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Cofinalities of Countable Ultraproducts: The Existence Theorem

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Abstract We show that there exists an ultrafilter U in the set $\mathbb{N}$ of natural numbers such that the cofinality of the U-ultrapower ${}^{\mathbb{N}}\mathbb{N}/U$ equals $\mathrm{cof}(d)$, where d is the minimal cardinality of a dominating subfamily of ${}^{\mathbb{N}}\mathbb{N}$. Moreover, the coinitiality of the family of finite-to-one functions in this ultrapower is also $\mathrm{cof}(d)$. If $c = d$, then U may be taken to be a P-point.

All reals are understood to be in ${}^{\omega}\omega$. EAL is the family of *eventually arbitrarily large* reals (also known as finite-to-one reals). If f, g are two reals, we say that f *majorizes* g and g *minorizes* f if $(\exists m \forall n > m) f(n) > g(n)$. Every real may be majorized by a nondecreasing real, and every EAL real may be minorized by a nondecreasing EAL real. A family of reals D is said to be a *majorizing* family if every real is majorized by a real from D. An EAL family of reals D' is said to be a *minorizing* family if every EAL real is minorized by a real from D'. D is said to be *unbounded* if there is no real which majorizes every real in D. Following [6], we let d be the minimal cardinality of a majorizing family of ${}^{\omega}\omega$ and let b be the minimal cardinality of an unbounded family. The cardinal b must be regular and uncountable, but it is consistent that d be singular. However, $\mathrm{cof}(d) \geq b$. (These and other interesting properties of these cardinals are discussed in [7] and [8].) It is easy to check that d is also the minimal cardinality of a minorizing family of EAL. (Proofs may be found in [3] and [4].)

All filters and ultrafilters are understood to be proper, nonprincipal, and on ω. If U is such an ultrafilter, then U induces a linear-ordering $<_U$ on ${}^{\omega}\omega$ where $f <_U g$ iff $\{n \mid f(n) < g(n)\} \in U$. We will write ${}^{\omega}\omega/U$ (respectively EAL$/U$) for the structure consisting of (U-equivalence classes of) reals (respectively EAL reals) together with this ordering. In this paper we will consider the possible cofinalities and coinitialities of EAL$/U$. (EAL is a final segment of ${}^{\omega}\omega$; hence the cofinality of ${}^{\omega}\omega/U$ equals that of EAL$/U$.) In the terminology of [10], these are precisely the cofinalities and coinitialities of skies in nonstandard

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models of complete arithmetic. Such cofinalities and coinitialities must be regular cardinals between b and d. This is the only restriction that can be proven in ZFC: in [3] it was shown that in the model obtained by adding λ Cohen reals to a model of CH there exists, for every pair (σ, τ) of uncountable, regular cardinals below λ, an ultrafilter U such that EAL/U has cofinality σ and coinitiality τ. (The cofinality part of this result was proven independently by Roitman in [11].) It is also consistent that there be only one possibility for the cofinality and coinitiality of EAL/U; in [1] Blass shows that the combinatorial principle NCF (Near Coherence of Filters) implies that the cofinalities and coinitialities of all EAL/U are d, which in turn was used to show that, under NCF, d is regular. This is remarkable in that NCF also implies that $b < d$ and that NCF itself is consistent, as proven in [2]. Upon reflection, the present author observed that that argument in [1] could be easily modified to obtain, in ZFC, a proof of:

Theorem (ZFC) *There exists an ultrafilter U such that the cofinality and coinitiality of* EAL/U *are* $\operatorname{cof}(d)$.

In light of the consistency results quoted above, this would appear to be the only possible existence theorem on this question which can be proven outright. This theorem tells us, for example, that one cannot hope to find a model in which all EAL/U have cofinality b and have $b < \operatorname{cof}(d)$.

Before proving the theorem, we need some preliminary lemmas and one definition: We let Q_λ be the collection of all filters which have a generating set of size $\leq\lambda$. By convention, Q_0 contains the filter of all cofinite sets.

Lemma 1 *If* $\omega \leq \lambda < d$, $F \in Q_\lambda$, $D \subseteq {}^\omega\omega$, *and* $|D| \leq \lambda$, *then there exist* $F' \in Q_\lambda$ *and* $H: \omega \to \omega$ *such that* $F \subseteq F'$, *and for every* $h \in D$, $\{n \mid h(n) < H(n)\} \in F'$.

Lemma 2 *If* $\omega \leq \lambda < d$, $F \in Q_\lambda$, $D \subseteq$ EAL, *and* $|D| \leq \lambda$, *then there exist* $F' \in Q_\lambda$ *and* $G \in$ EAL *such that* $F \subseteq F'$, *and for every* $g \in D$, $\{n \mid g(n) > G(n)\} \in F'$.

Lemma 1 is essentially proved within the proof of Theorem 16 of [1]. There Blass credits Peter Nyikos with noticing it independently. By making obvious minor modifications of that proof, we obtain:

Proof of Lemma 2: Let $\{A_\tau \mid \tau < \lambda\}$ be a generating set for F which is closed under finite intersections. We may assume that the family D consists of nondecreasing reals and is closed under finite minima: if $\{g_0, g_1, \ldots, g_m\} \subseteq D$ then the real which maps each n to $\min\{g_0(n), g_1(n), \ldots, g_m(n)\}$ is also in D. Let $D = \{g_\sigma \mid \sigma < \lambda\}$. For each $\tau < \lambda$ let $f_\tau: \omega \to \omega$ be any real for which $f_\tau(n) = \max\{a \in A_\tau \mid a < n\}$ whenever this set is nonempty. For each pair (σ, τ) define $G_{\sigma,\tau}: \omega \to \omega$ by $G_{\sigma,\tau}(n) = g_\sigma[f_\tau(n)]$. Since $\lambda < d$, we may find a $G: \omega \to \omega$ which is not minorized by any $G_{\sigma,\tau}$. We check that

$$F \cup \{\{n \mid G(n) < g(n)\} \mid g \in D\}$$

has the Finite Intersection Property (FIP), and then let F' be the filter thus generated.

In verifying that this has the FIP we are permitted, by our closure assumptions, to consider a single set A_τ from F and a single g_σ from D. Find a $k > \min(A_\tau)$ such that $G(k) < G_{\sigma,\tau}(k)$. We then have that $f_\tau(k) \in A_\tau$ and $f_\tau(k) <$

k. Since our reals are all nondecreasing, we may calculate that $G[f_\tau(k)] < G(k) < G_{\sigma,\tau}(k) = g_\sigma[f_\tau(k)]$. We thus have that $f_\tau(k) \in A_\tau \cap \{n | G(n) < g_\sigma(n)\}$, completing the proof.

Proof of Theorem: Let $\{h_\tau | \tau \in d\}$ and $\{g_\tau | \tau \in d\}$ be majorizing and minorizing families, respectively. (The g_τ's are assumed to be EAL.) We may construct a sequence of filters $\langle F_\tau | \tau \in d\rangle$, a sequence of reals $\langle H_\tau | \tau \in d\rangle$, and another sequence of EAL reals $\langle G_\tau | \tau \in d\rangle$ so that the following conditions are satisfied for each $\tau < d$:

(1) $\forall \sigma < \tau,\ F_\sigma \subseteq F_\tau$
(2) $F_\tau \in Q_{|\tau| + \omega}$
(3) $\{n | h_\tau(n) < H_\tau(n)\} \in F_\tau$
(4) $\forall \sigma < \tau,\ \{n | H_\sigma(n) < H_\tau(n)\} \in F_\tau$
(5) $\{n | g_\tau(n) > G_\tau(n)\} \in F_\tau$
(6) $\forall \sigma < \tau,\ \{n | G_\sigma(n) > G_\tau(n)\} \in F_\tau$.

To construct F_τ, we assume for the induction hypothesis that the preceding conditions have been satisfied at all the previous stages. Let $F_{(\tau)}$ be the union of $\{F_\sigma | \sigma \in \alpha\}$. $F_{(\tau)} \in Q_{|\tau|+\omega}$. Apply Lemma 1 with $F_{(\tau)}$ as F and $\{H_\sigma | \sigma < \tau\} \cup \{h_\sigma\}$ as D and obtain a filter F' and a real H_τ such that any filter extending F' will satisfy conditions (3) and (4). Apply Lemma 2 with F' as F and $\{G_\sigma | \sigma < \tau\} \cup \{g_\sigma\}$ as D to obtain a filter F_τ and an EAL real G_τ satisfying conditions (5) and (6). Conditions (1), (3), and (4) are met since $F_{(\tau)} \subseteq F' \subseteq F_\tau$ and an easy cardinality calculation shows that condition (2) is also met.

We now let U be any ultrafilter which includes $\cup\{F_\tau | \tau < d\}$. In EAL$/U$ the sequence $\langle G_\tau | \tau \in D\rangle$ is clearly decreasing. We now show that it is coinitial. Let f be an arbitrary EAL real. Find a $\tau < d$ such that $E = \{n | f(n) > g_\tau(n)\}$ is cofinite, hence in U. But $\{n | g_\tau(n) > G_\tau(n)\}$ is also in U, so $G_\tau <_U f$. From this we conclude that the coinitiality of EAL$/U$ is $\mathrm{cof}(d)$. A similar argument shows that $\langle H_\tau | \tau \in d\rangle$ is increasing and cofinal in ${}^\omega\omega$, telling us that the cofinality of ${}^\omega\omega/U$ is also $\mathrm{cof}(d)$.

In [9] Ketonen showed that if $c = d$ then a P-point may be constructed via a transfinite recursion of length d. If U is a P-point, then the reals which are nonstandard (not constant) over U are the EAL reals. By mixing that construction with the one given above we may obtain:

Corollary 3 *If $c = d$ then there exists a P-point U such that the nonstandard part of ${}^\omega\omega/U$ has cofinality and coinitiality* $\mathrm{cof}(d)$.

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