}$; $r_j^- \rightleftharpoons \{s \mid \neg R y_s y_j \text{ occurs in } \gamma_i\}$;
$e_j \rightleftharpoons \{s \mid y_s \neq y_j \text{ occurs in } \gamma_i\}$. Now put for each $j = 1, \dots, n$:
$\varphi_i^j \rightleftharpoons p_{3j} \wedge \boxplus p_{3j+1} \wedge \boxminus p_{3j+2} \wedge \bigwedge_{s \in r_j^+} \neg p_{3s+2} \wedge \bigwedge_{s \in r_j^-} \neg p_{3s+1} \wedge \bigwedge_{s \in e_j} \neg p_{3s}$;
$\varphi_j \rightleftharpoons \neg \bigwedge_{j=1}^{n} \blacklozenge \varphi_i^j$; $\varphi \rightleftharpoons \varphi_1 \wedge \dots \wedge \varphi_k$. We shall prove that $\mathrm{FR}(\alpha) = \mathrm{FR}(\varphi)$.

(1) Let $F \nvDash \alpha \Rightarrow$ for some i: $F \vDash \exists y_1 \dots \exists y_n \gamma_i \Rightarrow$ there exist points $w_1 \dots w_n$ such that $F \vDash \gamma_i [w_1, \dots, w_n]$. Define a valuation V as follows: $V(p_{3j}) = \{w_j\}$, $V(p_{3j+1}) = R(w_j)$, $V(p_{3j+2}) = -R(w_j)$. Let $\mathfrak{M} = \langle F, V \rangle$. Then $\mathfrak{M} \vDash \varphi_i^j [w_j] \rightarrow \mathfrak{M} \vDash \bigwedge_{j=1}^{n} \blacklozenge \varphi_i^j$, i.e., $\mathfrak{M} \vDash \neg \varphi_i \Rightarrow \mathfrak{M} \nvDash \varphi \Rightarrow F \nvDash \varphi$.

(2) For some model $\mathfrak{M}$ over a frame F and point w suppose that $\mathfrak{M} \nvDash \varphi[w] \Rightarrow$ for some i, $\mathfrak{M} \vDash \bigwedge_{j=1}^{n} \blacklozenge \varphi_i^j \Rightarrow$ there exist $w_1, \dots, w_n$ such that $\mathfrak{M} \vDash \varphi_i^j [w_j]$, $j = 1, \dots, n$. Then, if $R y_s y_j$ occurs in γ_i then $w_s \vDash \boxminus p_{3s+2}$ and $w_j \vDash \neg p_{3s+2} \Rightarrow R w_s w_j$. Analogously, if $\neg R y_s y_j$ occurs in γ_i then $\neg R w_s w_j$, and if $y_s \neq y_j$ occurs in γ_i then $w_s \vDash p_{3s}$, $w_j \vDash \neg p_{3s} \Rightarrow w_s \neq w_j$. So $F \vDash \gamma_i [w_1 \dots w_n] \Rightarrow F \nvDash \alpha$.

Note 10 The adduced algorithm was noted to be quite prodigious. An apparent step towards reducing the obtained formula is the following: If the propositional variable p_m has a single occurrence in the formulas $\varphi_i^1, \dots, \varphi_i^k$ for some i then it might be removed (together with the modality eventually prefixing it) from φ_i, since p_m does not bear any information there.

Example $\alpha = \forall y_1 \forall y_2 \forall y_3 \forall y_4 ((R y_1 y_2 \rightarrow R y_2 y_3) \vee y_2 = y_4 \vee (y_2 \neq y_3 \wedge R y_2 y_3))$. Transform:

$$
\begin{aligned}
\alpha \equiv\ & \forall y_1\forall y_2\forall y_3\forall y_4(\neg Ry_1y_2 \vee Ry_2y_3 \vee y_2 = y_4 \vee (y_2 \neq y_3 \wedge Ry_2y_3)) \equiv \\
& \forall y_1\forall y_2\forall y_3\forall y_4(\neg Ry_1y_2 \vee Ry_2y_3 \vee y_2 = y_4 \vee y_2 \neq y_3) \wedge \\
& \forall y_1\forall y_2\forall y_3\forall y_4(\neg Ry_1y_2 \vee Ry_2y_3 \vee y_2 = y_4 \vee Ry_2y_3) \equiv \\
& \forall y_1\forall y_2\forall y_4(\neg Ry_1y_2 \vee Ry_2y_2 \vee y_2 = y_4) \wedge \qquad (\beta_1) \\
& \forall y_1\forall y_2\forall y_3\forall y_4(\neg Ry_1y_2 \vee Ry_2y_3 \vee y_2 = y_4) \equiv \qquad (\beta_2) \\
& \neg\exists y_1\exists y_2\exists y_4(Ry_1y_2 \wedge \neg Ry_2y_2 \wedge y_2 \neq y_4) \wedge \\
& \neg\exists y_1\exists y_2\exists y_3\exists y_4(Ry_1y_2 \wedge \neg Ry_2y_3 \wedge y_2 \neq y_4).
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1^1 &= p_3 \wedge \boxplus p_4 \wedge \boxminus p_5; \\
\varphi_1^2 &= p_6 \wedge \boxplus p_7 \wedge \boxminus p_8 \wedge \neg p_5 \wedge \neg p_7; \\
\varphi_1^3 &= p_9 \wedge \boxplus p_{10} \wedge \boxminus p_{11}; \\
\varphi_1^4 &= p_{12} \wedge \boxplus p_{13} \wedge \boxminus p_{14} \wedge \neg p_6.
\end{aligned}
$$

Then, reduce according to Note 1:

$$
\begin{aligned}
& \varphi_1^1 = \boxminus p_5;\ \varphi_1^2 = p_6 \wedge \boxplus p_7 \wedge \neg p_5 \wedge \neg p_7;\ \varphi_1^3 = \top;\ \varphi_1^4 = \neg p_6. \\
& \varphi_1 = \neg(\blacklozenge \boxminus p_5 \wedge \blacklozenge(p_6 \wedge \boxplus p_7 \wedge \neg p_5 \wedge \neg p_7) \wedge \blacklozenge\top \wedge \blacklozenge\neg p_6) \equiv \\
& \qquad \blacksquare \diamondminus \neg p_5 \vee \blacksquare(\neg p_6 \vee \diamondplus \neg p_7 \vee p_5 \vee p_7) \vee \blacksquare p_6);
\end{aligned}
$$

analogously, $\varphi_2 = \blacksquare \diamondminus \neg p_5 \vee \blacksquare \diamondplus \neg p_7 \vee \blacksquare(p_5 \vee p_7) \vee \blacksquare p_6$ and $\varphi = \varphi_1 \wedge \varphi_2$.

Definability in the class of finite frames If we restrict ourselves to the class of finite frames, C_{fin}, we may ascertain that the language $\mathcal{L}(R,-R)$ is able to register each difference in the structures and by means of a modal formula to distinguish each finite frame F_1 from every other one not isomorphic to F_1, i.e., if $F_1, F_2 \in C_{\text{fin}}$ and $F_1 \not\cong F_2$ then $\text{Th}_{\text{mod}}(F_1) \neq \text{Th}_{\text{mod}}(F_2)$. (Let us remember that the classical modal language is not able to distinguish a given frame from an arbitrary one in its disjoint power.) Let us now examine the MDC of a finite frame F in $\mathcal{L}(R,-R)$. Two observations will help us:

(1) Each s-collapse of the frame F is simply a bi-morphism mapping the point x in the atom containing x, since each element of the finite universum W belongs to some atom.

(2) Each ultrapower of F is elementarily equivalent to F (see 4.1.10 in [7]) and therefore isomorphic to F (elementary equivalence coincides with isomorphism on the finite structures, see 1.3.19 in [7]).

Thus, $[F]_s$ consists of all bi-morphic images of F. Now let $F_1, F_2 \in C_{\text{fin}}$ and $F_1 \not\cong F_2$. If $F_1 \in [F_2]_s$ then $|W_1| < |W_2| \Rightarrow F_2 \notin [F_1]_s \Rightarrow \text{Th}_{\text{mod}}(F_1) \neq \text{Th}_{\text{mod}}(F_2)$.

Some concrete examples of modally definable properties of the relation R in $\mathcal{L}(R,-R)$ (but not in $\mathcal{L}(R)$)

(i) ([14], [17]) If the property $\mathcal{P}(R)$ is definable in $\mathcal{L}(R,-R)$ then $\mathcal{Q}(R) \rightleftharpoons \mathcal{P}(-R)$ is also definable in $\mathcal{L}(R,-R)$ and therefore in $\mathcal{L}(R,-R)$; in the formulas defining $\mathcal{P}(R)$ $\boxplus$ is replaced by $\boxminus$ and conversely. For example, the formula $\boxplus p \to p$ defines the property "R is reflexive" hence $\boxminus p \to p$ defines "$\neg R$ is reflexive", i.e., "R is irreflexive"; the formula $\boxplus\bot$ defines "R is the empty relation" $\Rightarrow$ $\boxminus\bot$ defines "R is the universal relation", etc.

(ii) *strict asymmetry* $\mathrm{ASYM_s}$: $\forall x,y(Rxy \to \neg Ryx)$ is defined by the formula $\varphi_1 = p \to \boxplus \diamondminus p$. Let $\langle F, V\rangle \nvDash \varphi_1[x]$, i.e., $x \vDash p$ and $x \vDash \diamondplus \boxminus \neg p \Rightarrow$ for some Rxy & $y \vDash \boxminus \neg p \Rightarrow Ryx \Rightarrow \neg \mathrm{ASYM_s}$. Conversely, let $F \nvDash \mathrm{ASYM_s}$, i.e., $\exists x, y \in W(Rxy$ & $Ryx)$. Then a valuation V, such that $V(p) = \{x\}$, refutes φ_1.

The condition $\mathrm{ASYM_s}$ for the relation $-R$, $\forall x,y(-Rxy \to Ryx)$ is equivalent to $\mathrm{CONN_s}$: $\forall x,y(Rxy \vee Ryx)$ which expresses the "strong connectedness" of R, and according to (i) it is defined by $\varphi_2 = p \to \boxminus \diamondplus p$. The following examples are verified analogously.

(iii) *antisymmetry* ASYM: $\forall x,y(Rxy \wedge Ryx \to x = y)$ is defined by $\varphi_3 = \diamondplus(\boxminus p \wedge p) \to p$;
(iv) *trichotomy* TRIH (dual to ASYM): $\forall x,y(Rxy \vee Ryx \vee x = y)$, is defined by $\varphi_4 = \diamondminus(\boxplus p \wedge p) \to p$;
(v) *complete antisymmetry* $\mathrm{ASYM_f}$: $\forall x,y(x \neq y \to (Rxy \leftrightarrow \neg Ryx))$ is defined by $\varphi_5 = (\diamondplus(\boxminus p \wedge p) \vee \diamondminus(\boxplus p \wedge p)) \to p$. ($\mathrm{ASYM_f}$ is the conjunction of ASYM and TRIH.)

Using these properties and the classically definable "reflexivity" REF, "symmetricity" SYM, and "transitivity" TRAN we can define some orderings, e.g.:

- *partial ordering* PO = REF + ASYM + TRANS
- *strict partial ordering* SPO = $\mathrm{ASYM_s}$ + TRANS
- *linear ordering* LO = PO + TRIH = PO + $\mathrm{ASYM_f}$
- *strict linear ordering* SLO = SPO + TRIH.

Various "tense principles" for the "point model" of time (the notion of time as a sequence of moments, $\langle \mathrm{T}, < \rangle$, where $<$ is a partial ordering (see [4])), are definable in $\mathcal{L}(R,-R)$. Here are some examples:

(vi) *left linearity* (determinism in the past, nondeterminism in the future) L-LIN: $\forall x,y,z((y < x \wedge z < x) \to (y < z \vee z < y \vee y = z))$ is defined by $\varphi_6 = (p \to \diamondminus q) \vee \boxminus(\diamondplus q \to (p \vee \diamondplus p))$.

Let us note that IRREF + TRANS + L-LIN defines a tree-ordering.

(vii) *right linearity* (determinism in the future) R-LIN: $\forall x,y,z((x < y \wedge x < z) \to (y < z \vee z < y \vee y = z))$ is defined by $\varphi_7 = \diamondminus p \vee \boxplus(q \to \boxminus(p \to (q \vee \diamondplus q)))$;
(viii) *existence of a $<$-maximal point* (end of the time) END: $\exists x \forall y \, \neg x < y$ is defined by $\varphi_8 = \blacklozenge \boxplus \bot$;
(ix) *left directedness* L-DIR: $\forall x \forall y \exists z(z < x \wedge z < y)$ is defined by $\varphi_9 = \blacklozenge p \wedge \blacklozenge q \to \blacklozenge(\diamondplus p \wedge \diamondplus q)$;
(x) *right directedness* R-DIR: $\forall x \forall y \exists z(x < z \wedge y < z)$ is defined by $\varphi_{10} = \boxplus p \to \blacksquare \diamondplus p$.

6 Modal definability in the base $\mathcal{L}(R,-R,R^{-1},-R^{-1})$

The general theory In this and the next section some enrichments of the bimodal base $\mathcal{L}(R,-R)$ will be briefly investigated, and analyzed with methods applied in the previous section using the (corresponding analogues of) results obtained there.

Let us begin with a modal base combining the advantages of the language for tense logics and the base $\mathcal{L}(R,-R)$, viz. $\mathcal{L}(R,-R,R^{-1},-R^{-1})$. It is a four-modal language $\mathcal{L} = \mathcal{L}(R_1,R_2,R_3,R_4)$ with a theory T having as its only axiom $(-,-1)$: $\forall x \forall y((R_1xy \Leftrightarrow R_3yx)$ & $(R_2xy \Leftrightarrow R_4yx)$ & $(R_1xy \Leftrightarrow -R_2xy))$. Now the basic frames are $\langle W,R_1,\dots,R_4\rangle$ such that

(i) $R_3 = R_1^{-1}$, $R_4 = R_2^{-1}$ and
(ii) $R_1 \cup R_2 = R_3 \cup R_4 =$ an equivalence relation.

$\mathbf{C}_b$ is modally defined by the axioms $p \to ([R_1]\langle R_3\rangle p \wedge [R_3]\langle R_1\rangle p)$ and $p \to ([R_2]\langle R_4\rangle p \wedge [R_4]\langle R_2\rangle p)$, defining (i) and the S5-axioms for $\blacksquare$, where $\blacksquare p = [R_1]p \wedge [R_2]p$. The proof is a slight modification of that for $\mathcal{L}(R,-R)$ in [14]. The total frames are those basic frames $\langle W,R_1,\dots,R_4\rangle$ in which $R_1 \cup R_2 = W^2$ and the standard ones are $\langle W,R,-R,R^{-1},-R^{-1}\rangle$. Furthermore, the theory of modal definability in $\mathcal{L}(R,-R,R^{-1},-R^{-1})$ is worked out in the same manner as for $\mathcal{L}(R,-R)$. In the end, the descriptions of the MDC's and the characterizations of the modal definability of arbitrary and Δ-elementary classes of total and standard frames will be obtained, and the assertions will literally repeat those from the previous sections, but the notions occurring in their formulations will already have definitions corresponding to the new base. Concretely:

- the definition of a general frame will require closure under the operations $[R_1],\dots,[R_4]$
- in the definition of s-collapse conditions (i) and (iii) will read:

  (i′) for each $a \in W'$ and $x,y \in a$, $R(x) = R(y)$ and $R^{-1}(x) = R^{-1}(y)$
  (ii′) for each $a \in W'$, $R[a] \in \mathbb{W}$ and $R^{-1}[a] \in \mathbb{W}$

- to the definition of p-morphism will be added clauses corresponding to R_3 and R_4.

Concrete examples

(1) *The natural order* The categorical description of the natural order $\langle \mathbb{N},<\rangle$ is out of reach both for the first-order language and for the language $\mathcal{L}(<,>)$ – it can define $\langle \mathbb{N},<\rangle$ up to disjoint powers (see 3.1.3 of [5]). It is doubtful if $\mathcal{L}(<,\geq)$ can also provide such a description.

Lemma 6.2 *The natural order $\langle \mathbb{N},<\rangle$ is categorically defined by the following formulas of the base $\mathcal{L}(R,-R,R^{-1},-R^{-1})$:*

(LF)	$[>]([>]p \to p) \to [>]p$
(TRIH)	$\langle \neg < \rangle([<]p \wedge p) \to p$
(SUCC)	$p \to \langle < \rangle [>](p \vee \langle < \rangle p)$
(PRED)	$(\langle > \rangle \top \wedge p) \to \langle > \rangle [<](p \vee \langle > \rangle p)$.

Proof: The Löb formula (LF) shows (see Chapter 3.9 of [6]) that $<$ is a transitive and well-founded relation and by (TRIH) (trichotomy for $<$) $<$ is a well-founded linear ordering. (SUCC) means that each element has an immediate successor (succ) $\forall x \exists y(x < y \wedge \forall z(z < y \to z = x \vee z < x))$. If $\langle F,V\rangle \nVdash \mathrm{SUCC}[x]$

then $x \vDash p$ and for each y, if $x < y$ then there exists a z such that $y > z$ & $z \nvDash p$ & $z \vDash [<]\neg p \Rightarrow z \neq x$ & $\neg z < x \Rightarrow F \nvDash$ (succ). Conversely, if (succ) is refuted in the point x from a frame F, then a valuation V such that $V(p) = \{x\}$ refutes SUCC in x: $x \vDash p$ but for each y, if $x < y$ then there exists a z such that $z < y$ & $z \neq x$ & $\neg z < x \Rightarrow z \nvDash p$ & $z \vDash [<]\neg p$. Analogously, it can be verified that (PRED) expresses the existence of an immediate predecessor of each element except the zero. Now, according to 3.35 and 3.36 of [8] each well-founded linear ordering, for which SUCC and PRED hold, is isomorphic to $\langle \mathbb{N}, < \rangle$.

Note 11

(i) The formula SUCC can be replaced by the simpler (R-DISC): $p \to \langle < \rangle [<] \langle \neg > \rangle p$, which means that $\forall x \exists y (x < y \wedge \forall z (z < y \to \neg x < z))$ and, together with TRIH, it implies SUCC; analogously, PRED can be replaced by (L-DISC$^+$): $\langle > \rangle \top \wedge p \to \langle > \rangle [>] \langle \neg < \rangle p$.

(ii) The existence of zero is already expressible by the formula (ZERO), $\blacklozenge [>] \bot$, but the induction axiom still remains out of reach; it requires, e.g., the presence of $[R^*]$.

(2) *Transitive* $\in$*-structures, ordinals, and* ZF Consider structures $\langle X, \in \rangle$ where $\in$ is a relation of belonging and X is an $\in$-transitive set, i.e., if $x \in X$ and $y \in x$ then $y \in X$. According to the well-known Mostowski lemma for collapses each extensional and well-founded structure $\langle W, R \rangle$ is isomorphic to a transitive $\in$-structure. So we can define such structures in the modal base $\mathcal{L}(R, -R, R^{-1}, -R^{-1})$, since extensionality EXT: $\forall y, z (\forall x (x \in y \leftrightarrow x \in z) \to y = z)$ is defined by $\varphi_{12} = p \to \blacksquare (\neg p \wedge q \to \blacklozenge ((\langle \in \rangle p \wedge \langle \neg \in \rangle q) \vee (\langle \neg \in \rangle p \wedge \langle \in \rangle q))$ and well-foundedness FUND: $\forall x \exists y \forall z (y \in x \wedge (z \in y \to \neg z \in x))$ is defined by $\blacklozenge p \to \blacklozenge (p \wedge [\ni] \neg p)$. By adding the axioms of linear ordering we can define the notion of an ordinal. The question arises: how far is it possible to translate the axioms of ZF by means of the considered modal base? Let us take the axiomatics of ZF suggested in Section 9.1 of [21]:

(1) EXT
(2) FUND
(3) Subset axiom scheme (SUB): $\forall y \exists z \forall x (x \in z \leftrightarrow (x \in y \wedge Px))$
(4) Power set axiom (POW): $\forall x \exists t \forall y (\forall z (z \in y \to z \in x) \to y \in t)$
(5) Replacement axiom scheme (REP): $\forall w (\forall x \exists z \forall y (R(x,y) \to y \in z) \to \exists t \forall y (\exists x (x \in w \wedge R(x,y)) \to y \in t))$
(6) Axiom of infinity (INF): $\exists x (\exists y (y \in x \wedge \forall z (z \notin y)) \wedge \forall y (y \in x \to \exists z (z \in x \wedge \forall w (w \in z \leftrightarrow (w \in y \vee w = y)))))$.

Besides EXT and FUND, in $\mathcal{L}(R, -R, R^{-1}, -R^{-1})$ the following axioms are also definable:

(SUB) with $\varphi_{\text{sub}} = \blacksquare (\langle \ni \rangle (p \to q) \vee \langle \neg \ni \rangle (p \wedge r)) \to (\langle \ni \rangle q \vee \langle \neg \ni \rangle r)$
(POW) with $\varphi_{\text{pow}} = p \to \blacklozenge [\neg \ni] \langle \ni \rangle \langle \neg \in \rangle p$.

The axioms (INF) and REP) will be considered in 6.3.

(3) *Discreteness and continuity* Here are some more examples of modally definable conditions, expressing some discreteness and continuity principles.

- *right discreteness* (R-DISC): $\forall x(\exists y(x<y)\to\exists y(x<y\wedge\forall z(z<y\to\neg x<z)))$ is defined by $\varphi_{\text{r-disc}}=p\wedge\langle<\rangle\top\to\langle<\rangle[<]\langle\neg>\rangle p$
- *left discreteness* (L-DISC): $\forall x(\exists y(x>y)\to\exists y(x>y\wedge\forall z(z>y\to\neg x>z)))$, *dually* $\varphi_{\text{l-disc}}=p\wedge\langle>\rangle\top\to\langle>\rangle[>]\langle\neg<\rangle p$
- *continuity principle* (see [4]) (CONT): $\forall P(((\forall x\forall y(Px\wedge\neg Py)\to x<y)\wedge\exists xPx\wedge\exists x\neg Px)\to\exists z(\forall t(z<t\to\neg Pt)\wedge\forall t(t<z\to Pt)))$ is defined by $\varphi_{\text{cont}}=\blacksquare(p\to[\neg<]p)\wedge\blacklozenge p\wedge\blacklozenge\neg p\to\blacklozenge([<]\neg p\wedge[>]p)$
- *ink spot principle* (two-dimensional analogue of the continuity principle, see [4]) (INK): $\forall P((\exists xPx\wedge\forall x(Px\to\exists y(x<y\wedge Py))\wedge\forall x(Px\leftrightarrow\forall y(y<x\to Py)))\to\forall x(Px\to\forall y(x<y\to Py)))$ is defined by $\varphi_{\text{ink}}=\blacklozenge p\wedge\blacksquare(p\to\langle<\rangle p)\wedge\blacksquare(p\leftrightarrow[>]p)\to\blacksquare(p\to[<]p)$.

It is worth mentioning that the modal equivalents, as a rule, are considerably shorter and use fewer (propositional) variables than the corresponding first-order formulas use individual ones.

(4) *Negative examples* Despite its great expressive possibilities the base $\mathcal{L}(R,-R,R^{-1},-R^{-1})$ can not overlay the first-order language $L_0(R)$. Here are some examples of first-order formulas, beyond these possibilities:

Lemma 6.3 *The following properties of R are not definable in* $\mathcal{L}(R,-R,R^{-1},-R^{-1})$:

(i) *each point can R-reach another one:* $\alpha_1\rightleftharpoons\forall x\exists y(Rxy\wedge x\neq y)$
(ii) *there exists an R-reflexive point:* $\alpha_2\rightleftharpoons\exists x\,Rxx$; *more generally, there exists an R-loop with length n, containing x*: $c(x,n)\rightleftharpoons\exists y_1\ldots\exists y_{n-1}(Rxy_1\wedge Ry_1y_2\wedge\ldots\wedge Ry_{n-1}x)$
(iii) *each point can R-reach a reflexive point:* $\alpha_3=\forall x\exists y(Rxy\wedge Ryy)$
(iv) *the axiom of infinity* INF
(v) *there exists a left R-compression point* (L-COMP): $\exists x\forall y(Ryx\to\exists z(Ryz\wedge Rzx))$
(vi) *the replacement axiom scheme* REP.

Proof: (i) α_1 is not preserved under p-morphisms. An example: $F=\langle\{x,y\},\{x,y\}^2\rangle\vDash\alpha_1$, $G=\langle\{u\},\{\langle u,u\rangle\}\rangle\nvDash\alpha_1$ and the mapping $f:F\to G$, defined by $g(x)=g(y)=u$ is a p-morphism in $\mathcal{L}(R,-R,R^{-1},-R^{-1})$. We shall examine in the next section an enrichment of this base, which will trivialize (in the standard frames) the p-morphisms and α_1 will then become definable.

(ii), (iii), (iv) We shall use essentially the construction of 4.15 (it is known that $\langle\omega,\in\rangle\cong\langle\mathbb{N},<\rangle$). None of the formulas α_2, α_3, INF is true in $\mathfrak{N}=\langle\mathbb{N},<,\geq,>,\leq\rangle$, whereas these formulas are true in $\text{ue}(\mathfrak{N})=\langle\mathbb{N}^*,<^*,\geq^*,>^*,\leq^*\rangle$. We shall construct a standard frame G such that:

(1) $\text{ue}(\mathfrak{N})$ is a p-morphic image of G
(2) $G\vDash\alpha_2\wedge\alpha_3\wedge\text{INF}$.

W and $<'$ are defined as in 4.15 and the other relations are defined such that $G=\langle W',<',\geq',>',\leq'\rangle$ is standard. The mapping $g:G\to\text{ue}(\mathfrak{N})$, defined by $g(\langle u,i\rangle=u$ is a p-morphism in $\mathcal{L}(R,-R,R^{-1},-R^{-1})$: if $u<g(\langle v,j\rangle)$ then, if

$u \in \mathbb{N}_p$ then $\langle u,0\rangle <' \langle v,j\rangle$ and otherwise $\langle u,j\rangle <' \langle v,j\rangle$. The other checks are analogous. Obviously $G \models \alpha_2 \wedge \alpha_3$; $G \models \mathrm{INF}$: let $v \in \mathbb{N}_f$. Then

(1) $\langle u_0,0\rangle <' \langle v,0\rangle$
(2) let $\langle u,i\rangle <' \langle v,0\rangle$.

- if $u \in \mathbb{N}_p$, $u = u_n$ then $\langle u_{n+1},0\rangle <' \langle v,0\rangle$ & $\forall w(w <' \langle u_{n+1},0\rangle \Leftrightarrow (w <' \langle u_n,0\rangle \vee w = \langle u_n,0\rangle)$)
- if $u \in \mathbb{N}_f$ then $\forall w(w <' \langle u,i\rangle \Leftrightarrow (w <' \langle u,i\rangle \vee w = \langle u,i\rangle))$.

Further reasonings repeat those in 4.15 and prove the undefinability of α_2, α_3, and INF. The formulas $c(x,n)$ are attacked analogously.

(v) A little modifying of the above construction: add to the carrier W of the frame G a new point $\langle v,2\rangle$ for some $v \in \mathbb{N}_f$ and extend the definition of $<'$: for each $x \in W$, $x <' \langle v,2\rangle$; $\langle v,2\rangle <' x$ iff $x = \langle v,2\rangle \vee (x = \langle u,1\rangle \,\&\, u \in \mathbb{N}_f)$. We thus obtain a frame G'. The mapping g, extended by $g(\langle v,2\rangle) = v$ remains a p-morphism and $G' \models \mathrm{COMP}$ while $\mathfrak{N} \not\models \mathrm{COMP}$.

(vi) The reason for the undefinability of REP seems to be technical: in REP a binary predicate occurs whereas the propositional variables are translated in L_2 as unary predicates. This difficulty can also be technically overcome, adding to the language a new binary relation R with a corresponding modality; then REP becomes an L_0-formula in the new language, though still undefinable. The reason is the heavy heaping of quantifiers $\exists\forall\exists$ in the consequent of REP. This rather heuristic argument will be generalized in the following conjecture.

Conjecture If a formula $\alpha \in L_0(R)$ is modally definable in $\mathcal{L}(R,-R,R^{-1},-R^{-1})$ then α is logically equivalent to a closed formula in a prenex form $\alpha' = Q_1x_1 \ldots Q_nx_n\beta$ such that if an atomic formula of the type $x_i = x_j$ or Rx_ix_j occurs in β and the quantifier Q_i is $\exists$ then

(1) $i \neq j$
(2) if i and j are not neighboring numbers then $j < i$ and Q_j is $\forall$.

If the conjecture is true then formulas of the type $\exists x\exists y\exists z(Rxy \wedge Ryz \wedge Rxz)$, $\exists x\exists y\forall z(Rxz \vee Ryz)$, $\exists x\forall y\exists z(Rxy \rightarrow Ryz)$ etc. remain out of reach for the base.

Note 12 It is necessary to pay attention to the following circumstance: Let $C \subseteq E \supseteq D$ be classes of frames. It can happen that C is not definable in E but $C \cap D$ is definable in D (and if D is definable in E then $C \cap D$ is also definable in E). For example, in the language $\mathcal{L}(R)$ the property "R^{-1} is a well-founded relation" is not definable but the Löb formula (LF) $\Box(\Box p \rightarrow p) \rightarrow \Box p$ defines the class of frames with a transitive and well-founded converse relation, so in the class of transitive frames the above property is definable. To this effect the following open question remains: is the formula INF (REP) definable in the class of frames $\langle W,\in\rangle$ ($\langle W,\in,R\rangle$, plus the corresponding modalities over R) which satisfy the other axioms of ZF?

7 Modal definability in other bases Further interesting examples of polymodal bases will be noted in this section. The theory of modal definability in them is developed in the known manner; here we shall only briefly illustrate the additional expressive possibilities which each of them possesses.

The base $\mathcal{L}(R,-R,\neq)$ We obtain this base as a four-modal language $\mathcal{L} = \mathcal{L}(R_1,R_2,R_3,R_4)$ with theory $T_{\neq}$ having an axiom $(-.\neq)$: $\forall x\forall y((R_1xy \Leftrightarrow -R_2xy)\ \&\ (R_3xy \Leftrightarrow -R_4xy)\ \&\ (R_4xy \Leftrightarrow x = y))$, so the standard frames are $\langle W,R,-R,\neq,=\rangle$. The basic frames are those $\langle W,R_1,\ldots,R_4\rangle$ in which $R_1 \cup R_2 = R_3 \cup R_4 =$ an equivalence relation and R_3 is $=$. The defining set of modal formulas consists of the S5-axioms for $\blacksquare'$, where $\blacksquare'p = [R_1]p \wedge [R_2]p$, and the axioms $[R_4]p \leftrightarrow p$ and $\blacksquare'p \leftrightarrow \blacksquare''p$, where $\blacksquare''p = [R_3]p \wedge [R_4]p$.

The modality $[=]$ can be omitted from the language since it is explicitly definable; it does not extend the expressive power. The presence of $[\neq]$ requires from the definition of s-collapse that all atoms consist of only one element, so the s-collapses are isomorphic to special substructures. Here is a demonstration of its expressive possibilities: in the obtained base the formula $\blacklozenge(\varphi \wedge [\neq]\neg\varphi)$ says that φ is true at exactly one point, i.e., syntactical objects playing the role of "constants" introduced by Passy (see [14] and [19]) can be constructed by prefixing the given formula φ with a conjunction of formulas of type $\blacklozenge(p \wedge [\neq]\neg p)$ for each variable p that is destined to play the role of constant in φ. Actually, the construction rather simulates than expresses the constants, since in nonstandard models the described antecedent can be trivialized; so this simulation cannot be used for axiomatizations. Nevertheless, the hope for improving the classical modal deductive machine by adding new rules for inference makes this construction potentially useful.

The base $\mathcal{L}(R_1,-R_1,R_2,-R_2,R_1 \cap R_2)$ The intersection of two relations (often discussed vis-à-vis dynamic logics) can be modeled by similar means: an underlying language $\mathcal{L}(R_1,R_2,R_3,R_4,R_5,R_6)$ and a theory $T_{\cap}$ with an axiom $(-.\cap)$: $\forall x\forall y((R_1xy \Leftrightarrow -R_2xy)\ \&(R_3xy \Leftrightarrow -R_4xy)\ \&\ (R_5xy \Leftrightarrow -R_6xy)\ \&\ (R_6xy \Leftrightarrow (R_2xy \vee R_4xy)))$. The standard frames are $\langle W,R_1,-R_1,R_2,-R_2,R_1 \cap R_2,-R_1 \cup -R_2\rangle$. The basic frames are those $\langle W,R_1,\ldots,R_6\rangle$ in which $R_1 \cup R_2 = R_3 \cup R_4 = R_5 \cup R_6 =$ an equivalence relation and $R_2 \cup R_4 = R_6$. Since the modality $[R_6] = [R_2 \cup R_4]$ is explicitly defined, it can be omitted. Now, applying the familiar techniques and results, a description of the modal definability in this base is obtained.

***The modal constant* loop** The languages under consideration, despite their power, cannot express the simple condition $\exists x\, Rxx$. The reason is that the validity of modal formulas is preserved under ultrafilter contractions while the above condition is not. In general, problems with definability arise with first-order formulas in which subformulas of the type Rxx occur. An effective solution is suggested in [14] by adding a "modal constant" **loop** to the base $\mathcal{L}(R,-R)$, with a semantics in $\langle W,R_1,R_2\rangle$: $x \models$ **loop** iff $\neg R_2xx$, which in a standard frame

$\langle W,R,-R\rangle$ becomes $x \models$ **loop** iff Rxx. Now, $\exists x\, Rxx$ is defined by $\blacklozenge$**loop** and the formula $\diamondplus$**loop** defines the condition $\forall x \exists y(Rxy \wedge Ryy)$, also undefinable up until now. The modal constant **loop** is not a traditional modal tool, but the general definability theory will here be developed in the familiar manner with some corrections in the definitions of a general frame and a p-morphism:

- general frame—the condition **loop** $\in \mathbb{W}$ is added
- p-morphism—the condition $R'f(x)f(x) \Rightarrow Rxx$ is added.

One more peculiarity: the ultrafilter extension is not already obliged to preserve the validity of formulas from $\mathcal{L}(R,-R,\mathbf{loop})$ and its extensions, which is why the results about definability of Δ-elementary classes are not directly transferred here. In this connection a question arises: which formulas from a language with **loop** are preserved under ultrafilter contractions? Also, the description of the modal definability of Δ-elementary classes and elementary formulas in such languages are open questions.

Actually, **loop** can be modeled by standard means in the following way. A relation S and its complement $-S$ are added to the base $\mathcal{L}(R,-R,\neq)$; the condition $-S = -R \cup \neq$ is imposed by the formula $[-S]p \leftrightarrow [-R]p \wedge [\neq]p$, whence (in standard frames) $S = (R \cap =) = \{\langle x,x\rangle | Rxx\}$. So in standard frames $x \models$ **loop** iff $x \models \langle S\rangle \top$. Now the problem, mentioned above, has disappeared, since in nonstandard frames (such as the ultrafilter extensions) the formula $\langle S\rangle \top$ will simply not be true—there it does not express the existence of an R-reflexive point but something stronger (and not ever true). So the preserving of the validity of modal formulas does not already contradict the condition $\exists x\, Rxx$. The question then arises whether the obtained language is stronger than $\mathcal{L}(R,-R,\neq,\mathbf{loop})$.

8 Concluding remarks The investigations in this paper present rather more problems than they solve. Here are some of them. The characterization of modal definability of arbitrary classes of frames in the examined bases does not seem to be quite satisfactory. The reason lies probably in the nature of the things, but the hope remains for more elegant results in this direction. Also, the open problem is whether or not natural analogues of the quoted theorem (8 in [12]) describing the modally definable $\Sigma\Delta$-classes exist. Some natural and powerful modal languages—the languages for dynamic logics and modal languages with constants (see [19]) (and possibly quantifiers over constants) beckon to be characterized with respect to modal definability. In this connection, the following may be asked: Does there exist a natural modal base, covering the expressive power of first-order languages? A useful investigation would be searching for partial syntactical characterizations of the modally definable (in a given base) first-order formulas in the spirit of Chapter 15 of [6]. Finally, the converse direction of the correspondence theory, viz. first-order definability, suggests a large field for investigation. It seems that positive results such as the syntactical criteria of Sahlqvist-type (see [6] and [20]) can be comparatively easily generalized in polymodal variants (at least in the treated cases), though the analogues are quite risky.

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