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"Pathologies" in Two Syntactic Categories of Partial Maps

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Introduction In [2], Di Paola and Heller introduce the *dominical categories* and the *recursion categories* in order to find an algebraic (or, better, category-theoretic) approach to recursion theory. The authors show that the basic part of recursion theory can be based on a few category-theoretic axioms and prove, by means of several relevant examples, that their approach is suitable not only for classical recursion theory, but also for a broad class of situations. The authors suggest that their approach could provide an algebraic version of Gödel-Rosser's incompleteness theorems, and provide a first, but relevant, step in this direction, showing that in any recursion category satisfying some natural conditions there are creative and effectively inseparable domains.

To carry their project one step further, it seems quite natural to study categories of partial maps from a syntactic point of view. This study has also been suggested by Di Paola and Heller. The most interesting category of this kind seems to me to be the one whose objects and morphisms are defined as follows:

1. The class *Ob* of *objects* is the smallest class $\mathcal{C}$ such that $\omega \in \mathcal{C}$, $A, B \in \mathcal{C}$ implies $A \times B \in \mathcal{C}$ and $A + B \in \mathcal{C}$ (where $A + B = A \times \{0\} \cup B \times \{1\}$ is the disjoint union of A and B).
2. The *morphisms* from A to B are the gödel numbers of partial recursive functions from A to B modulo provable equality in PA, i.e., the equivalence classes of natural numbers with respect to the equivalence $\sim$ defined by $n \sim m$ iff $\mathrm{PA} \vdash \forall x[(\varphi_n x \uparrow \leftrightarrow \varphi_m x\uparrow) \wedge (\varphi_n x\downarrow \rightarrow \varphi_m x = \varphi_n x)]$.

Here we are using notation from [8][1]; moreover, if $A \in Ob$, the elements of A can be naturally coded by natural numbers, and we can identify each $a \in A$ with its code in ω. *We denote this category by S.*

*We wish to express our gratitude to R. Di Paola and A. Heller, who introduced us to dominical categories; we also wish to thank Giuseppe Rosolini, who first suggested that S and S' are p-categories, and whose ideas contributed to improve the present paper. Lastly, we are indebted to Claudio Bernardi for useful discussions.

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Another category, a little less natural, but still useful, is defined similarly, with the exception that the equivalence $\sim$ is replaced by the equivalence $\sim'$ defined by $n \sim' m$ iff $\forall x \in A$, $\mathrm{PA} \vdash (\varphi_n \bar{x}\uparrow \leftrightarrow \varphi_m \bar{x}\uparrow) \wedge (\varphi_n \bar{x}\downarrow \rightarrow \varphi_m \bar{x} = \varphi_n \bar{x})$. *We denote this second category by* S'.

When studying these categories, a considerable difficulty is constituted by the difference between recursion theoretic notions taken "from the outside" and recursion theoretic notions considered "from the inside", i.e., with the eyes of PA. The most important difference is that the notion of totality suggested by Di Paola and Heller does not correspond to the expected one, and in particular S and S' are not dominical categories. It turns out that both S and S' satisfy the axioms of Rosolini's *pointed p-categories.* (These categories constitute an intuitionistic counterpart to dominical categories. For reference, cf. [9].)

We believe that S and S' are interesting in their own right. For example, when doing elementary recursion theory, we usually identify two (sets of instructions for) recursive functions iff we are able to prove in some metatheory, which is not far from PA, that the two sets of instructions determine the same function.

As regards Di Paola–Heller's project of finding a category theoretic counterpart to the incompleteness theorems, we find only *negative* results. We can however point out the following *positive* aspects of the present paper:

1. We provide two relevant examples of "models" of Rosolini's p-categories, which are not dominical categories, and prove that, in these examples, Rosolini's notion of totality (R) is more natural than Di Paola–Heller's (DPH).
2. We provide a syntactic classification of totality by means of category theoretic concepts.
3. We prove, by means of a trivial but useful conservation result, that many important properties of recursion categories extend to both S and S', and, more generally, to all p-categories with a Turing morphism.
4. We prove that the difference between p-categories with a Turing morphism and dominical categories with a Turing morphism is relevant, by showing that many important recursion theoretic theorems holding in all dominical categories with a Turing morphism are not valid in S and S', and thus are independent of the theory of p-categories with a Turing morphism.

Throughout the whole paper, a basic reference will be [2]. Also, knowledge of [9] would be very helpful.

1 Preliminary notions

Definition 1.1 (See [2]) A category is *pointed* iff for each pair X, Y of objects there is a morphism $0_{XY}: X \to Y$ such that for each morphism $\varphi: Y \to Z$ and $\psi: W \to X$, $\varphi 0_{XY} = 0_{XZ}$ and $0_{XY}\psi = 0_{WY}$. (In the following we omit subscripts.) A morphism φ is *DPH total* iff for all ψ whose target is the source of φ, $\varphi\psi = 0$ implies $\psi = 0$.

Definition 1.2 (See [2]) Let $\mathcal{C}$ be a pointed category. By a *near-product* is meant a bifunctor $\times: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the following conditions:

(N1) $\varphi \times \psi = 0$ *iff* $\varphi = 0$ *or* $\psi = 0$

(N2) $\times$ restricts to $\mathcal{C}_T \times \mathcal{C}_T \to \mathcal{C}_T$ (here $\mathcal{C}_T$ denotes the subcategory of DPH total morphisms), where it becomes a product, accompanied by projections $X_1 \xleftarrow{p} X_1 \times X_2 \xrightarrow{q} X_2$, and thus a diagonal Δ_X, viz. the unique total morphism $\varphi: X \to X \times X$ such that $p\varphi = q\varphi = \mathrm{Id}_X$

(N3) The associativity and symmetry isomorphisms of this restriction are natural on $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ and $\mathcal{C} \times \mathcal{C}$, so that $\times$ is coherently associative and symmetric

(N4) For all $\varphi: X \to X'$ and all Y, $p(\varphi \times \mathrm{Id}_Y) = \varphi p$, $q(\mathrm{Id}_Y \times \varphi) = \varphi q$ and $(\varphi \times \varphi)\Delta_X = \Delta_{X'}\varphi$.

Definition 1.3 (See [2]) A *dominical category* is a pointed category $\mathcal{C}$ with a near-product satisfying (N1)–(N4).

Definition 1.4 (See [9]) A *p-category* is a category $\mathcal{C}$ endowed by a bifunctor $\times: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called (near) *product*, a natural transformation Δ: Id $\to \times \cdot$(Id,Id), called *diagonal*, and for each object A in $\mathcal{C}$, natural transformations $p_{-,A}: - \times A \to \mathrm{Id}$ and $q_{A,-}: A \times - \to \mathrm{Id}$, called *projections*, such that

$$\begin{array}{ll}
p_{A,A}\Delta_A = \mathrm{Id}_A = q_{A,A}\Delta_A & (p_{A,B} \times q_{A,B})\Delta_{A\times B} = \mathrm{Id}_{A\times B} \\
p_{A,B}(\mathrm{Id}_A \times p_{B,C}) = p_{A,B\times C} & p_{A,C}(\mathrm{Id}_A \times q_{B,C}) = p_{A,B\times C} \\
q_{A,C}(p_{A,B} \times \mathrm{Id}_C) = q_{A\times B,C} & q_{B,C}(q_{A,B} \times \mathrm{Id}_C) = q_{A\times B,C}
\end{array}$$

and the isomorphisms α and μ for associativity and commutativity defined by $\alpha = ((p \times pq)\Delta \times qq)$ and $\mu = (q \times p)\Delta$ are natural in all variables.

A *pointed p-category* is a p-category which is pointed in the sense of Definition 1.1 and such that, for every morphism φ, $\varphi \times 0 = 0 \times \varphi = 0$.

Note that dominical categories are pointed p-categories (cf. [9]) but, as we shall see, *not* vice versa.

Definition 1.6 In both dominical categories and p-categories, the *domain* dom φ of a morphism φ is defined by dom $\varphi = q\langle\varphi,\mathrm{Id}\rangle = p\langle\mathrm{Id},\varphi\rangle$, where $\langle\varphi, \psi\rangle = (\varphi \times \psi)\Delta$.

Definition 1.7 A morphism φ is *R total* iff dom φ = Id.

In a dominical category, R total and DPH total morphisms coincide (cf. [2]). In a pointed p-category, R total morphisms are DPH total (cf. [9]), but not vice versa, as we shall see in Section 2.

Definition 1.8 A *dominical semigroupoid (p-semigroupoid)* is a dominical category (a pointed p-category) where all objects are isomorphic.

Definition 1.9 A *Turing morphism* of a dominical semigroupoid (a p-semigroupoid) is a morphism $\tau: X \times Y \to Z$ such that, for all $\varphi: X \times Y \to Z$, there is a total $g: X \to X$ for which $\varphi = \tau(g \times \mathrm{Id}_Y)$. A *recursion category* (a *p-recursion category*) is a dominical semigroupoid (a p-semigroupoid) endowed with a Turing morphism.

In the following, we limit ourselves to morphisms from ω to ω. This restriction is not relevant, since all objects of $S(S')$ are isomorphic to ω. Furthermore, we shall omit subscripts, thus writing Δ instead of Δ_ω, Id instead of Id_ω, etc.

Now, let n, m, and d be such that PA $\vdash \forall x \varphi_n x\uparrow$, PA $\vdash \forall x \varphi_m x = x$, and PA $\vdash \forall x \varphi_d x = \langle x,x \rangle$, where $\langle x,y \rangle$ is a PA term representing the pairing function (see [8]); let Id $= [m]$, $0 = [n]$, $\Delta = [d]$ where, for $i \in \omega$, $[i]$ is the equivalence class of i modulo $\sim$ or $\sim'$ according as we are working in S or in S'. Let $\times$ be the bifunctor induced by the pairing function, and let p, q be the natural transformations induced by the projection functions (see [8]). It is easily seen that the axioms of pointed p-categories are satisfied. Note that the domain of a morphism $\varphi = [n]$ is the equivalence class $[m]$ where m is such that PA $\vdash$ $\forall x[(\varphi_m x\uparrow \leftrightarrow \varphi_n x\uparrow) \wedge (\varphi_m x\downarrow \rightarrow \varphi_m x = x)]$. A Turing morphism τ is the one induced by the function $\alpha(e,x) = \varphi_e x$. The defining property of Turing morphisms follows from the formalization of the s-m-n Theorem.

It is easily seen that in both S and S' *Axiom (N1)* of dominical categories is *not verified*: let α, β be undecidable Σ_1^0 sentences such that PA $\vdash \alpha \rightarrow \neg\beta$. Let $\varphi = [e]$, $\psi = [i]$, where e, i are such that PA $\vdash \forall x(\varphi_e x\downarrow \leftrightarrow \alpha)$, and PA $\vdash$ $\forall x(\varphi_i x\downarrow \leftrightarrow \beta)$. Clearly, $\varphi \neq 0$, $\psi \neq 0$, but $\varphi \times \psi = 0$.

Note that *Axiom (N2) is not verified* in $S(S')$ if "total" is understood as "DPH total". This can be seen indirectly, by showing that some theorems of the theory of recursion categories which can be proved without the use of (N1) are not valid in S and S' (cf. for instance, Proposition 5.3 of [2] and Example 1.1 of the present paper).

On the other hand, (N2) becomes true in both S and S' if "total" is understood as "R total". In fact, this property holds in all p-categories (cf. [9]). Lastly, (N3) and (N4) hold in any p-category. This leads to the following trivial, but useful, result.

Proposition 1.1 *Let Φ be a statement provable from the axioms of recursion categories, and suppose that in some proof of Φ Axiom (N1) is never used and in all occurrences of (N2) the word "DPH total" can be replaced by "R total". Then Φ is true in S and in S'. (More generally, Φ is true in every pointed p-recursion category).*

2 Totality As we said before, DPH and R totality coincide in dominical categories. In this section, we show that this is not true in S and S' and, therefore, in p-categories in general. This result will follow from a syntactic classification of totality by category-theoretic means. First, it is easily seen that *R totality* in $S(S')$ corresponds to "*provable totality*" in PA (to *totality* in the real world respectively). Thus, we have a category-theoretic counterpart to the two most common notions of totality. The interpretation of the notion of DPH totality in S and S' is much more surprising, and is closely related to interpretability. First of all let us recall the following:

Proposition 2.1 (See [7]) *For every PA sentence α, the following are equivalent:*
(1) *α is Π_1^0 conservative (i.e., for all $\beta \in \Pi_1^0$, if PA $\vdash \alpha \rightarrow \beta$, then PA $\vdash \alpha$)*
(2) *PA + α is relatively interpretable in PA*
(3) *For all n, PA $\vdash$* $\mathrm{Con}(PA \restriction \bar{n} + \alpha)$, *where* $\mathrm{Con}(PA \restriction \bar{n} + \alpha)$ *is a formula which naturally expresses the consistency of the theory whose nonlogical axioms are α and the axioms of PA whose gödel number is $\leq n$.*

We note that if α,β are Σ_1^0 and Π_1^0 conservative then $\alpha \wedge \beta$ is also.

In the following, $\Pi(v)$ denotes a standard Δ_0 formula binumerating the set of nonlogical axioms of PA; we write $A(\ulcorner B(x)\urcorner)$ for $A(Sb^x_{nm_x}\overline{B(x)})$, where Sb is defined as in [3]; moreover, if $\gamma(x)$ has exactly x free, then we write $\mathrm{Con}(\mathrm{PA} \restriction n + \gamma(x))$ for Con_β, where β is the formula $[\Pi(v) \wedge v \leq \bar{n}] \vee v = Sb^x_{nm_x}\,\overline{\gamma(x)}$. Recall that PA is essentially reflexive, whence, if $\alpha(x)$ is any formula and $n \in \omega$, one has $\mathrm{PA} \vdash \alpha(x) \to \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x))$.

In the sequel, we make use of the following fact: if $\alpha(x)$ is a Σ_1^0 formula, there is an index e such that

(*) $\mathrm{PA} \vdash (\varphi_e x\!\downarrow\ \leftrightarrow \alpha(x)) \wedge (\varphi_e x\!\downarrow\ \to \varphi_e x = x)$.

To prove this, note that, by Theorem 2.13 of [4], if $\beta(z,x,v)$ is Σ_1^0 and

$$\mathrm{PA} \vdash \beta(z,x,v) \wedge \beta(z,x,v) \to v = w,$$

then there is an index e such that

$$\mathrm{PA} \vdash \varphi_e x = v \leftrightarrow \beta(e,x,v).$$

To get the claim, it suffices to apply the above theorem, with $\beta(z,x,v)$ replaced by $z = z \wedge \alpha x \wedge x = v$, thus getting an index e such that $\mathrm{PA} \vdash \varphi_e x = v \leftrightarrow \alpha x \wedge (x = v)$. From this and $\mathrm{PA} \vdash \varphi_e x\!\downarrow\ \leftrightarrow \exists v(\varphi_e x = v)$ we get (*).

Theorem 2.1 *Let $\varphi = [e]$ be a morphism of S. The following are equivalent:*
(1) *φ is DPH total*
(2) *for all n, $PA \vdash \forall x \mathrm{Con}(PA \restriction n + \varphi_e x\!\downarrow)$*
(3) *for all n, $PA \vdash \forall x \mathrm{Con}(PA \restriction n + \forall y \leq x\ \varphi_e x\!\downarrow)$*
(4) *for all $\alpha x \in \Pi_1^0$, if $PA \vdash \forall x[\varphi_e x\!\downarrow\ \to \alpha x]$, then $PA \vdash \forall x \alpha x$.*

Proof: First, we prove that (2) and (3) are equivalent. (3) $\Rightarrow$ (2) is trivial. To prove (2) $\Rightarrow$ (3), one uses formal induction on x; for $x = 0$, the claim is trivial; to prove the induction step, let q be such that $\mathrm{PA} \restriction q$ contains Robinson's Q. If $\alpha(x),\beta(x)$ are Σ_1^0 formulas and $n \geq q$, one has:

(i) $\mathrm{PA} \vdash \alpha(x) \wedge \mathrm{Con}(\mathrm{PA} \restriction \mathrm{n} + \beta(x)) \to Pr_{\mathrm{PA}\restriction n}(\ulcorner\alpha(x)\urcorner) \wedge \mathrm{Con}(\mathrm{PA} \restriction n + \beta(x)) \to \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x) \wedge \beta(x))^2$

(ii) $\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA} \restriction n + \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x))) \leftrightarrow \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x))$.

Moreover, if $\gamma(x)$, $\delta(x)$ are arbitrary formulas, then

(iii) If $\mathrm{PA} \vdash \forall \mathrm{x}[\gamma(x) \to \delta(x)]$, then $\exists m \forall n > m$ $\mathrm{PA} \vdash \forall x[\mathrm{Con}(\mathrm{PA} \restriction n + \gamma(x)) \to \mathrm{Con}(\mathrm{PA} \restriction n + \delta(x))]$.

Now, let $\alpha(x),\beta(x)$ be $\forall y \leq x\ \varphi_e y\!\downarrow$ and $\varphi_e(x+1)\!\downarrow$ respectively; by (2) and (i),

$$\mathrm{PA} \vdash \alpha(x) \to \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x) \wedge \beta(x)).$$

By (iii), there is an $m > q$ such that for all $n > m$

$$\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x)) \to$$
$$\mathrm{Con}(\mathrm{PA} \restriction n + \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x) \wedge \beta(x))).$$

By (ii), there is an m such that for all $n > m$

$$\mathrm{PA} \vdash \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x)) \to \mathrm{Con}(\mathrm{PA} \restriction n + \alpha(x) \wedge \beta(x)).$$

Thus, for all m there is an $n > m$ such that PA $\vdash \forall x[\mathrm{Con}(\mathrm{PA} \upharpoonright n + \forall y \leq x\, \varphi_e y\downarrow) \to \mathrm{Con}(\mathrm{PA} \upharpoonright n + \forall y \leq x + 1 \varphi_e y\downarrow)]$. Using induction we get: $\exists m \forall n > m$ PA $\vdash \forall x \mathrm{Con}(\mathrm{PA} \upharpoonright n + \forall y \leq x\, \varphi_e y\downarrow)$; from this we easily get: $\forall n$ PA $\vdash \forall x \mathrm{Con}(\mathrm{PA} \upharpoonright n + \forall y \leq x \varphi_e y\downarrow)$.

(1) ⇒ (4): Assume that PA $\vdash \forall x[\varphi_e x\downarrow \to \alpha(x)]$, $\alpha(x) \in \Pi^0_1$. Let i be such that

$$\mathrm{PA} \vdash \forall x[(\varphi_i x\downarrow \leftrightarrow \neg\alpha(x)) \wedge (\varphi_i x\downarrow \to \varphi_i x = x)] \text{ (Cf.}(*)).$$

We get:

$$\begin{aligned} \mathrm{PA} \vdash \forall x[\varphi_e \varphi_i x\downarrow &\to \exists y(\varphi_i x = y \wedge \varphi_e y\downarrow)] \\ &\to \varphi_i x = x \wedge \varphi_e x\downarrow] \\ &\to \alpha(x) \wedge \neg\alpha(x)]. \end{aligned}$$

We conclude:

$$\mathrm{PA} \vdash \forall x \varphi_e \varphi_i x\uparrow.$$

Letting $\psi = [i]$, we get $\varphi\psi = 0$. Since φ is DPH total, we conclude that $\psi = 0$.

Thus, PA $\vdash \forall x\, \varphi_i x\uparrow$. By the definition of i, PA $\vdash \forall x\, \alpha(x)$.

(4) ⇒ (2): Since, by the essential reflexiveness of PA, for all n, PA $\vdash \forall x[\varphi_e x\downarrow \to \mathrm{Con}(\mathrm{PA} \upharpoonright n + \varphi_e x\downarrow)]$, condition (4) yields PA $\vdash \mathrm{Con}(\mathrm{PA} \upharpoonright n + \varphi_e x\downarrow)$ (this sentence being Π^0_1).

(2) ⇒ (1): Let q be such that PA $\upharpoonright q$ contains Robinson's Q. Then for any Σ^0_1 formula $\alpha(x)$, PA $\vdash \forall x[\alpha(x) \to Pr_{\mathrm{PA}\upharpoonright q}\ulcorner \alpha(x) \urcorner]$. Let $\psi = [i]$ be such that $\varphi\psi = 0$. Then, PA $\vdash \forall x \varphi_e \varphi_i x\uparrow$, whence, for some $n > q$, PA $\vdash Pr_{\mathrm{PA}\upharpoonright n}\overline{\forall x \varphi_e \varphi_i x\uparrow}$. *A fortiori*, PA $\vdash \forall x Pr_{\mathrm{PA}\upharpoonright n}\ulcorner \varphi_e \varphi_i x\uparrow \urcorner$.

Now,

$$\begin{aligned} \mathrm{PA} \vdash \psi_i x\downarrow &\to \exists y(\varphi_i x = y) \\ &\to \exists y Pr_{\mathrm{PA}\upharpoonright n}(\ulcorner \varphi_i x = y \urcorner) \wedge \forall z Pr_{\mathrm{PA}\upharpoonright}(\ulcorner \varphi_e \varphi_i z\uparrow \urcorner) \\ &\quad \exists u Pr_{\mathrm{PA}\upharpoonright n}\ulcorner \varphi_e u\uparrow \urcorner. \end{aligned}$$

Since $\forall k$, PA $\vdash \forall u \mathrm{Con}(\mathrm{PA} \upharpoonright k + \varphi_e u\downarrow)$, we conclude that PA $\vdash \forall x \varphi_i x\uparrow$; i.e., $\psi = 0$.

DPH total morphisms of S also have a model-theoretic characterization. Let $\varphi = [e]$ be DPH total, M be a model of PA; by (3), for all n, $M \models \forall x \mathrm{Con}(\mathrm{PA} \upharpoonright n + \forall y \leq x\, \varphi_e y\downarrow)$. Let T be PA $\upharpoonright n + \{\varphi_e b\downarrow : b \in M\}$. In M, there is no proof of any contradiction from the axioms of T (otherwise, by the least number principle, we would get an inconsistency in M of PA $\upharpoonright n + \forall y \leq b\, \varphi_e y\downarrow$, for some $b \in M$). By Overspill, we get an M-definable model M' *of* PA such that M possesses a truth-definition (satisfying the obvious conditions for truth) for M' and, for all $b \in M$, $M' \models \varphi_e b\downarrow$; i.e., the domain of φ_e in M' contains M. Conversely, this condition implies that for every model M of PA, and for every $n \in \omega$, one has $M \models \forall x \mathrm{Con}(\mathrm{PA} \upharpoonright \mathrm{n} + \varphi_e x\downarrow)$, whence by Theorem 2.1, φ is DPH total.

It is clear that condition (3) of Theorem 2.1 is satisfied if PA + "$\forall x \varphi_e x\downarrow$" is relatively interpretable in PA. This suggests the following problem.

Problem 2.1 *Is there a DPH total morphism $\varphi = [e]$ of S such that PA + "$\forall x \varphi_e x\downarrow$" is not interpretable in PA?*

A different characterization of DPH total morphisms in S and S', and, more generally, in any p-category is obtained by means of the notion of a dense domain.

Definition 2.1 A domain ϵ is called *dense* iff for every nonzero domain δ, $\epsilon\delta \neq 0$.

Trivially, Id is a dense domain; a dense domain which does not coincide with Id will be called *nontrivial*. By Proposition 5.3 of [2], in a dominical category such that each morphism φ has a *range* (that is, a domain $\mathrm{ran}\varphi$ such that $(\mathrm{ran}\varphi)\varphi = \varphi$ and if $\psi\varphi = \psi'\varphi$, then $\psi\mathrm{ran}\varphi = \psi'\mathrm{ran}\varphi$ (cf. [2]), there are no nontrivial dense domains. On the contrary, even though in both S and S' each morphism $\varphi = [i]$ has a range (it suffices to consider an n such that $\mathrm{PA} \vdash \forall x[(\varphi_n x{\downarrow} \leftrightarrow \exists y(\varphi_i y = x)) \wedge (\varphi_n x{\downarrow} \to \varphi_n x = x)])$ and to define $\mathrm{ran}\varphi = [n]$), we will prove in Example 1 below that in both categories there are nontrivial dense domains. This constitutes another difference between dominical categories and p-categories. We also observe that dense domains constitute a filter if the composition is assumed as infimum of two domains and, consequently, inclusion is defined by $\mathrm{dom}\varphi \subseteq \mathrm{dom}\psi$ iff $\mathrm{dom}\varphi\mathrm{dom}\psi = \mathrm{dom}\varphi$.

Theorem 2.2 *Let $\mathcal{C}$ be any pointed p-category where each morphism has a range. Then a morphism φ is DPH total iff $\mathrm{dom}\varphi$ is dense.*

Proof: First of all, we show, using some properties of domains proved in [2] for dominical categories and extended in [9] to p-categories, that φ is DPH total iff $\mathrm{dom}\varphi$ is. This follows from the fact that $\varphi\psi = 0$ iff $\mathrm{dom}(\varphi\psi) = 0$ iff $\mathrm{dom}((\mathrm{dom}\varphi)\psi) = 0$ iff $(\mathrm{dom}\varphi)\psi = 0$. Now, it is clear that, if $\mathrm{dom}\varphi$ is DPH total, it is dense. Vice versa, suppose that $\mathrm{dom}\varphi$ is dense; if $(\mathrm{dom}\varphi)\psi = 0$, then $\mathrm{dom}\varphi(\mathrm{ran}\psi)\psi = 0 = 0\psi$. By the definition of $\mathrm{ran}\psi$, this entails that $\mathrm{dom}\varphi\mathrm{ran}\psi$ $\mathrm{ran}\psi = 0$ *$\mathrm{ran}\psi$* $= 0$, whence $\mathrm{dom}\varphi\mathrm{ran}\psi = 0$. Since any range is a domain, and $\mathrm{dom}\varphi$ is dense, $\mathrm{ran}\psi = 0$, therefore $\psi = 0$. Thus, $\mathrm{dom}\varphi$ is DPH total.

We are now in a position to characterize DPH total morphisms of S'.

Theorem 2.3 *Let $\varphi = [e]$ be a morphism of S'. The following are equivalent:*
(1) *φ is DPH total*
(2) *for all n, the sentence "$\varphi_e n{\downarrow}$" is Π^0_1 conservative*
(3) *for all n, PA + "$\varphi_e n{\downarrow}$" is interpretable in PA*
(4) *PA + {"$\varphi_e n{\downarrow}$": $n \in \omega$} is interpretable in PA*
(5) *$\mathrm{dom}\varphi$ is dense*
(6) *every model M of PA can be extended to an M-definable model M' such that M has a truth definition for M' and for all $b \in \omega$, $M' \vDash \varphi_e b{\downarrow}$.*

Proof: The equivalence of (2) and (3) follows from Proposition 2.1 and the equivalence of these and (4) follows from the compactness of interpretability and from the fact that, if α, β are Σ^0_1 sentences such that both PA + α and PA + β are interpretable in PA, then α and β are Σ^0_1 and Π^0_1 conservative, whence $\alpha \wedge \beta$ is and PA + $\alpha \wedge \beta$ is interpretable in PA. The equivalence of (1) and (5) follows from Theorem 2.2, and the proofs of (1) ⇒ (2) and (2) ⇒ (1) are very similar to the proofs of (1) ⇒ (4) and (2) ⇒ (1) of Theorem 2.1. Lastly, (6) is easily shown to be equivalent to (4).

Remark 2.1 We have seen that S and S' are not dominical. One could ask whether it is possible to define another near product with respect to which the above categories become dominical. It turns out that, if such a near product exists, it does not satisfy some very natural conditions; for example, there cannot be a near product making S dominical and for which there exists a recursive function f such that, for all n, $[f(n)] = \mathrm{dom}[n]$. Indeed, since in a dominical category a morphism φ is DPH total iff $\mathrm{dom}\varphi = \mathrm{Id}$, such an f would reduce the set $T = \{n\colon [n] \text{ is DPH total}\}$ to the recursively enumerable set $\{i\colon [i] = \mathrm{Id}\}$. This is a contradiction, because, using the fact that the set of Σ_1^0 sentences α such that PA + α is interpretable in PA is Π_2^0 complete (see [5] or [10]), one easily proves T to be Π_2^0 complete too. We conclude this section with two examples.

Example 1 Let n be such that $\mathrm{PA} \vdash \forall x[(\varphi_n x\downarrow \leftrightarrow \neg\mathrm{Con}(\mathrm{PA})) \wedge (\varphi_n x\downarrow \rightarrow \varphi_n x = x)]$ (where Con(PA) is the usual consistency statement for PA). Since $\neg$Con(PA) is Π_1^0 conservative (see [3]), $[n]$ is DPH total in both S and S'. Since φ_n is everywhere divergent in the real world, $[n]$ is not R total. Also, by Theorem 2.2, $[n]$ constitutes a nontrivial dense domain.

Example 2 Let e be such that $\mathrm{PA} \vdash \forall x[(\varphi_e x\downarrow \leftrightarrow \neg Prf_{\mathrm{PA}}(x,\overline{\bar{0} = \bar{1}})) \wedge (\varphi_e x\downarrow \rightarrow \varphi_e x = x)]$. φ_e is total in the real world, whence $[e]$ is R total (and DPH total) in S'; however, $[e]$ is not DPH total (whence it is not R total) if it is thought of as a morphism of S. Indeed, let i be such that

$$\mathrm{PA} \vdash \forall x[(\varphi_i x\downarrow \leftrightarrow Prf_{\mathrm{PA}}(x,\overline{\bar{0} = \bar{1}})) \wedge (\varphi_i x\downarrow \rightarrow \varphi_i x = x)].$$

Clearly, in S, $[i] \neq 0$, $[e][i] = 0$.

3 Creative domains By Proposition 1.1, many classical theorems of recursion theory, which hold in all recursion categories, can be extended to p-recursion categories, and to S and S' in particular. For example, the Recursion theorem as stated in Theorem 4.5.1 of [2] holds in any p-recursion category. In this section, we are particularly interested in creative domains. We shall see that in both S and S', there are creative and effectively inseparable domains, but, unlike the classical case, there are creative domains which are not complete. The presence of such "pathologies" is a typical aspect of S and S'.

Definition 3.1 A *c-p (recursion) category* is a p-(recursion) category where each morphism φ has a section (i.e., a morphism σ such that $\varphi\sigma = \mathrm{dom}\sigma$ and $\varphi\sigma\varphi = \varphi$). A c-p (recursion) category having *coproducts* compatible with the near product in the sense of [2] is called a *c^+-p (recursion) category*. *c^+-dominical* and *c^+-recursion categories* are defined similarly (cf. [2]).

Definition 3.2 A *constant* is an R total morphism c such that, for all R total f,g, $cf = cg$. An *atom* is an R total morphism a such that, for every domain ϵ, either $\epsilon a = 0$ or $\epsilon a = a$.

Both S and S' are c^+-p recursion categories: coproducts are defined in the obvious way by means of the disjoint union and a section σ of a morphism $[n]$ is defined as the equivalence class of a number e such that the condition

$$\varphi_e x = \begin{cases} \Pi_1 \text{ (first } z \text{ in the list of the graph of } \varphi_e \text{ such that} \\ \qquad \Pi_2 z = x) \text{ if such a } z \text{ exists (see [8])} \\ \text{divergent otherwise} \end{cases}$$

is provable in PA.

The constants of $S(S')$ are the equivalence classes of gödel numbers of provably constant (respectively, constant) recursive functions. Even though in c-dominical categories any constant is an atom, S and S' have *no atoms at all.* Indeed, let $a = [e]$ be an R total morphism, $k = \varphi_e(0)$, α be an undecidable Σ_1^0 sentence; let n be such that $\mathrm{PA} \vdash \forall x[(\varphi_n x\downarrow \leftrightarrow (x \neq \bar{k} \vee \alpha)) \wedge (\varphi_n x\downarrow \rightarrow \varphi_n x = x)]$. It is easily seen that in both S and S', $[n]$ is a domain, $[n]\cdot a \neq 0$, $[n]\cdot a \neq a$.

Definition 3.3 A category has *sufficiently many atoms* iff for all morphisms φ, ψ, if $\varphi \neq \psi$, there is an atom a such that $\varphi a \neq \psi a$. A category has *sufficiently many constants* iff for all φ, ψ, $\varphi \neq \psi$ implies that $\varphi c \neq \psi c$ for a suitable constant c.

Clearly neither S nor S' has sufficiently many atoms. Moreover, S' has sufficiently many constants, where S does not.

Definition 3.4 A domain ϵ is *creative* with respect to the constants iff for some (hence for each) Turing morphism τ: $X \times X \rightarrow X$, there is an R total morphism h such that, for all constants c, if $\tau(c \times \epsilon) = 0$, then $\epsilon hc = \tau\langle c, hc\rangle = 0$. In this case, h is said to be a productive morphism for ϵ.

Letting $\epsilon = [e]$, $h = [i]$, the above condition becomes equivalent in $S(S')$ to the following one: For $n \in \omega$, if $\mathrm{PA} \vdash \forall x(\varphi_n x\downarrow \rightarrow \varphi_e x\uparrow)$ (respectively, if for all i, $\mathrm{PA} \vdash \varphi_n i\downarrow \rightarrow \varphi_e i\uparrow$), then $\mathrm{PA} \vdash \varphi_e \varphi_i n\uparrow \wedge \varphi_n \varphi_i n\uparrow$.

By Proposition 1.1, the proof of Theorem 8.7 of [2] can be extended to any c-p recursion category. Thus, $\mathrm{K} = \mathrm{dom}\tau\Delta$ is creative in both S and S'. It follows from Theorem 8.13 of [2] and from the fact that in any c^+-dominical category every constant is an atom, that in any c^+-recursion category with sufficiently many constants and in which Δ has an inverse, each creative domain ϵ is *complete*: i.e., for each domain δ, there is an R total morphism h such that $\delta = \mathrm{dom}(\epsilon h)$. We shall see that this property fails to hold in a very strong way in both S and S', even though S' has sufficiently many constants and Δ has an inverse in both S and S'. More precisely, we shall prove that in both S and S' there are infinitely many creative domains which are incomparable with respect to a very weak kind of reducibility.

Definition 3.5 Let ϵ, δ be domains. δ is said to be *reducible* to ϵ (abbreviated as $\delta \leq \epsilon$) iff there is an R total morphism h such that $\delta = \mathrm{dom}(\epsilon h)$.

Clearly, ϵ is complete iff, for all δ, $\delta \leq \epsilon$.

Definition 3.6 We say that δ is *weakly reducible* to ϵ (abbreviated as $\delta \leq_w \epsilon$) iff there is an R total morphism h such that, for each constant c, δc is total iff ϵhc is.

It is clear that weak reducibility is a much weaker condition than reducibility and corresponds to reducibility in the real world.

Theorem 3.1 *In both S and S' there are infinitely many creative domains which are mutually incomparable with respect to $\leq_w$ and all have Id as productive morphism.*

Proof: Let $X = \{n\colon \mathrm{PA} \vdash \varphi_n n\uparrow\}$, and let Y be any recursively enumerable set disjoint from X. By a result proved in both [5] and [10], there is a Σ^0_1 formula $\alpha(x)$ such that the following conditions hold:

(1) If $n \in Y$, then $\mathrm{PA} \vdash \alpha(\bar{n})$
(2) If $n \in X$, then $\mathrm{PA} \vdash \neg\alpha(n)$
(3) If $n \notin X \cup Y$, then $\alpha(\bar{n})$ is undecidable and Π^0_1 conservative.

Now, let e be such that

$$\mathrm{PA} \vdash \forall x[(\varphi_e x\downarrow \leftrightarrow \alpha(x)) \wedge (\varphi_e x\downarrow \to \varphi_e x = x)]$$

and let $\epsilon^Y = [e]$.

We wish to prove that ϵ^Y is creative (in both S and S') with Id as production morphism. To see this, first note that, for any $n \in \omega$, if $n \in X$, then $\mathrm{PA} \vdash \neg\alpha(\bar{n})$, therefore $\mathrm{PA} \vdash \varphi_e n\uparrow$; moreover, in this case, by the definition of X, $\mathrm{PA} \vdash \varphi_n n\uparrow$. Observe also that, if $n \notin X$, then $\alpha(\bar{n})$ is Π^0_1 conservative (this is true also if $n \in Y$, because provable sentences are trivially Π^0_1 conservative). Furthermore, PA + "$\varphi_n n\downarrow$" is consistent, therefore, by the Π^0_1 conservativeness of $\alpha(\bar{n})$, PA + "$\varphi_n n\downarrow$" + $\alpha(\bar{n})$ = PA + "$\varphi_n n\downarrow$" + "$\varphi_e n\downarrow$" is in turn consistent. Now, let m be given; if, for all i, $\mathrm{PA} \vdash \varphi_m i\downarrow \to \varphi_e i\uparrow$ (*a fortiori*, if $\mathrm{PA} \vdash \forall x(\varphi_m x\downarrow \to \varphi_e z\uparrow)$), then $\mathrm{PA} \vdash \varphi_m m\downarrow \to \varphi_e m\uparrow$, and PA + "$\varphi_m m\downarrow$" + "$\varphi_e m\downarrow$" is not consistent. By what we proved above, this implies that $m \in X$ and $\mathrm{PA} \vdash \varphi_m m\uparrow \wedge \varphi_e m\uparrow$. By the remark following the definition of creativeness, this is sufficient to show that ϵ^Y is creative with Id as productive morphism. Now, let $K = \{m\colon \varphi_m m\downarrow\}$. Since $K \cap X = \emptyset$, for all recursively enumerable $Y \subseteq K$, the corresponding ϵ^Y defined as above is creative. Moreover, it is an easy exercise to show that K (and in general every infinite recursively enumerable set) contains infinitely many recursively enumerable sets mutually incomparable with respect to m-reducibility. Let $Y_1, \ldots, Y_n, \ldots$ be such recursively enumerable subsets. It is readily seen that the corresponding domains $\epsilon^{Y_1}, \ldots, \epsilon^{Y_n}, \ldots$, defined in a similar way as ϵ^Y but with $Y_1, \ldots, Y_n \ldots$ in place of Y, are creative and mutually incomparable with respect to $\leq_w$.

Remark 3.1 The above-introduced creative domains are "pathological", in the sense that they are not creative in the real world. We do not know whether there are pathological pairs of effectively inseparable (e.i.) domains of $S(S')$. (Recall that ϵ, δ are said to be e.i. iff $\epsilon\delta = 0$ and there is a total morphism f such that, whenever h, k are indices (in the sense of [2]) of domains ϵ', δ' such that $\epsilon' \supseteq \epsilon$, $\delta' \supseteq \delta$, $\epsilon'\delta' = 0$, then $\epsilon' f\langle h,k\rangle = \delta' f\langle h,k\rangle = 0$.) Note that, by Proposition 1.1, Theorem 8.18 of [2] can be extended to S and S', therefore, there are pairs of e.i. domains in both S and S'.

Concluding remark We have not been able to find a reasonable category-theoretic version of the incompleteness theorems (i.e., one which is not a pedestrian transposition of well-known proofs in the language of categories). Our contribution to this problem could be the following: in this paper, we have seen that many concepts of recursion theory have a different meaning according as they are interpreted in the classical recursion category of partial recursive functions or in a syntactical category like S or S'. This reflects the difference between truth and provability. Since both these concepts play a fundamental role in the incompleteness theorems, a reasonable conjecture could be that a good analysis of the incompleteness theorems by category theoretic means should require consideration of both a recursion category, reflecting true arithmetic (or, equivalently, recursive functions in the real world) and of a p-recursion category, reflecting formal arithmetic (or, equivalently, recursive functions considered with the eyes of PA). Of course, the problem is how to connect these categories together. Even if we are interested in this problem, we think that it is better for it to be dealt with by mathematicians who are better situated than the author with respect to the theory of categories.

NOTES

1. More precisely, we fix once and for all two Δ_0 formulas $\tilde{T}(x,y,z)$ binumerating Kleene's predicate and $\tilde{U}(x,y)$ representing the primitive recursive function $U(x)$ defined by

$$U(x) = \begin{cases} \text{the final element of the computation coded by } x \\ \quad \textit{if } x \text{ codes a computation} \\ 0 \text{ otherwise} \end{cases}$$

and define $\varphi_n x$, $\varphi_n x = y$, $\varphi_n x = \varphi_m x$ etc. by means of $\tilde{T}(x,y,z)$ and $\tilde{U}(x,y)$ in the obvious way; thus, e.g., $\varphi_n x\!\downarrow$ is an abbreviation for $\exists z \tilde{T}(n,x,z)$, $\varphi_n x = y$ is an abbreviation for $\exists z[\tilde{T}(n,x,z) \wedge \tilde{U}(z,y)]$, and so on.

2. Here, of course, $Pr_{\mathrm{PA}}(x)$, $Pr_{\mathrm{PA}\restriction n}(x)$, denote $Pr_{\Pi(v)}(x)$ and $Pr_{\Pi(v)\restriction n}(x)$ respectively (cf. [3]), where $\Pi(v)$ is the abovementioned binumeration of the axioms of PA, and $\Pi(v) \restriction n$ is $\Pi(v) \wedge v \leq \bar{n}$. $Prf_{\mathrm{PA}}(x,y)$ and $Prf_{\mathrm{PA}\restriction n}(x,y)$ are defined similarly.

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