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On an Unsound Proof of the Existence of Possible Worlds

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Abstract In this paper, an argument of Alvin Plantinga's for the existence of abstract possible worlds is shown to be unsound. The argument is based on a principle Plantinga calls "Quasicompactness", due to its structural similarity to the notion of compactness in first-order logic. The principle is shown to be false.

The concept of possible worlds has been of great value to the development of modal logic, metaphysics, and the philosophy of language over the past thirty years or so. However, there is still a great deal of controversy regarding the nature of these entities. Are they concrete? Abstract? Indeed, are there really any such things at all, or is the concept merely heuristically useful?

Much of this controversy stems from the fact that, typically, accounts of possible worlds simply *postulate* their existence, usually offering by way of justification no more (and no less) than the theoretical elegance and usefulness of the account. The question of the nature and existence of possible worlds thus generally comes down to one's assessment of a given account's theoretical power and appeal.

In his reply [5] to Pollock's article [6] in the *Profiles* volume dedicated to his work, Plantinga offers a *proof* of the existence of abstract possible worlds from relatively weak premises. More specifically, he argues for the proposition that

(*) For any possible state of affairs S, there is a possible world in which S obtains.

If sound, this argument would provide a strong justification for believing in possible worlds, and would give Plantinga's conception of worlds a forceful

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advantage over competing paradigms. But it is not to be. I will show that the argument fails as it stands, and suggest that its prospects for repair are dim.

We begin with some definitions and an assumption used in the proof.

D1 A state of affairs S *includes* another state of affairs S' just in case, necessarily, S obtains only if S' does.
D2 A *possible world* S is a possible state of affairs such that for any state of affairs S', either S includes S' or S includes *not-S'*.[1]
D3 A state of affairs S obtains *in* another state of affairs S' iff S' includes S (see (*)).
D4 *S properly includes S'* iff S includes S', and S' does not include S.
A1 For any set M of states of affairs, there is a state of affairs $\bigwedge M$ that obtains iff every member of M obtains.

Now let us review some first-order metatheory which will be useful in clarifying the ideas that lie behind Plantinga's argument. Let $\mathcal{L}$ be a language with some infinite number of terms and predicates, and let Σ be the set of all sentences of $\mathcal{L}$. A set $\Sigma' \subseteq \Sigma$ is *saturated* iff, for every sentence σ of $\mathcal{L}$, either $\sigma \in \Sigma'$ or $\neg\sigma \in \Sigma'$.[2] We prove the following restricted version of Lindenbaum's Lemma:

LL *For every sentence $\sigma \in \Sigma$, if $\{\sigma\}$ is consistent, then $\{\sigma\}$ can be extended to a consistent saturated set.*

Proof: Let $\sigma \in \Sigma$. Consider the set $\Gamma \subseteq \mathrm{Pow}(\Sigma)$ (the power set of Σ) such that $s \in \Gamma$ iff $\sigma \in s$ and s is consistent. Γ is partially ordered by the set theoretic proper inclusion relation $\subset$. By the Hausdorff maximal principle, Γ has a maximal linearly ordered (MLO) subset Γ', i.e., a subset that is linearly ordered (by $\subset$) and that is not a proper subset of any linearly ordered subset of Γ. *Claim: $\bigcup\Gamma'$ is a consistent saturated set.* It is easy to show from the maximality of Γ' that $\bigcup\Gamma'$ is saturated. So suppose $\bigcup\Gamma'$ is not consistent. Then, by the definition of consistency, there is a proof P of a contradiction from $\bigcup\Gamma'$. Since proofs are finite, and since Γ' is linearly ordered by $\subset$, this means that there is some set $s \in \Gamma'$ containing all the members of $\bigcup\Gamma'$ used in P. But then s is inconsistent, contrary to our assumption that the members of $\Gamma' \subseteq \Gamma$ are consistent.

A consistent saturated set like $\bigcup\Gamma'$ can be thought of as the result of starting with $\{\sigma\}$ and consistently choosing a new sentence to add to the set of sentences one has constructed up to any given stage (taking unions at limits).[3] A possible world in the sense of D2 might be thought of in a similar (though informal) way as the result of starting with a possible state of affairs S and consistently "conjoining" a new state of affairs to the state of affairs one has constructed up to any given "stage". Plantinga exploits this similarity to construct an argument for (*) parallel to the one above for LL. I'll present the argument with commentary interspersed.

Let S be a possible state of affairs, and let A be the set of states of affairs that are possible and include S. (A is thus analogous to the set Γ above. As Plantinga notes, the existence of A itself is hardly uncontroversial; see below.) A is partially ordered by proper inclusion (in the sense of D4). By the Hausdorff maximal principle, A has an MLO subset B, i.e., a subset that is linearly ordered (by proper inclusion) and that is not a proper subset of any linearly ordered subset of A. (B is thus analogous to our Γ'.) The natural move to make at this point, analogous to the *Claim* in the proof above, would be to show that $\bigwedge B$ (the analogue of $\cup\Gamma'$) is a possible world, i.e., that it is possible and "saturated"—that for every state of affairs S', either $\bigwedge B$ includes S' or $\bigwedge B$ includes *not-S'*. The proof that B, and indeed the "conjunction" of any MLO subset of A, is "saturated" and includes S is relatively simple (see [5], p. 329). The problem comes in showing that MLO subsets are in fact possible. In the proof for LL, the saturated set $\cup\Gamma'$ is shown to be consistent in virtue of the finiteness of *proofs*. There is no analogous notion for Plantinga to draw on with respect to states of affairs. Hence, he appeals a priori to the following principle of "Quasicompactness" ([5], p. 329):

A2 For any set M of possible states of affairs, if M has an MLO subset (ordered by proper inclusion), then M has an MLO subset M^* which is such that if $\bigwedge M^{**}$ is possible for every finite subset M^{**} of M^*, then so is $\bigwedge M^*$.

Since B is an MLO subset of A, by A2 it follows that A has an MLO subset B^* which is such that if $\bigwedge B^{**}$ is possible for every finite subset B^{**} of B^*, then so is $\bigwedge B^*$. But then we can show that $\bigwedge B^*$ is possible. For consider any finite subset C of B^*. Since B^* is linearly ordered by proper inclusion, C is as well. Hence, since C is finite, it has a unique "most inclusive" member, i.e., a member T that properly includes every other member of C. Since $T \in A$, T is possible. It is easy to see that T is equivalent to $\bigwedge C$, and so $\bigwedge C$ too is possible. Since C was arbitrary, it follows that $\bigwedge B^{**}$ is possible for every finite subset B^{**} of B^*. Therefore, $\bigwedge B^*$ is possible as well.

The problem with the argument is that the principle of Quasicompactness is false. To see this, note first that the concept of finiteness is perfectly acceptable within Plantinga's framework; since the framework is designed fully to describe all of reality, he places no limitations on the concepts that can be used to pick out a state of affairs. Consider, then, for each natural number n the state of affairs P_n: *The number of stars being finite and greater than n*. I assume that each P_n is possible, for all $n \in N$. Let $D = \{P_n | n \in N\}$; so D is a set of possible states of affairs. For each n, P_n properly includes P_m, for all $m < n$, so D itself is an MLO subset of D, and it is the only such subset. Furthermore, for every finite subset D^{**} of D, D^{**} is possible, since the most inclusive member P_m of D^{**} includes every member of D^{**} and, as assumed, each such P_m is possible. But of course $\bigwedge D$ is not itself possible, since it includes both *There being only finitely many stars* and *There being infinitely many stars*.

Quasicompactness, of course, is drawn from the structurally similar compactness theorem of first-order model theory, which states that if every finite subset of a given set Φ of sentences of a first-order language is model theoretically consistent (i.e., has a model), then so is Φ. The reason that first-order logic is immune from a proof analogous to the one in the previous paragraph is that finiteness is not first-order expressible: no first-order sentence is true in all and only finite models.[4] It *is* expressible in higher-order logics; which is just to say that they are not compact. Since Plantinga's rich framework is analogous to a higher-order logic (insofar as it contains the concept of finiteness), it exhibits an analogue of noncompactness.

There are two ways to patch up the proof that spring to mind, neither of them particularly attractive. First, one could put expressive limitations on Plantinga's framework that would not allow the construction of the counterexample above by disallowing the use of the concept of finiteness in picking out states of affairs. But then one would be hard pressed to justify calling the impoverished possible worlds of the resulting framework "possible worlds", since intuitively there would still be states of affairs which no such worlds would include, e.g., *Edgar's believing that there are finitely many stars.*[5]

Second, one might try to avoid the counterexample to Quasicompactness by tightening up the conditions on M in A2. Specifically, one might postulate that the cardinality of A is greater than $\aleph_0$, and then restrict M in A2 to uncountable sets of states of affairs; that is, one could replace A2 with

A2′ For any set M of possible states of affairs, and any uncountable cardinal κ, if $\mathrm{card}(M) = \kappa$ and M has an MLO subset (ordered by proper inclusion), then M has an MLO subset M^* which is such that if $\bigwedge M^{**}$ is possible for every subset M^{**} of M^* such that $\mathrm{card}(M^{**}) < \kappa$, then so is $\bigwedge M^*$.

However, there are problems here as well. Where κ is a limit cardinal, an analogous counterexample arises. Benardete [2] suggests the possibility of other physical universes spatio-temporally unrelated to our own (not unlike David Lewis's picture *sans* extensional analysis of modality). There seems no a priori reason to deny that there could be κ such universes for any infinite cardinal κ. So let κ' be a limit cardinal, and for every cardinal $n < \kappa'$, let Q_n be the state of affairs, *the number of stars being less than* κ' *and greater than* n. A counterexample to A2′ goes through for κ' just as in the counterexample to A2.

Suppose then we restrict the cardinals in A2′ to infinite successor cardinals. Then there are no obvious counterexamples along the same lines. But Plantinga's proof surely loses its luster if it must be assumed *ad hoc* that the cardinality of the set A of possible states of affairs that include S is a successor cardinal. Indeed, insofar as there is anything to be said on the matter, it seems to me that A is not a set at all[6] but a proper class, so long as we grant: (i) that for each cardinal n there is the state of affairs R_n: *There being n stars*, and (ii) that if $m \neq n$, then $R_m \neq R_n$.[7] However that may be, there is no happy move in the offing here either. The proof, I think, cannot be salvaged.

NOTES

1. A more adequate definition must bring temporality into the picture, as Pollock notes ([6], pp. 121–122) and as Plantinga agrees ([5], p. 327). This added subtlety, although important in a complete account, is inessential here. Irrespective of the issue of temporality, Pollock's own definition ([6], p. 122) doesn't assume that every state of affairs has a complement, and so is perhaps to be preferred. But the definition here, which is Plantinga's ([7], pp. 88–89), helps to clarify the motivations for his proof of (*).

2. Such sets are of course usually called *maximal*, but I use 'saturated' so as to avoid confusing maximality in this sense with maximality in the sense of Hausdorff's principle, introduced below.

3. The usual proof relies more overtly on such a construction. In that proof (for infinite languages generally) Hausdorff's principle appears in the guise of the well-ordering principle.

4. More strongly still, no set of first-order sentences has arbitrarily large finite models but no infinite models (cf. [3], p. 143).

5. In reply, one might claim that this state of affairs could be captured simply by a world that includes, for some n, 10^{26} say, the state of affairs *Edgar's believing there are n stars.* But, of course, Edgar needn't believe such a thing for any particular n in order to believe there are finitely many stars. So that won't work. One might then claim that, by giving the numerical quantifier narrow scope, we can capture Edgar's belief via the state of affairs *Edgar's believing that, for some n, there are n stars.* But this is ambiguous. If the variable 'n' here is ranging over all numbers, then the state of affairs will not do the job, since it is compatible with Edgar's believing, e.g., that there are $\aleph_0$ stars. What is needed is the state of affairs *Edgar's believing that, for some* finite *number n, there are n stars.* But that's just the sort of thing that can't be expressed on the proposal in question.

6. At least, not a ZF set. For a set theory that allows sets which are "too big" to be ZF sets, see [4]. This theory would be no help to Plantinga.

7. A last gasp here might be to formulate a Hausdorff principle for classes generally and try to run the argument through with a proper class size analogue of A2, i.e.,

   A2″ For any proper class M of possible states of affairs, if M has an MLO subclass (ordered by proper inclusion), then M has an MLO subclass M^* which is such that if $\bigwedge M^{**}$ is possible for every subset M^{**} of M^*, then so is $\bigwedge M^*$.

   But a counterexample is still available. Say that the number of a class K is *determinable* just in case K is a set. Then, for any cardinal κ, let P_κ be *the number of stars being determinable and greater than* κ, and let $D' = \{P_\kappa \mid \kappa \in \text{CARD}\}$ (where CARD is the class of all cardinals). Assuming that each P_κ is possible, D' then provides a counterexample parallel to D in the counterexample to A2.

   What we really need in order to get a handle on these questions is a thoroughly developed, axiomatic theory of states of affairs. Progress in this direction is found in [1] and in Chapter 4 of [8].

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