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Isomorphisms of Finite Cylindric Set Algebras of Characteristic Zero

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Abstract The basic fact of cylindric algebraic model theory according to which any pair of isomorphic finite-dimensional cylindric set algebras of positive characteristic are base-isomorphic (J. D. Monk) can be extended in a natural way to some algebras of characteristic zero. Moreover, no further improvement is possible in any obvious way.

1 The theorem proved by J. D. Monk, which is the algebraic version of the logical result stating that any two elementarily equivalent finite models are isomorphic, is in fact a generalization of the logical theorem since it includes the algebraic counterpart of the case of languages with finitely many variables. That is, it partly claims that *if α is finite, then any two isomorphic α-dimensional cylindric set algebras are base-isomorphic whenever one of the algebras has a base of power* $< \alpha$ (Henkin et al. [8] I.3.6, Henkin et al. [7] 3.1.38 (1)). (Recall from [7] that for any set U and ordinal α, an *α-dimensional cylindric set algebra* (Cs_α) with base U is a Boolean algebra of subsets of ${}^\alpha U$ supplemented by distinguished elements $d_{ij} = \{p \in {}^\alpha U : p_i = p_j\}$ for any $i, j \in \alpha$, and operations defined for any $i \in \alpha$ as follows: $c_i X = \{p \in {}^\alpha U : (\exists q \in X)(\forall k \in \alpha,\ k \neq i)\ q_k = p_k\}$ for any $X \subseteq {}^\alpha U$. On the other hand, f is a *base-isomorphism* of a Cs_α $\mathfrak{A}$ if there is a one-to-one function g such that $fa = \{g \circ x : x \in a\}$ for any $a \in A$.) The aim of this paper is to investigate what happens when the power of the base reaches and even exceeds the bound set by the dimension. Actually, we shall prove the following theorem.

Theorem *If α is finite, then*

(i) *Any isomorphism between α-dimensional cylindric set algebras is a base-isomorphism if one of the algebras is not minimal and has a base of power $<$ $\alpha + 1$ or the bases of both algebras have the same power $< \alpha + 2$.*

(ii) *For any finite $\alpha \geq 2$, $n \geq m \geq \alpha + 2$ or $n > m \geq \alpha + 1$, there are nonminimal isomorphic but not base-isomorphic (not even lower base-isomorphic) α-dimensional cylindric set algebras with bases n and m, respectively.*

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As a matter of fact, as we shall see, our proof of (i) will also work for all (not only for nonminimal) algebras if one of them has a base of power $< \alpha$ and indeed it will directly imply the truth of the infinite dimensional case as well (assuming then, of course, the regularity and local finiteness of the algebras concerned) giving an alternative proof of Monk's original theorem. Actually, Monk's theorem can also be given in a slightly strengthened form guaranteeing not only the existence of a base-isomorphism between the algebras concerned but stating that *all* isomorphisms between them are in fact base-isomorphisms.

Our theorem above is related to some results of Andréka, Comer, and Németi concerning various properties of finite-dimensional Cs_α's such as, for example, the minimal number of generators, the homogeneity of full set algebras, or the term definability of substitutions (cf. Comer [6] Theorem 2.5 and 3.7, Andréka and Németi [3]). In most of these cases, as in ours, on the one hand, the positive results are established for algebras with base of power $< \alpha + 2$; on the other hand, this bound is proved to be the best possible.

As far as applications of the Theorem are concerned, the proposition generalizing Monk's original result to the algebraic version of classes of models (due to Andréka and Németi, see [8] Prop. 3.4 on p. 157) can be strengthened so as also to include some algebras of characteristic zero. Further, the model theoretic fact according to which compact models for languages with an infinite sequence of variables of length α are $|\alpha|^+$-universal (cf. Németi [10] Theorem 2, Chang and Keisler [5] Ex. 4.3.24 on p. 211, Serény [11]) can *without any change* be extended to models for languages with finitely many variables. The homogeneity of full Cs_α's with bases of power $< \alpha + 2$ (cf. [6] Th. 3.7) can also be directly derived from our result. Finally, Andréka, Düntsch, and Németi [2] have recently used both parts of the Theorem to show that, in the case of a finite dimensional Cs_α with base of cardinality μ, every so-called permutation invariant function is term definable whenever $\mu < \alpha + 2$ and otherwise this fails to remain true.

The second part of the Theorem shows that the first part cannot be improved in any obvious way. (The case of minimal algebras is taken care of in [7] 3.1.38 and in Lemma 14 below.) In addition, it improves the counterexample of Biró [4] showing that the bases of algebras needed to prove the existence of isomorphic but not lower base-isomorphic Cs_α's are not necessarily different.

2 Now, let us turn to the proof of the Theorem. First, a few words about notation. *α is always a finite ordinal* except when explicitly stated otherwise. We use the terminology and notation of [7] on cylindric algebras, among others $\bar{d}^{\mathfrak{A}}(R) = \prod_{\substack{\langle i,j\rangle \in R \\ i \neq j}} - d_{ij}^{\mathfrak{A}} (\mathfrak{A} \in Cs_\alpha,\ R \subseteq \alpha \times \alpha)$. Moreover, we set $\bar{d}^{\mathfrak{A}} = \bar{d}^{\mathfrak{A}}(\alpha \times \alpha)$ and omit superscripts when no confusion is likely. We follow the notation of [7] even when that notation is not the most widely used. For example, $H \upharpoonright f$ will denote the restriction of a function f to a set H (cf. [7], p. 28). f_u^k is the function differing from a given function f only at place k where its value is u (cf. [7] 3.1.1). Trying to keep the exposition self-contained we shall give the definitions of the key notions when they first appear.

In order to prove the first part of the Theorem itself, we should make some preparations. The first fact we need is related to the description of atoms of simple minimal cylindric algebras in [7] 2.4.68 (see also [6]).

Lemma 1 *If $\bar{d}$ is an atom of a $Cs_\alpha\mathfrak{A}$, then $\mathfrak{A}$ is minimal.*

Proof: Suppose that $\mathfrak{A}$ is not minimal. By [7] 1.9.2(iii), we may assume that $\alpha \geq 2$. Let $Mn\mathfrak{A}$ be the universe of the minimal algebra of $\mathfrak{A}$, and let U be the base of $\mathfrak{A}$. We may suppose that $|U| \geq \alpha$ since otherwise $\bar{d} = 0$. For any $q \in {}^\alpha U$, let $q^+ = \langle \min\{j \in \alpha : q_j = q_i\} : i \in \alpha \rangle$, $H_q = Rg\, q^+$, $P_q = \bar{d}(H_q \times H_q) \cdot (\prod_{i \in \alpha \sim H_q} d_{iq_i^+})$. Now, there is an $X \in A \sim Mn\mathfrak{A}$ and $q \in X$ such that $P_q X \notin Mn\mathfrak{A}$ since otherwise for all $X \in A$, $X = \sum_{q \in X} P_q X \in Mn\mathfrak{A}$. Thus $c_{(\alpha \sim H_q)}(P_q X) \notin Mn\mathfrak{A}$ since $P_q X = c_{(\alpha \sim H_q)}(P_q X) \cdot (\prod_{i \in \alpha \sim H_q} d_{iq_i^+})$. Therefore $0 < c_{(\alpha \sim H_q)}(P_q X) < \bar{d}(H_q \times H_q)$, which in turn, by $|U| \geq \alpha$, implies that $0 < c_{(\alpha \sim H_q)} P_q X \cdot \bar{d}(H_q \times (\alpha \sim H_q)) \cdot \bar{d}((\alpha \sim H_q) \times \alpha) < \bar{d}$ showing that $\bar{d}$ is not an atom of $\mathfrak{A}$.

By [7] 3.1.38(8), we obtain the next Lemma.

Lemma 2 *The power of bases of isomorphic Cs_α's are the same if one of these cardinalities is $<\alpha$.*

Now, we shall show that

Lemma 3 *The power of bases of isomorphic nonminimal Cs_α's are the same if one of these cardinalities is $\leq\alpha$.*

Proof: We may suppose that $\alpha \geq 2$. The claim follows directly from the two facts below, where $\mathfrak{C}$ is a Cs_α with base T, and for any $i \in \alpha$, $e_i(x)$ is the following equation: $\bar{d}(\{i\} \times \alpha) \cdot c_i x = x$.

(3.1) If $|T| = \alpha$, then

$$(\forall x \in C)\,[0 < x \leq \bar{d} \Rightarrow (\forall i \in \alpha) e_i(x)].$$

(3.2) If $|T| > \alpha$, then

$$(\forall x \in C)\,[(0 < x \leq \bar{d} \wedge (\forall i \in \alpha) e_i(x)) \Rightarrow x = \bar{d}].$$

Indeed, if $\mathfrak{A}$ and $\mathfrak{B}$ are nonminimal Cs_α's with bases U and W, respectively, f is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then, by Lemma 2, we may suppose that $|U| = \alpha$ and $|W| \geq \alpha$. Further, by Lemma 1, there is an $X \in A$ such that $0 < X < \bar{d}^{\mathfrak{A}}$, so that $0 < fX < \bar{d}^{\mathfrak{B}}$. Thus, by (3.1), $(\forall i \in \alpha) e_i(X)$ holds implying the truth of $(\forall i \in \alpha) e_i(fX)$. So, if $|W| > \alpha$, then, by (3.2), we have $fX = \bar{d}$, which is a contradiction. Since (3.1) is nothing more than the formal expression of the trivial fact that any repetition-free sequence with the same finite length as the power of its range is uniquely determined by all but one of its elements, to establish Lemma 3, only (3.2) remains to be proved. Let us therefore suppose that $|T| > \alpha$ and assume the remaining hypotheses of (3.2) for an arbitrary $X \in C$. Then

(*) For any $i \in \alpha$, $x \in X$ and $t \in T \sim Rg(i \upharpoonright x)$,
there is a $y \in X$ such that $i \upharpoonright x = i \upharpoonright y$ and $y_i = t$.

In fact, if $t \notin Rg((\alpha \sim i) \upharpoonright x)$ holds, then $y = x_t^i \in X$ satisfies the requirements. If not, let $j = x^{-1}t$. Then $j > i$. Moreover, there is a $u \in T \sim Rg\, x$ since $|Rg\, x| = \alpha < |T|$. Now, $x_u^j \in X$ since $e_j(X)$ holds, thus $y = (x_u^j)_t^i \in X$ also, which proves (*).

An induction argument based on (*) shows that for any $x \in \bar{d}$ and $0 \leq i \leq \alpha$, there is a $y \in X$ such that $i \upharpoonright y = i \upharpoonright x$, which in turn, with $i = \alpha$, yields $\bar{d} \leq X$.

This concludes the proof of (3.2) since $X \leq \bar{d}$ by the hypotheses. Consequently the proof of Lemma 3 is also complete.

Turning to the case when the power of the base is just $\alpha + 1$, we introduce some notation and prove a simple technical proposition.

Definition 4 Let $\mathfrak{C}$ be a Cs_α with base T, $|T| = \alpha + 1$.

(i) For any $i \in \alpha$ and $q \in \bar{d}$, let $m_i q = q_t^i$, where t is the unique element of $T \sim Rg\, q$ (cf. [6]).

(ii) For any $i \in \alpha$ and $X \in C$, let

$$M_i^{\mathfrak{C}} X = -X \cdot c_i(X \cdot \bar{d}) + -c_i(-X \cdot \bar{d}),$$

where the superscript will often be omitted.

Proposition 5 *Under the conditions above, for any $i \in \alpha$, $q \in \bar{d}$, and $X \in C$,*

$$q \in M_i X \quad \text{iff} \quad m_i q \in X.$$

Proof: First, let $q \in M_i X$. If $q \in -X \cdot c_i(X \cdot \bar{d})$, then there is a $t \in T$ such that $q_t^i \in X \cdot \bar{d}$. Thus $t \notin Rg\, q$ since $q \notin X$, that is $m_i q \in X$. If $q \in -c_i(-X \cdot \bar{d})$, then $m_i q \notin -X \cdot \bar{d}$. That is, $m_i q \in X$ again since $m_i q \in \bar{d}$ by definition. Now, let $m_i q \in X$. If $q \in X$ also, then $q \notin c_i(-X \cdot \bar{d})$, so $q \in M_i X$. If $q \notin X$, then $q \in -X \cdot c_i(X \cdot \bar{d})$ since $m_i q \in \bar{d}$ by definition. Consequently, $q \in M_i X$.

As a last step in our preparations, we recall from [3] a result of Andréka and Németi on the existence of substitution operators.

Lemma 6 *Let K be the class of Cs_α's with base of power $<\alpha + 2$. There is a function s defined on $K \times {}^\alpha\alpha$ such that s assigns to any $\mathfrak{A} \in K$ and $\tau \in {}^\alpha\alpha$ a function $s_\tau^{\mathfrak{A}} \in {}^A A$ such that*

(a) *for any $X \in A$ and $x \in 1^{\mathfrak{A}}$, $x \in s_\tau^{\mathfrak{A}} X$ iff $x \circ \tau \in X$*

(b) *for any $\mathfrak{B} \in K$, $f \in Is(\mathfrak{A},\mathfrak{B})$, $X \in A$, and $\tau \in {}^\alpha\alpha$, $f s_\tau^{\mathfrak{A}} X = s_\tau^{\mathfrak{B}} f X$.*

The original version of this result regarding the case $K = {}_{<\alpha}Cs_\alpha$ was proved by Monk (cf. [8] I.3.6). Comer (and, independently, the present author) proved it to be true in case $K = {}_\alpha Cs_\alpha$. For that matter, Lemma 6 also holds with K as the class of *all* minimal Cs_α's (see Serény [13]).

Now, we are ready to prove the first part of the Theorem itself. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Cs_α's with bases U and W, respectively, and let f be an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. Let us suppose that one of the following conditions holds:

(i) $|U| < \alpha$
(ii) $|U| = \alpha$ and $\mathfrak{A}$ is not minimal (therefore $\mathfrak{B}$ is not minimal either)
(iii) $|U| = \alpha + 1$ and $|W| = |U|$.

By Lemmas 2 and 3, in any one of these cases the powers of bases of $\mathfrak{A}$ and $\mathfrak{B}$ are the same, so in what follows $|W| = |U|$.

Let $a \in {}^\alpha U$ be such that

$$Rg\, a = U \text{ if } |U| < \alpha + 1$$

and

$$a \in \bar{d}^{\mathfrak{A}} \text{ if } |U| = \alpha + 1.$$

Let $Q = \{X \in A : a \in X\}$. Since $a \in \cap Q$ and $\mathfrak{A}$ is finite, $\cap f^*Q = f(\cap Q) \neq 0$ (where $g^*H = Rg(H \upharpoonright g)$ for any function g and set H). Let $b \in \cap f^*Q$ be arbitrary, and let $Q^+ = \{Y \in B : b \in Y\}$.

Now, by the definitions, $f^*Q \subseteq Q^+$. On the other hand, if $Y \notin f^*Q$, then $f^{-1}Y \notin Q$, that is, $a \in f^{-1} - Y$. Therefore $-Y \in f^*Q \subseteq Q^+$, which implies that $Y \notin Q^+$. Consequently,

$$f^*Q = Q^+. \tag{7.1}$$

Since we would like to treat the two essentially different cases $|U| = \alpha + 1$ and $|U| < \alpha + 1$ together, it will be useful to supplement the notation introduced in Definition 4. So if $|U| = \alpha + 1$, then we use the notation of Definition 4 together with $m_\alpha a = a$, $m_\alpha b = b$, $M_\alpha^{\mathfrak{A}} = A \upharpoonright Id$, $M_\alpha^{\mathfrak{B}} = B \upharpoonright Id$, while if $|U| < \alpha + 1$, then we set

$$m_i a = a, \quad m_i b = b \qquad \text{for any } i \in \alpha + 1$$

and

$$M_i^{\mathfrak{A}} = A \upharpoonright Id, \quad M_i^{\mathfrak{B}} = B \upharpoonright Id \qquad \text{for any } i \in \alpha + 1.$$

Further, by (7.1),

$$a \in d_{ij}^{\mathfrak{A}} \text{ iff } d_{ij}^{\mathfrak{A}} \in Q \text{ iff } d_{ij}^{\mathfrak{B}} \in Q^+ \text{ iff } b \in d_{ij}^{\mathfrak{B}} \qquad \text{for any } i, j \in \alpha. \tag{7.2}$$

Therefore, if $|U| = \alpha + 1$, then $a \in \bar{d}^{\mathfrak{A}}$ implies $b \in \bar{d}^{\mathfrak{B}}$. Consequently, in this case the conditions of Definition 4 are satisfied. We set

$$H = \langle \{X \in A : m_i a \in X\} : i \in \alpha + 1 \rangle$$

$$H^+ = \langle \{Y \in B : m_i b \in Y\} : i \in \alpha + 1 \rangle$$

$$G = \langle\langle \{X \in A : (m_i a) \circ \tau \in X\} : \tau \in {}^\alpha\alpha \rangle : i \in \alpha + 1 \rangle$$

$$G^+ = \langle\langle \{Y \in B : (m_i b) \circ \tau \in Y\} : \tau \in {}^\alpha\alpha \rangle : i \in \alpha + 1 \rangle.$$

From (7.1) and Proposition 5 we obtain

$$f^*H_i = H_i^+ \qquad \text{for any } i \in \alpha + 1. \tag{7.3}$$

Indeed, $Y \in f^*H_i$ iff $m_i a \in f^{-1}Y$ iff $a \in M_i^{\mathfrak{A}} f^{-1} Y = f^{-1} M_i^{\mathfrak{B}} Y$ iff $M_i^{\mathfrak{B}} Y \in f^*Q = Q^+$ iff $b \in M_i^{\mathfrak{B}} Y$ iff $m_i b \in Y$ iff $Y \in H_i^+$.

Similarly, by Lemma 6 and (7.3), we get

$$f^*G_i(\tau) = G_i^+(\tau) \qquad \text{for any } \tau \in {}^\alpha\alpha \text{ and } i \in \alpha + 1. \tag{7.4}$$

In fact, $Y \in f^*G_i(\tau)$ iff $(m_i a) \circ \tau \in f^{-1} Y$ iff $m_i a \in s_\tau^{\mathfrak{A}} f^{-1} Y = f^{-1} s_\tau^{\mathfrak{B}} Y$ iff $s_\tau^{\mathfrak{B}} Y \in f^* H_i = H_i^+$ iff $m_i b \in s_\tau^{\mathfrak{B}} Y$ iff $(m_i b) \circ \tau \in Y$ iff $Y \in G_i(\tau)$.

Now, we define the function that induces the desired base-isomorphism. First, let $g = \{\langle a_k, b_k \rangle : k \in \alpha\}$. g is one-one by (7.2). Moreover, g maps U onto W if $|U| < \alpha + 1$ since in this case $Rg\, a = U$ by our stipulation. Now, we set $h = g \cup \{\langle u, w \rangle\}$, where u and w are the unique elements of $U \sim Rg\, a$ and $W \sim Rg\, b$, respectively, if there are any. If not, let $h = g$. Clearly, h is a one-one function from U onto W whether $|U| < \alpha + 1$ or $|U| = \alpha + 1$.

Further, let $n = \langle \min\{j \in \alpha + 1 : Rg\, x \subseteq Rg\, m_j a\} : x \in {}^\alpha U \rangle$ and $\sigma = \langle\langle \min\{j \in \alpha : (m_{nx} a)_j = x_k\} : k \in \alpha \rangle : x \in {}^\alpha U \rangle$. (Note that n is well defined therefore so is σ.) Then for any $x \in {}^\alpha U$ and $k \in \alpha$, $x_k = (m_{nx} a)_{\sigma_x k}$, that is

$$x = (m_{nx} a) \circ \sigma_x \qquad \text{for any } x \in {}^\alpha U. \tag{7.5}$$

Thus for any $k \in \alpha$,

$$(h \circ x)_k = h((m_{nx}a) \circ \sigma_x)_k = h(m_{nx}a)_{\sigma_x k}$$

$$= \begin{cases} h(a_{\sigma_x k}) = b_{\sigma_x k} = (m_{nx}b)_{\sigma_x k} = ((m_{nx}b) \circ \sigma_x)_k \\ \qquad\qquad \text{if } \sigma_x k \neq nx \text{ or } |U| < \alpha + 1 \\ h(u) = w = (m_{nx}b)_{\sigma_x k} = ((m_{nx}b) \circ \sigma_x)_k \\ \qquad\qquad \text{if } \sigma_x k = nx \text{ and } |U| = \alpha + 1. \end{cases}$$

Consequently,

$$h \circ x = (m_{nx}b) \circ \sigma_x \qquad \text{for any } x \in {}^{\alpha}U. \tag{7.6}$$

Now we are ready to show that f is in fact a base-isomorphism induced by h. Let $X \in A$, $x \in {}^{\alpha}U$ be arbitrary and let $Y = fX$. By (7.4), (7.5), and (7.6), we obtain $x \in X = f^{-1}Y$ iff $(m_{nx}a) \circ \sigma_x \in f^{-1}Y$ iff $f^{-1}Y \in G_{nx}(\sigma_x)$ iff $Y \in G^{+}_{nx}(\sigma_x)$ iff $(m_{nx}b) \circ \sigma_x \in Y$ iff $h \circ x \in Y = fX$, which proves part (i) of the Theorem.

Before turning to the second part, we would like to point out that, in addition to what is claimed in part (i) of the Theorem, we have in effect proved a strengthened version of the finite dimensional part of Monk's original theorem referred to in the introductory remarks. What is more, the infinite dimensional part (also in this improved form) follows from our proof immediately as well, using the fact that any infinite dimensional regular and locally finite cylindric set algebra with a finite base can be generated by one of its finite neat reducts (cf. (*) in Serény [12]).

3 We begin the proof of the second part of the Theorem by giving the basic definition, which is a variant of the notion of cylindrically equivalent sets introduced in Andréka et al. [1] and used in [4].

Let $\alpha \geq 2$ and $\mathfrak{A}$ be a Cs_α. $0 \neq P \subseteq \bar{d}$ is a *cut* (of $\bar{d}$) if $c_i P = c_i(\bar{d} \sim P) = c_i \bar{d}$ for any $i \in \alpha$.

In order to describe the idea behind the proof, we should recall from [7] and [8] that an isomorphism f of a $Cs_\alpha \mathfrak{A}$ is an *ext-isomorphism* if $f = \langle X \cap {}^{\alpha}W : X \in A \rangle$ for some subset W of the base of $\mathfrak{A}$. Further, a Cs_α is *base-minimal* if it is not ext-isomorphic to any Cs_α except itself. Now, as we first shall see, Cs_α's generated by single cuts are all isomorphic. Since Biró [4] proved that the Cs_α's generated by the special single cuts defined in [1] (which, for the sake of completeness, will be described before Lemma 16 below) are base-minimal, in order to prove part (ii) of the Theorem, we should only construct Cs_α's generated by single cuts that are not base-minimal, which is just what we shall do in the concluding part of this paper.

Now, let us first describe the simple structure of Cs_α's generated by single cuts. ($\mathfrak{Sb}\,{}^{\alpha}U$ is the cylindric algebra of *all* subsets of ${}^{\alpha}U$ for any set U.)

Lemma 8 *Let $\alpha \geq 2$, $n \in \omega$, $n > \alpha$, and let P be a cut of $\bar{d} = \bar{d}^{\mathfrak{Sb}\,{}^{\alpha}n}(\alpha \times \alpha)$. Further, let $C = \{P, \bar{d} \sim P\}$. If $\mathfrak{A} = \mathfrak{Sg}^{\mathfrak{Sb}\,{}^{\alpha}n}\{P\}$, and $\mathfrak{M}$ is the minimal algebra of $\mathfrak{A}$, then for any $X \in A$,*

$X \notin M$ iff there are unique $Q \in M$ and $Z \in C$ such that $Q \cdot \bar{d} = 0$ and $X = Q + Z$.

Proof: First observe that, by [7] 2.1.17(i),

$$\mathfrak{M} = \mathfrak{Sg}^{\mathfrak{Bl}\mathfrak{M}}\{d_{ij} : i, j \in \alpha\},$$

which in turn implies directly that

(8.1) $\bar{d}$ is an atom of $\mathfrak{M}$

(8.2) for any $Z \in C$, $Q \in M$, if $Q \cdot Z \neq 0$, then $Q \cdot Z = Z$.

Thus $S = \{Q + Z : Q \in M,\ Z \in C\} \cup M$ is a subuniverse of $\mathfrak{Sb}^{\alpha}U$ since it is obviously closed with respect to addition; for any $Q \in M$, $Z \in C$,

$$-(Q+Z) = -Q \cdot -\bar{d} + -Q \cdot (\bar{d} \sim Z) = \begin{cases} -Q \cdot -\bar{d} + (\bar{d} \sim Z) & \text{if } -Q \cdot (\bar{d} \sim Z) \neq 0 \\ -Q \cdot -\bar{d} & \text{otherwise,} \end{cases}$$

and $c_i Z = c_i \bar{d}$ for any $i \in \alpha$, $Z \in C$. Consequently, $A \subseteq S$ since $M \cup C \subseteq S$. Moreover, for any $Q \in M$ and $Z \in C$, $Q + Z \notin M$ iff $Q \cdot \bar{d} = 0$. Indeed, if $Q \cdot \bar{d} \neq 0$ for some $Q \in M$, then, by (8.1), $Q \cdot \bar{d} = \bar{d}$, so $Q \supseteq \bar{d} \supseteq Z$ for any $Z \in C$, that is $Q + Z = Q \in M$. If, on the other hand, $Q + Z \in M$ for some $Q \in M$, $Z \in C$, then $Q \cdot Z \neq 0$ since otherwise $Q \in M$ implies $Z \in M$ contradicting (8.1). Therefore, in this case $Q \cdot Z = Z$ by (8.2), that is $Q \cdot \bar{d} \neq 0$.

Finally, turning to the unique expressibility, let $Q + Z = Q' + Z' \notin M$, $Q, Q' \in M$, $Z, Z' \in C$. If $Z \neq Z'$, then $Q' + Z' = Q + Z + Z' = Q + \bar{d} \in M$. Hence $Z = Z'$. By (8.2), there are three possibilities. If $Q \cdot Z = Q' \cdot Z = 0$, then $Q = Q \cdot -Z = (Q + Z) \cdot -Z = (Q' + Z) \cdot -Z = Q' \cdot -Z = Q'$. If $Q \cdot Z = Q' \cdot Z = Z$, then $Q = Q + Z = Q' + Z = Q'$. At last, without loss of generality, let $Q \cdot Z = Z$, $Q' \cdot Z = 0$. Then $Q = Q + Z = Q' + Z$, so $Z = Q \cdot -Q' \in M$ contradicting (8.1).

Lemma 8 leads directly to the following.

Lemma 9 *All Cs_α's generated by single cuts are isomorphic.*

Proof: Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras generated by single cuts P and R, respectively, and let $\mathfrak{M}$ and $\mathfrak{N}$ be their respective minimal algebras. Now, for any $Cs_\alpha\mathfrak{C}$, if there is a cut of $\bar{d}^{\mathfrak{C}}$, then $\mathfrak{C}$ must be of characteristic zero (cf. [8] I.5.3). Therefore, $Is(\mathfrak{M},\mathfrak{N}) \neq 0$ by [7] 2.5.30. Let $h \in Is(\mathfrak{M},\mathfrak{N})$ and $g = \{\langle P, R\rangle, \langle \bar{d}^{\mathfrak{A}} \sim P, \bar{d}^{\mathfrak{B}} \sim R\rangle\}$. Then, by Lemma 8, $f = h \cup \{\langle Q + Z, hQ + gZ\rangle : Q \in M, Z \in \{P, \bar{d}^{\mathfrak{A}} \sim P\}, Q \cdot \bar{d}^{\mathfrak{A}} = 0\}$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

Now we shall construct our special cuts that generate non-base-minimal Cs_α's. In order to do this, we first describe a more general way to define cuts.

Definition 10 Let $\alpha \geq 2$, $n \in \omega$, $n > \alpha$, and $\mathfrak{C} = \mathfrak{Sb}^{\alpha}n$. For any $p, q \in \bar{d}$ such that $Rg\,p = Rg\,q$, considering $\tau = q^{-1} \circ p \in {}^{\alpha}\alpha$ as an element of the symmetric group S_α of degree α, τ can always be expressed as a product of transpositions. Though the number of these transpositions are not unique itself, its parity is in fact unique, that is the number of transposition factors in any such product yielding a given element of S_α is either always even or always odd (see e.g. Ledermann [9] Theorem 21). Consequently, we can define a function mapping S_α onto 2 as follows:

for any $\tau \in S_\alpha$, let $\pi(\tau) = 0$ if the number of transpositions expressing τ as their product is even, and let $\pi(\tau) = 1$ otherwise.

Further, for any $X, Y \subseteq \bar{d}$, let

$$E(X,Y) = \{p \in \bar{d} : (\exists q \in X)(\exists \tau \in S_\alpha)(p = q \circ \tau \text{ and } \pi(\tau) = 0)$$
$$\text{or } (\exists q \in Y)(\exists \tau \in S_\alpha)(p = q \circ \tau \text{ and } \pi(\tau) = 1)\},$$
$$E^+(X,Y) = \bar{d} \sim E(X,Y).$$

A $Q \subseteq C$ is a *kernel* (of a cut) if it consists of two disjoint elements being subsets of $\bar{d}$; and, using from now on the notation $X^d = (\cup Q) \sim X$ for any $X \in Q$,
(a) $Rg\, q = Rg\, q'$ implies $q = q'$ in case of every $q, q' \in \cup Q$,
(b) $\{Rg\, q : q \in \cup Q\} = Sb_\alpha n$, where $Sb_\alpha n = \{x : x \subseteq n, |x| = \alpha\}$ that is the set of all subsets of n with α elements,
(c) If $H \subseteq n$, $|H| = \alpha - 1$, and $X \in Q$, then there are $i \in \alpha$, $q \in E(X,X^d)$, $q^+ \in E^+(X,X^d)$ such that $(\alpha \sim \{i\}) \upharpoonright q = (\alpha \sim \{i\}) \upharpoonright q^+$ and $Rg((\alpha \sim \{i\}) \upharpoonright q) = H$.

Our construction will be based on the following fact.

Fact 11 *If Q is a kernel, $X \in Q$, then $E(X,X^d)$ is a cut.*

Proof: Let us assume that Q is a kernel, $X \in Q$, $i \in \alpha$, and let us introduce the following abbreviations: $\bar{E} = E(X,X^d)$, $\bar{E}^+ = E^+(X,X^d)$. Since, from the definition, $c_i\bar{E}, c_i\bar{E}^+ \subseteq c_i\bar{d}$, we have to prove only the other inclusion. So let $p \in c_i\bar{d}$. By hypotheses, there are $j \in \alpha$, $q \in \bar{E}$, $q^+ \in \bar{E}^+$ such that $(\alpha \sim \{j\}) \upharpoonright q = (\alpha \sim \{j\}) \upharpoonright q^+$ and $Rg((\alpha \sim \{j\}) \upharpoonright q) = Rg((\alpha \sim \{i\}) \upharpoonright p)$. Let $r = p^i_{q_j}$, $r^+ = p^i_{q_j^+}$. Then $Rg\, r = Rg\, q$, $Rg\, r^+ = Rg\, q^+$. Moreover, $r, r^+ \in \bar{d}$. Now, let $\tau = q^{-1} \circ r$ and $\tau^+ = (q^+)^{-1} \circ r^+$. We shall show that $\tau = \tau^+$. Indeed, $\tau(i) = j = \tau^+(i)$, and if $k \neq i$, then there is a $t \in \alpha$, $t \neq j$ such that $q(t) = p(k)$. Hence if $k \neq i$, then $\tau(k) = q^{-1}(p(k)) = q^{-1}(q(t)) = t = (q^+)^{-1}(q^+(t)) = (q^+)^{-1}(q(t)) = (q^+)^{-1}(p(k)) = \tau^+(k)$. The other case being completely analogous, we may suppose that $\pi(\tau) = 0$. Since $E(\bar{E},\bar{E}^+) = \bar{E}$ and $E^+(\bar{E},\bar{E}^+) = \bar{E}^+$ by the first two conditions in the definition of a kernel, we have $r \in \bar{E}$, $r^+ \in \bar{E}^+$. Therefore $p = r^i_{p_i} = (r^+)^i_{p_i}$ implies $p \in c_i\bar{E} \cap c_i\bar{E}^+$, which was to be proved.

Now, for any $\alpha \geq 2$, $n \in \omega$, $n > \alpha + 1$, we shall define a pair of sets which we shall show to be a kernel of a cut that generates the desired non-base-minimal Cs_α with base n. From now on n is always a natural number.

Definition 12 First, for any $k, j \in \omega \sim 1$, and $H \subseteq {}^k\omega$, $L \subseteq {}^j\omega$, we denote the set obtained by concatenation of elements of H and L by $H^\frown L$, that is, if $H, L \neq 0$, then we set $H^\frown L = \{x \in {}^{k+j}\omega : \langle x_i : i \in k\rangle \in H, \langle x_{k+i} : i \in j\rangle \in L\}$ (cf. [7], p. 33), and for any H we set $H^\frown 0 = 0^\frown H = H$. Further, for any $k \in \omega \sim 1$ and $H \subseteq \omega$, let $R_k(H) = \{x \in {}^kH : (\forall i,j \in k)(i < j \Rightarrow x_i < x_j)\}$ and we set $R_0(H) = 0$ for any $H \subseteq \omega$. We define the sets $G(n,\alpha)$ and $G^+(n,\alpha)$ by induction for any $\alpha \geq 0$ (resp. $\alpha > 0$) and $n > \alpha$.
(a) $G(n,0) = 0$ for any $n > 0$,
(b) $G(n,1) = \{\langle 0,0\rangle\}$, $G^+(n,1) = \{\langle 0,k\rangle : 1 \leq k < n\}$ for any $n > 1$,
(c) $G(\alpha+1,\alpha) = \{s_\alpha\}$, $G^+(\alpha+1,\alpha) = \{(s_\alpha)^i_\alpha : i \in \alpha\}$, where $s_\alpha = \langle i : i \in \alpha\rangle$, for any $\alpha \geq 2$.

(d) Let $\alpha \geq 2$, $n > \alpha + 1$. Suppose that for any $\beta < \alpha$ and $n > \beta$, $G(n,\beta)$ has already been defined, and for any $0 < \beta < \alpha$ and $n > \beta$, $G^+(n,\beta)$ also has already been defined. Let $I = \min\{\alpha, n - (\alpha + 1)\}$ and $I^+ = \min\{\alpha - 1, n - (\alpha + 1)\}$. Then

$$G(n,\alpha) = \bigcup_{i \in I} G(\alpha + 1, \alpha - i)^\frown R_i(n \sim (\alpha + 1)),$$

$$G^+(n,\alpha) = \bigcup_{i \in I^+} G^+(\alpha + 1, \alpha - i)^\frown R_i(n \sim (\alpha + 1)).$$

Lemma 13 *For every $\alpha \geq 2$, $n > \alpha$, $\{G(n,\alpha), G^+(n,\alpha)\}$ is a kernel of a cut of $\bar{d}^{\mathfrak{Sb}^\alpha n}$.*

Proof: Proof is by a straightforward induction based on the facts that for any $n > 1$, $\{G(n,1), G^+(n,1)\}$ obviously satisfies conditions (a), (b), (c) in Definition 10 and for any $\alpha \geq 2$, $\{G(\alpha + 1,\alpha), G^+(\alpha + 1,\alpha)\}$ is in fact a kernel. Even in this second case, only the last condition of Definition 10 needs proof. Let $H \subseteq \alpha + 1$, $|H| = \alpha - 1$. There are two cases. If $H \subseteq \alpha$, then there is a unique $i \in \alpha$ such that $i \notin H$; therefore $Rg((\alpha \sim \{i\}) \upharpoonright s_\alpha) = \alpha \sim \{i\} = H$ and $(\alpha \sim \{i\}) \upharpoonright s_\alpha = (\alpha \sim \{i\}) \upharpoonright (s_\alpha)^i_\alpha$. If, on the contrary, $\alpha \in H$, then there are $i,j \in \alpha$, $i \neq j$ such that $i,j \notin H$. Hence $Rg((\alpha \sim \{i\}) \upharpoonright (s_\alpha)^j_\alpha) = (\alpha + 1) \sim \{i,j\} = H$ and, using (ij) to denote the transposition interchanging i and j, $(\alpha \sim \{i\}) \upharpoonright (s_\alpha)^j_\alpha = (\alpha \sim \{i\}) \upharpoonright ((s_\alpha)^i_\alpha \circ (ij))$. Moreover, $(s_\alpha)^j_\alpha \in G^+(\alpha + 1,\alpha) \subseteq E^+(G(\alpha + 1,\alpha), G^+(\alpha + 1,\alpha))$ and $(s_\alpha)^i_\alpha \circ (ij) \in E(G(\alpha + 1,\alpha), G^+(\alpha + 1,\alpha))$ since $(s_\alpha)^i_\alpha \in G^+(\alpha + 1,\alpha)$ and $\pi((ij)) = 1$.

Though we do not need it, a remark on the special case when the power of the base is just $\alpha + 1$ may be of some interest. As it can be proved, the *only* pair of cuts of $\bar{d} = \bar{d}^{\mathfrak{Sb}^\alpha(\alpha+1)}$ is just $E = E(G(\alpha + 1,\alpha), G^+(\alpha + 1,\alpha))$, $\bar{d} \sim E$.

In order to prove that the Cs_α's generated by cuts whose kernels are defined in Definition 12 are in fact not base-minimal, we should first prove a simple lemma (which, however, would have a place in our discussion anyway since it nicely fills in the gap (concerning the case of minimal algebras) in our main theorem).

Lemma 14 *Let α be arbitrary (finite or infinite). For any isomorphism between minimal Cs_α's, the isomorphism itself or its inverse is an ext-base-isomorphism (that is the composition of a base- and an ext-isomorphism).*

Proof: Let $\mathfrak{M}$ and $\mathfrak{N}$ be isomorphic minimal Cs_α's with bases U and W, respectively. We may suppose $W \subseteq U$, $W \neq U$, and, by [7] 3.1.38(1), $|W| \geq \alpha \cap \omega$. Let $g = \langle X \cap {}^\alpha W : X \in M \rangle$. By [7] 0.2.14(iii), it is enough to prove that $g \in Is\mathfrak{M}$. Since $g \in Ho\mathfrak{Bl}\mathfrak{M}$ and $\mathfrak{M}$ is simple (cf. [7] 3.1.70(i), 3.1.64), we only have to check that g preserves cylindrifications. This in turn follows from the fact (implied by [7] 2.1.17(i), 2.3.14, and the simplicity of $\mathfrak{M}$) that $\mathfrak{M} = \mathfrak{Sg}^{\mathfrak{BJ}\mathfrak{M}}\{d_{ij} : i,j \in \alpha\}$ since on the one hand in any CA_α, $c_i d_{ij} = 1$ for any $i,j \in \alpha$ and, on the other hand, $c_i^{\mathfrak{M}} \prod_{j \in H} - d_{ij}^{\mathfrak{M}} = 1^{\mathfrak{M}}$, $c_i^{\mathfrak{N}} \prod_{j \in H} - d_{ij}^{\mathfrak{N}} = 1^{\mathfrak{N}}$ for any finite $H \subseteq \alpha$ and all $i \in \alpha \sim H$ because $|U| \geq |W| \geq \alpha \cap \omega$.

Now we have the property we need.

Lemma 15 *For any $n,m \in \omega$, $n > m > \alpha \geq 2$, the Cs_α with base n generated by $E(G(n,\alpha), G^+(n,\alpha))$ is ext-isomorphic to that with base m generated by $E(G(m,\alpha), G^+(m,\alpha))$.*

Proof: If $\mathfrak{A}_k = \mathfrak{Sg}^{\mathfrak{Sb}^{\alpha}k}\{E(G(k,\alpha),G^+(k,\alpha))\}$ for any $k > \alpha \geq 2$, $k \in \omega$, and $g_{nm} = \langle X \cap {}^{\alpha}m : X \in A_n \rangle$ for any $n > m > \alpha \geq 2$, then, by Lemmas 8 and 14, g_{nm} preserves all the cylindrifications since $E(G(n,\alpha),G^+(n,\alpha)) \cap {}^{\alpha}m = E(G(m,\alpha),G^+(m,\alpha))$ for all $n > m > \alpha \geq 2$, which in turn, adding [7] 0.2.18(i) to the same argument that we used in Lemma 14, is enough to guarantee that $g_{nm} \in Is(\mathfrak{A}_n,\mathfrak{A}_m)$. (As a matter of fact, the ${}_{\alpha}Cs_{\alpha}$ generated by $E(G(\alpha+1,\alpha), G^+(\alpha+1,\alpha))$ *is* base-minimal since in the case of a $Cs_{\alpha}\mathfrak{A}$ with base U, there are no cuts of $\bar{d}^{\mathfrak{A}}$ if $|U| \leq \alpha$.)

Before concluding our proof, for the sake of completeness, we cite the main result of [4] (Lemma 2 and Lemma 5). For any $\alpha \geq 2$ and $i \in \omega$, $i \geq 1$, a set $X^i_{\alpha} \subseteq {}^{\alpha}(\alpha + i)$ is defined by induction on α as follows:

$$X^i_2 = \{s \in {}^2(2+i) : s_1 = s_0 + 1(\mathrm{mod}(2+i))\}$$

and using the abbreviation $\bar{d} = \bar{d}^{\mathfrak{S}}$, where $\mathfrak{S} = \mathfrak{Sb}^{\alpha+1}(\alpha+1+i)$,

$$X^i_{\alpha+1} = \{s \in \bar{d} : \alpha + i \notin Rg\, s, \alpha \upharpoonright s \in X^i_{\alpha}\} \cup \{s \in \bar{d} : s_{\alpha} = \alpha + i,\ \alpha \upharpoonright s \notin X^i_{\alpha}\}$$
$$\cup \left(\bigcup_{j \in \alpha} \{s \in \bar{d} : s_j = \alpha + i, \alpha \upharpoonright s^j_{s_{\alpha}} \in X^i_{\alpha}\} \right).$$

Lemma 16 *For any $i \in \omega$, $i \geq 1$, and $\alpha \geq 2$,*
(a) $\mathfrak{A}^i_{\alpha} = \mathfrak{Sg}^{\mathfrak{Sb}^{\alpha}(\alpha+i)}\{X^i_{\alpha}\}$ *is base-minimal,*
(b) X^i_{α} *is a cut of* $\bar{d}^{\mathfrak{A}^i_{\alpha}}$.

Finally, recalling from [8] (Def. 3.1, p. 156) that f is a *lower base-isomorphism* between two Cs_{α}'s if $f = e_1^{-1} \circ b \circ e_2$ for some ext-isomorphisms e_1, e_2 and base-isomorphism b, we can finish the proof. In fact, if $\alpha \geq 2$ and $n \geq m \geq \alpha + 2$ or $n > m \geq \alpha + 1$, then, by Lemma 9, Fact 11, Lemma 13, and Lemma 16(b), $\mathfrak{Sg}^{\mathfrak{Sb}^{\alpha}n}\{X^{n-\alpha}_{\alpha}\}$ is isomorphic to $\mathfrak{Sg}^{\mathfrak{Sb}^{\alpha}m}\{E(G(m,\alpha),G^+(m,\alpha))\}$. Yet, by Lemma 15 and Lemma 16(a), they cannot be base-isomorphic—not even lower base-isomorphic—which proves the second part of the Theorem, completing the whole proof.

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