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The Boolean Spectrum of an o-Minimal Theory

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Abstract We show that the number of isomorphism types of Boolean algebras of definable subsets of countable models of an *o*-minimal theory is either 1 or $2^{\aleph_0}$. We also show that the number of such isomorphism types is 1 if and only if no countable model of the *o*-minimal theory contains an infinite discretely ordered interval.

A structure $\mathfrak{M}$ linearly ordered by $<$ is said to be *o-minimal* if its definable subsets are exactly those that can be obtained by using only quantifier-free formulas involving $<$, i.e., unions of finitely many points and intervals. A complete theory **T** of linearly ordered structures is said to be *o*-minimal if all models of **T** are *o*-minimal. We note that in [2] and [5] it is shown that "all models" may be replaced by "some model" in the definition of an *o*-minimal theory. Model theoretically, *o*-minimal structures are the simplest linearly ordered structures, playing the same role with respect to $<$ as minimal structures do with respect to $=$. Carrying this analogy further, *o*-minimal theories correspond to strongly minimal theories.

o-minimal theories were studied extensively in [4]. Here we wish to consider a particular question about such theories. Let **T** be a theory and $\mathfrak{M}$ a model of **T**. Denote by $B(\mathfrak{M})$ the Boolean algebra of the definable subsets of $\mathfrak{M}$, and define the Boolean spectrum of **T**, Spec**T**, to be the set of isomorphism types of the algebras $B(\mathfrak{M})$ as $\mathfrak{M}$ ranges over the countable models of **T**. It is well known that the Boolean spectrum of a strongly minimal theory **T** contains only one element: the isomorphism type of the countable superatomic algebra of CB-type (2,1). That is, a strongly minimal theory **T** is p-$\aleph_0$-categorical (see [7]). Thus we are interested in examining the corresponding problem in the *o*-minimal case.

The most obvious question to raise is whether all *o*-minimal theories are

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p-$\aleph_0$-categorical. The answer, just as obviously, is no; one need only consider the theory of discrete linear order without endpoints. Nevertheless, there are many examples of p-$\aleph_0$-categorical o-minimal theories:

the theory of dense linear order without endpoints
the theory of divisible ordered abelian groups
the theory of real closed fields.

Observe that in each case the underling order is dense.

Our aim then is to classify the p - $\aleph_0$-categorical o-minimal theories, or, more generally, to analyze the Boolean spectrum of any o-minimal theory. We will show that

- an o-minimal theory **T** is p-$\aleph_0$-categorical iff no (countable) model of **T** contains an infinite discrete interval, and that
- if **T** is o-minimal but not p-$\aleph_0$-categorical, then $|\text{Spec}\mathbf{T}| = 2^{\aleph_0}$.

Here is the plan of the article. In Section 1 we recall some basic facts concerning o-minimal theories and Ketonen's classification of countable atomic Boolean algebras (see [1]). We also show that, for every theory **T** of linearly ordered structures, Spec**T** contains the isomorphism type of the countable atomic Boolean algebra B_0 with the property that for every infinite $b \in B_0$, there exist infinite $b_1, b_2 \in B_0$ such that $b_1 \vee b_2 = b$ and $b_1 \wedge b_2 = 0$. Section 2 is devoted to some technical lemmas. Finally, in Section 3, we prove the main theorems.

We assume throughout that **T** is a complete theory without finite models. We also adopt the usual convention that all models of **T** are elementary submodels of some large saturated model $\mathfrak{U}$ of **T**. If $\mathfrak{M} \models \mathbf{T}$ and $X \in B(\mathfrak{M})$, we will sometimes identify X with any formula defining it. Lastly, we say that X is 0-definable if X is definable without parameters.

1 Some preliminaries We first review some facts about countable atomic Boolean algebras. Ketonen classified the isomorphism types of countable atomic Boolean algebras in [1]. In what follows, however, we adopt the terminology of [7].

We first recall the Cantor-Bendixon (hereafter CB) analysis of a countable atomic Boolean algebra. For an atomic Boolean algebra B, we denote by $I(B)$ the ideal generated by the atoms of B. Then, if B is a countable Boolean algebra, we define the sequence $\langle I_\nu(B): \nu \in \text{On}\rangle$ of ideals of B as follows:

$$I_0(B) = (0),\ I_1(B) = I(B),$$

$$I_\lambda(B) = \bigcup_{\nu<\lambda} I_\nu(B) \text{ for limit } \lambda,$$

$$I_{\nu+1}(B) = \text{the preimage of } I(B/I_\nu(B)) \text{ under the canonical homomorphism of } B \text{ onto } I(B/I_\nu(B)).$$

Clearly, there is a (smallest) countable ordinal ν_0 for which $I_{\nu_0+1}(B) = I_{\nu_0}(B)$, which we designate the CB-rank (B). For $b \in B - I_{\nu_0}(B)$, we define CB-rank $(b) = \infty$, and for all other $b \in B$, we let CB-rank $(b) = \min\{\nu: b \in I_{\nu+1}(B)\}$.

We recall that a countable atomic Boolean algebra is called *superatomic* if, with ν_0 as above, $B/I_{\nu_0}(B) \cong (0)$. It is known that the CB-analysis of a countable superatomic Boolean algebra determines its isomorphism type. Let us turn now to the case where a countable atomic Boolean algebra is not superatomic. For a Boolean algebra B and $b \in B$, we denote by $B[b]$ the Boolean algebra whose domain is $\{x \in B: x \leq b\}$. The key to the classification of countable atomic Boolean algebras that are not superatomic is provided by the classification of such algebras B that are *uniform*, i.e., which satisfy CB-rank $(B[b]) \leq$ CB-rank $(B[b^*])$ for all $b \in B$ with CB-rank $(b) < \infty$, where b^* denotes the complement of b in B (see [1] or [7]). We first observe that if $B \neq (0)$ is a uniform, countable atomic algebra and CB-rank $(B) = \nu_0$, then $B/I_{\nu_0}(B)$ is a countable atomless algebra. The isomorphism type of such an algebra then can be described by a function f_B: $B/I_{\nu_0}(B) \to \omega_1$, which we now define. Let $S(B)$ be the dual space of B, and for $p \in S(B)$, let CB-rank $(p) = \min\{\text{CB-rank } (b): b \in p\}$. Also, for $b \in B$, let

$$U(b) = \{p \in S(B): b \in p \wedge \text{CB-rank } (p) = \infty\}$$

and, for every $p \in S(B)$ with CB-rank $(p) = \infty$, let

$$\mathrm{r}(p) = \min\{\text{CB-rank } (B[c]): c \in p\}.$$

Then we define

$$f_B(b/I_{\nu_0}(B)) = \sup\{\mathrm{r}(p): p \in U(b)\}.$$

For details we refer the reader to [1] or [7]. We only wish to point out that f_B is strictly additive if we let $\nu + \xi = \max\{\nu, \xi\}$.

Finally, we denote by B_0 the uniform countable atomic algebra satisfying, for each infinite $b \in B_0$, that there exist infinite $b_1, b_2 \in B_0$ with $b = b_1 \vee b_2$ and $b_1 \wedge b_2 = 0$.

We assume that the reader is familiar with the principal results of [4], but nonetheless we wish to recall the characterization of o-minimal linear orders from that paper. Let $\mathfrak{M} = (M,<,\dots)$ be o-minimal. Then there is an $m < \omega$ such that $(M,<)$ can be written as the ordered sum

$$(M,<) = C_1 + \cdots + C_m$$

where, for each $i \leq m$, $(C_i,<)$ is elementarily equivalent to one of

$$(\omega,<),(\omega^*,<),(\mathbf{Z},<),(\omega + \omega^*,<),(\mathbf{Q},<),$$
or a finite linear order,

and for all $i \leq m$, if C_i does not have a last element, then C_{i+1} has a first element. Moreover, if m is minimal, then C_i is 0-definable in $(M,<)$ for each $i \leq m$. That is, $C_i = \phi_i(M)$ for some formula $\phi_i(v)$ without parameters in the language $\{<\}$. So if $\mathfrak{N} \equiv \mathfrak{M}$ then $(N,<)$ can be written as the ordered sum

$$(N,<) = \phi_1(N) + \cdots + \phi_m(N)$$

where $(\phi_i(N),<) \equiv (C_i,<)$ for each $i \leq m$. In particular, if such an $\mathfrak{M}$ contains an infinite discrete interval I, then we may assume without loss of generality that there is an $i \leq m$ such that $I = \phi_i(M)$. Hence, we may suppose that I is 0-definable and, for any $\mathfrak{N} \equiv \mathfrak{M}$, that $\phi_i(N)$ is an infinite discrete interval in $(N,<)$.

We conclude this section with the following straightforward fact.

Proposition 1.1 *For any countable linearly ordered structure* $\mathfrak{M}$, *there is an* $\mathfrak{N} > \mathfrak{M}$ *such that* $\mathfrak{N}$ *is countable and* $B(\mathfrak{N}) \cong B_0$.

Proof: An easy compactness argument shows that there exists a countable $\mathfrak{M}_1 > \mathfrak{M}$ such that for every formula $\phi(v)$ with parameters from $\mathfrak{M}$, if $\phi(M)$ is infinite, then $\phi(M_1)$ can be split into two infinite, disjoint $\mathfrak{M}_1$-definable subsets. Taking $\mathfrak{N}$ to be the union of an increasing ω elementary chain of models built in this way gives the result.

From Proposition 1.1 we immediately obtain

Proposition 1.2 *If a theory* **T** *extending the theory of linear order has a countable saturated model* $\mathfrak{M}$, *then* $B(\mathfrak{M}) \cong B_0$.

2 Basic lemmas This section is devoted to proving some technical lemmas. Throughout, we assume that:

(a) **T** is an o-minimal theory
(b) $\mathfrak{M} \models \mathbf{T}$
(c) $I = \phi(\mathfrak{M})$ is an infinite, discrete, 0-definable interval in $\mathfrak{M}$ such that $(\phi(\mathfrak{M}),<)$ is elementarily equivalent to $(\omega,<)$, $(\omega^*,<)$, $(\mathbf{Z},<)$, or $(\omega + \omega^*,<)$, and S denotes the successor function on I.

Definition 2.1 Let $a_1,\ldots,a_n \in \phi(\mathfrak{U})$. The sequence $(a_1,\ldots,a_n)$ is *S-independent* if for each $i = 1,\ldots,n$ and $j \in \omega$, $S^j(a_i) < a_{i+1}$.

Let $m \in I$ and suppose that there exists some $m' \in I$ such that (m,m') is S-independent. Also, let $p \in S_1(\mathfrak{M})$ be the type over $\mathfrak{M}$ corresponding to the cut

$$\{v > S^j(m) : j \in \omega\} \cup \{v < m' : m' \in \mathfrak{M} \text{ and } m' > S^j(m) \text{ for all } j \in \omega\}.$$

Lastly, if a realizes p, we denote by $\mathfrak{M}(a)$ the model of **T** prime over $\mathfrak{M} \cup \{a\}$, as guaranteed by [4].

Lemma 2.2 *With notation as above*, $p(\mathfrak{M}(a)) = \{S^j(a) : j \in \mathbf{Z}\}$.

Proof: Suppose for a contradiction that there is some $a' \in \mathfrak{M}(a) - \mathfrak{M}$ that realizes p but is different from $S^j(a)$ for all $j \in \mathbf{Z}$. Then $\mathrm{tp}(a',\mathfrak{M} \cup \{a\})$ is isolated by some formula $\theta(v,a)$ in the language $L(\mathfrak{M})$ having constants for each element in $\mathfrak{M}$. By replacing a' by an endpoint of $\theta(\mathfrak{M}(a),a)$, if necessary, we can assume that $a' = f(a)$, where f is an $\mathfrak{M}$-definable function. Without loss of generality, by using f^{-1} if required, we may also suppose that $a' > a$. Then, by the Monotonicity Theorem (see [4]), there is an interval $J = (a,b)$ in $\mathfrak{M}$ such that the formula $a < v < b$ is in p, $f|_J$ is a strictly monotone bijection between J and another interval $J' = (a',b')$ in $\mathfrak{M}$, the formula $a' < v < b'$ is in p, and $f(c) > c$ for all $c \in J$.

It is easy to see that we may assume that $m \in J \cap J'$, where m is as in the definition of p. Since $f(c) > c$ for all $c \in J$, it follows that f must be order-preserving. If there were some $j \in \omega$ for which $f(m) = S^j(m)$, then it would have to be the case that $a' = f(a) = S^j(a)$, contrary to our hypothesis. Hence, $f(m)$ must be greater than any realization of p, and so, since f is order-preserving, $a' = f(a)$ could not realize p. Hence the lemma is proved.

Lemma 2.3 *Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be S-independent sequences of realizations of p. Then $tp(\bar{a}, \mathfrak{M}) = tp(\bar{b}, \mathfrak{M})$.*

Proof: We proceed by induction on n. There is nothing to prove in the case where $n = 1$. So let $n > 1$, and suppose for a contradiction that $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ are sequences of S-independent realizations of p such that $\mathrm{tp}(\bar{a}, \mathfrak{M}) \neq \mathrm{tp}(\bar{b}, \mathfrak{M})$. By the induction hypothesis, we may assume that $\bar{b} = (b_1, a_2, \ldots, a_n)$.

Let $\theta(v)$ be an $L(\mathfrak{M} \cup \{a_2, \ldots, a_n\})$-formula such that

$$\models \theta(a_1) \wedge \neg\theta(b_1).$$

It then follows for all $k, l < \omega$ that the formulas

$$(\exists v)(\exists w) S^k(m) < v < S^{-l}(v_2) \wedge S^k(m) < w < S^{-l}(v_2) \wedge \theta(v, v_2, \ldots, v_n) \wedge \neg\theta(w, v_2, \ldots, v_n)$$

are in $\mathrm{tp}(a_2, \ldots, a_n; \mathfrak{M})$. However, applying Lemma 2.2 $n - 1$ times, it is easy to see that there is no $c \in \mathfrak{M}' = \mathfrak{M}(a_n) \cdots (a_2)$ satisfying $S^j(m) < c$ and $S^j(c) < a_2$ for all $j < \omega$, and hence by o-minimality, for some $k, l < \omega$, we have either

$$\models (\forall v) S^k(m) < v < S^{-l}(a_2) \rightarrow \theta(v)$$

or

$$\models (\forall v) S^k(m) < v < S^{-l}(a_2) \rightarrow \neg\theta(v).$$

Either alternative immediately yields a contradiction, so the lemma is proved.

Lemma 2.4 *Suppose that $\bar{a} = (a_1, \ldots, a_n)$ is an S-independent sequence of realizations of p. Then $p(\mathfrak{M}(\bar{a})) = \bigcup_{1 \leq i \leq n} \{S^j(a_i) : j \in \mathbf{Z}\}$.*

Proof: For a contradiction, suppose that there is some $b \in p(\mathfrak{M}(\bar{a}))$ satisfying $b \neq S^j(a_i)$ for all $j \in \mathbf{Z}$ and $i = 1, \ldots, n$. Suppose that $a_i < b < a_{i+1}$ for some $1 \leq i < n$ (the cases $b < a_0$ and $b > a_n$ are handled similarly). Let $\sigma(v, \bar{a})$ be an $L(\mathfrak{M})$ formula isolating $\mathrm{tp}(b, \mathfrak{M} \cup \{\bar{a}\})$. By Lemma 2.3, it follows that $\models \sigma(c, \bar{a})$ holds for all $c \in p(\mathfrak{M}(\bar{a}))$ satisfying $S^j(a_i) < c$ and $S^j(c) < a_{i+1}$ for all $j < \omega$. Hence, by o-minimality, there must be some $k, l < \omega$ such that $\models (\forall v) S^k(a_i) < v < S^{-l}(a_{i+1}) \rightarrow \sigma(v, \bar{a})$. But this is clearly impossible.

Lemma 2.5 *Suppose that X is a set of realizations of p (in $\mathfrak{U}$) that is closed under S and S^{-1}(so, in particular, $(X,<)$ is a discrete linear order without endpoints). Then $p(\mathfrak{M}(X)) = X$.*

Proof: Since any $b \in p(\mathfrak{M}(X))$ must be isolated over $\mathfrak{M}$ by a finite, S-independent sequence $\bar{a}$ of elements in X, the result follows immediately from Lemma 2.4.

3 The principal theorems

We now prove the main results of the paper.

Theorem 3.1 *An o-minimal theory is p-$\aleph_0$-categorical iff no (countable) model of* **T** *contains an infinite discrete interval.*

Proof: Let $\mathbf{T}$ be a p-$\aleph_0$-categorical o-minimal theory. For a contradiction, suppose that there is a countable $\mathfrak{M} \models \mathbf{T}$ that contains an infinite discrete interval I. By the remarks made about o-minimal linear orders in Section 1, we may assume that I is 0-definable and elementarily equivalent to either $(\omega,<)$, $(\omega^*,<)$, $(\mathbf{Z},<)$, or $(\omega + \omega^*,<)$. Moreover, by Proposition 1.1 we may also assume that $B(\mathfrak{M}) \cong B_0$, and hence that for every $n < \omega$, I contains an S-independent sequence of length n. Let $m, m' \in I$ so that (m,m') is S-independent, and let p be the type over $\mathfrak{M}$ determined by the cut

$$\{v > S^j(m): j < \omega\} \cup \{v < b: b \in \mathfrak{M} \wedge (\forall j < \omega) b > S^j(m)\}.$$

Now, let a realize p. It follows by Lemma 2.2 that $p(\mathfrak{M}(a)) = \{S^j(a): j \in \mathbf{Z}\}$. But then the interval $[m,a] \subseteq \mathfrak{M}(a)$ is infinite and cannot be split into two disjoint infinite definable subsets. This implies that $B(\mathfrak{M}(a)) \not\cong B_0$, a contradiction, since $\mathbf{T}$ was assumed to be p-$\aleph_0$-categorical.

Conversely, suppose that $\mathbf{T}$ is an o-minimal theory and that no countable model of $\mathbf{T}$ contains an infinite discrete interval. Then by the remarks on o-minimal linear orders made in Section 1, if $\mathfrak{M}$ is a countable model of $\mathbf{T}$, we can write $(M,<)$ as the ordered direct sum

$$(M,<) = C_1 + \cdots + C_m$$

where, for each $i \leq m$, $(C_i,<)$ is a 0-definable interval in $\mathfrak{M}$ and is either finite or a dense linear order without endpoints. Then it is easy to see that $B(\mathfrak{M}) \cong B_0$, and hence that $\mathbf{T}$ is p-$\aleph_0$-categorical.

Let us also observe that an o-minimal theory $\mathbf{T}$ is p-$\aleph_0$-categorical iff there exists a (countable) $\mathfrak{M} \models \mathbf{T}$ that does not contain an infinite discrete interval. This follows from the proof of Theorem 3.1 and the remarks in Section 1 about the ordered sum decomposition of an o-minimal linear order.

Theorem 3.2 *If* $\mathbf{T}$ *is an o-minimal theory that is not* p-$\aleph_0$*-categorical, then* $|\mathrm{Spec}\mathbf{T}| = 2^{\aleph_0}$.

(In particular, if $\mathbf{T}$ has fewer than $2^{\aleph_0}$ non-isomorphic countable models, then $\mathbf{T}$ is p-$\aleph_0$-categorical.[1] The theory of real closed fields demonstrates that the converse obviously is false.)

Proof: For $n \geq 3$, let $(D(n),<)$ be the discrete linear order without endpoints where

$$D(3) = \omega \times \mathbf{Z}$$

$$D(n+1) = \omega \times D(n) \quad \text{for all } n \geq 3$$

and $<$ is given by the lexicographic order. It is an easy matter to verify by induction on n that $B((D(n),<))$ is a superatomic Boolean algebra of CB-rank n.

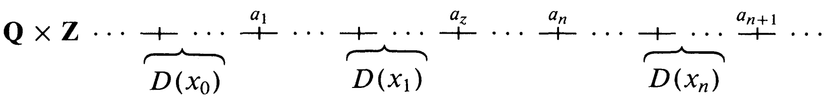
Next, let $(\mathbf{Q} \times \mathbf{Z},<)$ be the discrete linear order in which $<$ is given by lexicographic order. We now fix a sequence $\langle a_n: 0 < n < \omega\rangle$ of elements from this structure satisfying $S^j(a_n) < a_{n+1}$ for all $j < \omega$ and all $0 < n < \omega$, and let $I_0 = (-\infty, a_1)$ and $I_n = (a_n, a_{n+1})$ for each $n > 0$.

For each infinite $X \subseteq \omega - \{0,1,2\}$, we now define a discrete linear order without endpoints, $(D(X),<)$, as follows. Enumerate X as $\{x_0, x_1, \ldots\}$ with

$x_0 < x_1 < \cdots$. Then we obtain $(D(X),<)$ by inserting into each cut in $(\mathbf{Q} \times \mathbf{Z},<)$ of the form

$$\{S^j(a) < v: j < \omega\} \cup \{v < b: b \in \mathbf{Q} \times \mathbf{Z} \wedge (\forall j < \omega) b > S^j(a)\},$$

where $a \in I_n$ and $S^j(a) < a_{n+1}$ for all $j < \omega$, a copy of $(D(x_n),<)$. Pictorially,



We now assert that the CB-rank of $B((D(X),<)) = \omega$, and that $B((D(X),<))$ is not superatomic but is uniform. For the first of these assertions, it suffices to consider an interval $I = (a,b)$ in $D(X)$, with $a,b \in D(X) \cup \{\pm\infty\}$ and $S^j(a) < b$ for all $j < \omega$. Because the CB-rank of $B((\mathbf{Q} \times \mathbf{Z},<))$ is ∞, it is easy to see that CB-rank $(I) = \infty$ if any one of the following hold: (i) b lies in a copy of $\mathbf{Z}$ in $\mathbf{Q} \times \mathbf{Z}$ or is ∞, or (ii) a is $-\infty$, or (iii) a lies in a copy of $\mathbf{Z}$ in $\mathbf{Q} \times \mathbf{Z}$ and b lies in a copy of $D(x_n)$ that is not in the cut determined to the left by $\{S^j(a) < v: j < \omega\}$, or (iv) a lies in a copy of $D(x_n)$ and b lies in a copy of $D(x_m)$ where $D(X_n)$ and $D(x_m)$ are distinct. In the remaining cases, namely if a lies in a copy of $\mathbf{Z}$ in $\mathbf{Q} \times \mathbf{Z}$ and b lies in the copy of $D(x_n)$ that is in the cut determined to the left by $\{S^j(a) < v: j < \omega\}$, or a and b lie in the same copy of $D(x_n)$, we easily see that CB-rank $(I) \leq x_n - 2$. Hence, the CB-rank of $B((D(X),<)) = \omega$, as asserted. It is obvious from the analysis just presented that $B((D(X),<))$ is not superatomic and also that it is uniform. We leave the details to the reader.

Now suppose that $\mathbf{T}$ is an o-minimal theory. We complete the proof of the theorem by defining countable models $\mathfrak{M}_X$ for each infinite $X \subseteq \omega - \{0,1,2\}$ such that

(*) if $X \neq Y$, then $B(\mathfrak{M}_X) \not\cong B(\mathfrak{M}_Y)$.

Let $\mathfrak{M}$ be a countable model of $\mathbf{T}$ satisfying $B(\mathfrak{M}) \cong B_0$. It then follows that $\mathfrak{M}$ contains an infinite discrete 0-definable interval I, which, ignoring a possible initial copy of $(\omega,<)$ and final copy of $(\omega^*,<)$, is isomorphic to $(\mathbf{Q} \times \mathbf{Z},<)$. Let $m \in I$. For any X as above, let $\mathfrak{M}_X$ be the model of $\mathbf{T}$ that is prime over $\mathfrak{M} \cup D(X)$, where each element of $D(X)$ realizes the type $p \in S(\mathfrak{M})$ determined by the cut in $\mathfrak{M}$ given by

$$\{v > S^j(m): j < \omega\} \cup \{v < b: b \in \mathfrak{M} \wedge (\forall j < \omega) b > S^j(m)\}.$$

By Lemma 2.5, we see that $p(\mathfrak{M}_X) = D(X)$.

Before proving (*), we show that if q is a nonalgebraic type over $\mathfrak{M}$ that is realized in $\mathfrak{M}_X$, then $q(\mathfrak{M}_X)$ as an order is isomorphic to $D(X)$ with either its usual or reversed order. Given such a q, we will prove this assertion by finding $\mathfrak{M}$-definable intervals $I_p = (a_1,a_2)$ and $I_q = (b_1,b_2)$ such that $a_1 < v < a_2$ is in p and $b_1 < v < b_2$ is in q, and an $\mathfrak{M}$-definable order-preserving or reversing bijection f between I_p and I_q.

Let $b \in \mathfrak{M}_X$ realize q, and let $\phi(v,y_1,\ldots,y_n)$ be an $L(\mathfrak{M})$-formula so that there are $a_1,\ldots,a_n \in p(\mathfrak{M}_X)$ such that $\phi(v,a_1,\ldots,a_n)$ isolates $\mathrm{tp}(b,\mathfrak{M} \cup D(X))$. Assuming now that we have taken n above to be minimal, we observe that we will be done if we show that $n = 1$. Toward a contradiction, suppose that $n > 1$. Notice that the minimality of n allows us to assume that $\bar{a} = (a_1, \ldots, a_n)$ is an S-independent sequence of realizations of p. Let $\mathfrak{M}'$ be a model of $\mathbf{T}$ that is prime over $\mathfrak{M} \cup \{a_2, \ldots, a_n\}$. We assume that $\mathfrak{M}'$ is elementarily embedded into $\mathfrak{M}(\bar{a})$ over $\mathfrak{M} \cup \{a_2, \ldots, a_n\}$. By Lemma 2.4, we see that $p(\mathfrak{M}') = \bigcup_{2 \leq i \leq n} \{S^j(a_i) : j \in \mathbf{Z}\}$, and since n is minimal, we have that $q(\mathfrak{M}') = \emptyset$. Let p' be the type over $\mathfrak{M}'$ determined by the cut

$$\{v > S^j(m) : j < \omega\} \cup \{v < S^{-j}(a_2) : j < \omega\}.$$

Clearly, a_1 realizes p'. Let $\mathfrak{M}''$ be the model of $\mathbf{T}$ prime over $\mathfrak{M}' \cup \{a_1\}$. We notice that $\mathfrak{M}''$ contains a realization b_0 of q, and also, by Lemma 2.2, that $p'(\mathfrak{M}'') = \{S^j(a_1) : j \in \mathbf{Z}\}$. Without loss of generality, $b_0 = f(\bar{a})$, where f is an $\mathfrak{M}$-definable function. Now let $g(v)$ be the $\mathfrak{M} \cup \{a_2, \ldots, a_n\}$-definable function given by $g(v) = f(v, a_2, \ldots, a_n)$. Then there exists an interval $J = [c_1, c_2]$ in $\mathfrak{M}'$, with $c_1 = S^k(m)$ and $c_2 = S^{-1}(a_1)$ for some $k, l \in \omega$, on which g is a strictly monotone bijection of intervals in $\mathfrak{M}'$. Since $\mathfrak{M}'$ contains no realizations of q, it follows that $g(J) \subseteq \mathfrak{M}$. But $g(J)$ evidently is an interval in $\mathfrak{M}$ that cannot be split into two infinite definable subsets, contradicting the assumption that $B(\mathfrak{M}) = B_0$. Hence, $n = 1$, as claimed.

Using what we have just proved concerning realizations of types in $\mathfrak{M}_X$, and that $B(\mathfrak{M}) = B_0$, an analysis similar to one we gave in order to calculate the CB-rank $(B(D(X)))$ shows that CB-rank $(B(\mathfrak{M}_X)) = \omega$ for any infinite $X \subseteq \omega - \{0,1,2\}$. Similarly, we also observe that $B(\mathfrak{M}_X)$ is not superatomic, but is uniform. Details are left to the reader. In particular, we now have that $B(\mathfrak{M}_X)/I_\omega(B(\mathfrak{M}_X))$ is a countable free Boolean algebra.

Finally, we can move to the proof of (*). Suppose that $X \neq Y$. Let $x \in X \triangle Y$, say $x \in X - Y$, and let $x = x_n$. We define the interval $I \subseteq \mathfrak{M}_X$ by

$$I = \begin{cases} [m, x_1), & \text{if } n = 0 \\ [x_n, x_{n+1}), & \text{if } n > 0. \end{cases}$$

(Here, we identify the set of realizations of p in $\mathfrak{M}_X$ with $D(X)$ itself.) A now routine analysis establishes that CB-rank $(I) = \infty$.

Let $f_X = f_{B(\mathfrak{M}_X)}: B(\mathfrak{M}_X)/I_\omega(B(\mathfrak{M}_X)) \to \omega_1$ be as defined in Section 1. We claim that

$$f_X(I/I_\omega(B(\mathfrak{M}_X))) = x - 1.$$

For this, we first observe that a case-by-case analysis shows that the only types q over $\mathfrak{M}_X$ containing the formula $v \in I$ which are members of $U(I)$ are, if $n > 0$, those whose cut in $\mathfrak{M}_X$ is the "limit" from either the right or left, or both sides, of copies of $D(x)$. If $n = 0$, then $U(I)$ contains, in addition, the type q determined by the cut $\{v > S^j(m) : j < \omega\} \cup \{v < b : b \in D(X)\}$. Since CB-rank $(D(x)) = x$, and

$$\begin{aligned} \text{CB-rank } (B(\mathfrak{M}_X)[S]) = \sup\{&\text{CB-rank } (J) + 1 : \\ &J \in B(\mathfrak{M}_X) \wedge J \subseteq S \text{ an interval} \\ &\wedge \text{CB-rank } (J) < \infty\} \end{aligned}$$

for any definable set $S \subseteq \mathfrak{M}_X$, it follows (identifying the definable set S and the formula $v \in S$) that $\mathrm{r}(q) = \min\{\text{CB-rank } (B(\mathfrak{M}_X)[S]) : S \in q\} = x - 1$ for $q \in U(I)$. Hence, $f_X(I/I_\omega(B(\mathfrak{M}_X))) = x - 1$, as claimed.

By the Ketonen analysis of uniform Boolean algebras (see [1]), we complete the proof of the theorem if we show that there is no $S \in B(\mathfrak{M}_Y)$ for which CB-rank $(S) = \infty$ and $f_Y(S/I_\omega(B(\mathfrak{M}_Y))) = x - 1$. Since f_Y is strongly additive, the o-minimality of **T** allows us to assume that $S = (a,b)$ for some $a, b, \in \mathfrak{M}_Y \cup \{\pm\infty\}$ with $a < b$. The analysis now proceeds through four cases. Using the strict additivity of f_Y and what we proved about realizations of types over $\mathfrak{M}$ in $\mathfrak{M}_Y$, we leave it to the reader to verify that all other cases can be analyzed using the four below.

(i) $a, b \in \mathfrak{M}$ and $(a,b) \cap (\mathfrak{M}_Y - \mathfrak{M}) = \emptyset$. Here, since $B(\mathfrak{M}) = B_0$, it is a simple matter to verify that $f_Y(S/I_\omega(B(\mathfrak{M}_Y))) = 1$.

(ii) $a, b \in \mathfrak{M}$ and $(a,b) \cap (\mathfrak{M}_Y - \mathfrak{M}) \neq \emptyset$. Arguing as we did above to show that $f_X(I) = x - 1$, we can show for every $n < \omega$ that there is an interval $J_n \subseteq S$ satisfying $f_Y(J_n/I_\omega(B(\mathfrak{M}_Y))) = y_n - 1$. Hence, the strict additivity of f_Y implies that $f_Y(S/I_\omega(B(\mathfrak{M}_Y))) \geq \omega$.

(iii) $a = m$ and b realizes p. Using the strict additivity of f_Y once again, it is routine to show that $f_Y(S/I_\omega(B(\mathfrak{M}_Y))) = y - 1$ for some $y \in Y$.

(iv) a realizes p and b realizes p. In this case also, one easily sees that $f_Y(S/I_\omega(B(\mathfrak{M}_Y))) = y - 1$ for some $y \in Y$.

Since we have shown that $f_Y(S/I_\omega(B(\mathfrak{M}_Y))) \neq x - 1$, it follows that (*), and so Theorem 3.2, is proved.

NOTE

1. The referee has pointed out that this remark can be deduced from Theorem 3.1 using the fact that an o-minimal theory whose models all contain an infinite discrete order has $2^{\aleph_0}$ nonisomorphic countable models. This fact has an easy proof due to Marker—see [3]. It also follows from a general result of [6], asserting that a theory which extends the theory of linear order and has Skolem functions has $2^{\aleph_0}$ nonisomorphic countable models.

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