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Extensions of the $\aleph_0$-Valued Łukasiewicz Propositional Logic

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Abstract MV-algebras were introduced by Chang in 1958 preliminary to his providing an algebraic completeness proof for the $\aleph_0$-valued Łukasiewicz propositional logic, $\mathbf{Ł}_{\aleph_0}$. In this paper a method is given for determining, for an arbitrary normal extension of $\mathbf{Ł}_{\aleph_0}$, an MV-algebra characteristic for the extension. The characteristic algebras are finite direct products of two sets of linearly ordered MV-algebras identified by Komori. Conversely, it is shown that, given an algebra which is isomorphic to the direct product of elements from these two sets of linearly ordered MV-algebras, a single axiom can be determined which, when added to the axioms for $\mathbf{Ł}_{\aleph_0}$, yields an axiomatization sound and weakly complete for the given algebra. As a consequence of the lattice ordering of these products of MV-algebras the cardinal and ordinal degrees of completeness of any normal extension of $\mathbf{Ł}_{\aleph_0}$ can be determined.

1 Introduction In [4] Komori proves that any proper extension of $\mathbf{Ł}_{\aleph_0}$ has a characteristic matrix isomorphic to a finite direct product of elements of two fundamental types of MV-algebras, thus providing a type of characterization for any axiomatic extension of $\mathbf{Ł}_{\aleph_0}$. We will provide a somewhat stronger result in Theorem 4.4 below as the principal result of this paper. Our result improves on Komori's by giving a method for determining which MV-algebras occur in the finite products. Using the notion of the genus of a formula, given in Rose [11], we show that a formula is satisfied in the Komori characterization if and only if it is of a corresponding genus. By connecting Komori's characterization with Rose's notion of genus it is possible to link characteristic matrices with extensions of $\mathbf{Ł}_{\aleph_0}$. This result amounts to a soundness and weak completeness theorem for the characterizations provided by Komori.

2 Łukasiewicz matrices The n-valued Łukasiewicz propositional calculi (for n either an integer greater than 1 or $\aleph_0$) were defined with the aid of logical

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matrices by Łukasiewicz in the 1920s. The matrix given by Łukasiewicz for the $\aleph_0$-valued system was the algebra

$$[S_{\aleph_0} = \langle Q_{[0,1]}, \{1\}, \rightarrow, \neg \rangle]$$

of type $\langle 2,1 \rangle$, where $Q_{[0,1]}$ designates the set of rational numbers in the interval $[0,1]$, that is, the matrix is composed of the rational numbers in the unit interval $[0,1]$, with the set of designated elements being $D = \{1\}$, having a binary operator $\rightarrow$ and a unary operator $\neg$. Łukasiewicz specified that $a \rightarrow b$ is to be evaluated by $\min(1, 1 - a + b)$ and $\neg a$ is to be evaluated by $1 - a$.

The matrices for the finite systems are gotten by taking subsets of $Q_{[0,1]}$ closed under the operators. The standard examples of these are algebras of the form

$$[S_{n+1} = \langle \{0, 1/n, 2/n, \ldots, 1\}, \{1\}, \rightarrow, \neg \rangle]$$

for n some integer greater than 1. Let $S_{\aleph_0}$ denote $Q_{[0,1]}$, for n a positive integer let S_{n+1} denote $\{0, 1/n, 2/n, \ldots, 1\}$ and let $S_1 = \{1\}$.

$S_{\aleph_0}$ can be axiomatized with the rules modus ponens and substitution with the following axioms:

A1. $p \supset (q \supset p)$.
A2. $(p \supset q) \supset ((q \supset r) \supset (p \supset r))$.
A3. $((p \supset q) \supset q) \supset ((q \supset p) \supset p)$.
A4. $(\sim q \supset \sim p) \supset (p \supset q)$.

The first published proof of the weak completeness of axioms 1–4 for $S_{\aleph_0}$ is found in Rose and Rosser [12] . Chang [2] gives an algebraic proof of completeness using MV-algebras. The Lindenbaum algebra of a logic L is the algebra of equivalence classes of formulas of the logic determined by provable equivalence. Given a soundness and completeness result for a logic **L** with respect to a matrix **S**, we say that **S** is characteristic for **L**. We say that the logic L_1 is an extension of the logic L_2 if L_1 and L_2 have the same sentences and every theorem of L_2 is a theorem of L_1. An extension is normal when closed under modus ponens and substitution.

$S_{\aleph_0}$ is characteristic for $\mathbf{Ł}_{\aleph_0}$, so $\mathbf{Ł}_{\aleph_0}$ has an infinite characteristic matrix, but there is no finite characteristic matrix for $\mathbf{Ł}_{\aleph_0}$. For a proof of this fact see Urquhart [16].

Chang developed MV-algebras in an attempt to parallel the treatment of classical two-valued propositional logic with Boolean algebras, i.e., as the algebras "that would correspond in a natural fashion" to the logic $\mathbf{Ł}_{\aleph_0}$. We give here not Chang's axioms but rather a simpler set provided in Mangani [8] and shown to be equivalent to Chang's axioms in Mundici [9].

Definition 2.1 The algebra $S = \langle S, +, \circ, -, 0, 1 \rangle$ is an MV-algebra if S is a nonempty set, S is of type $\langle 2,2,1,0,0 \rangle$ with 0 and 1 distinct such that:
P1. $(x + y) + z = x + (y + z)$
P2. $x + 0 = x$
P3. $x + y = y + x$
P4. $x + 1 = 1$
P5. $x^{=} = x$

P6. $0^- = 1$
P7. $x + x^- = 1$
P8. $(x^- + y)^- + y = (x + y^-)^- + x$
P9. $x \circ y = (x^- + y^-)^-$.

Theorem 1.18 of Chang [1] shows that MV-algebras are a variety. Chang goes on to show that the Lindenbaum algebra of an extension of $\text{Ł}_{\aleph_0}$ is an MV-algebra, and furthermore every MV-algebra satisfies the translation of the axioms of $\text{Ł}_{\aleph_0}$. "Hence we conclude that the two sets of axioms (those of MV-algebras and those of the $\aleph_0$-valued Łukasiewicz propositonal logic) are equivalent under an appropriate relationship between C and N (that is, between $\supset$ and $\sim$) and $+$, $\circ$, and $-$." (See [1], p. 473.) Chang then gives an MV-algebra, which he denotes by C, composed of the following elements: $[0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots,$ and $\ldots, 1-3\varepsilon, 1-2\varepsilon, 1-\varepsilon, 1.]$ The algebra is linearly ordered, the order being exemplified by the above sequence or more formally:

$$x \leq y \text{ if and only if } \begin{cases} \text{either} & x = n\cdot\varepsilon \text{ and } y = 1 - m\cdot\varepsilon \\ \text{or} & x = n\cdot\varepsilon \text{ and } y = m\cdot\varepsilon \text{ and } n \leq m \\ \text{or} & x = 1 - n\cdot\varepsilon \text{ and } y = 1 - m\cdot\varepsilon \text{ and } m \leq n. \end{cases}$$

The table for this MV-algebra can be considered the result of adding a band of "infinitesimals" adjacent to the values of the two-valued system. Komori generalizes Chang's C by, in effect, for each n, adding bands of infinitesimals around each of the elements of the n-valued algebra S_n. Komori defines the CN-algebras $\langle S_n^\omega, \rightarrow, \neg, (1,0), (0,0)\rangle$ for n an integer greater than 1 as follows:

$$S_{n+1}^\omega = \left\{(x,y) \mid x \in \left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}, y \in Z\right\}$$
$$\cup \{(0,y) \mid y \in N\} \cup \{(1,-y) \mid y \in N\}$$

where Z and N are the set of integers and the set of natural numbers. We have changed Komori's indices to accord with conventions adopted. Komori's CN-algebras are categorically equivalent to Chang's MV-algebras. The operators $\neg$ and $\rightarrow$ are defined on S_n^ω by:

$$\neg(x,y) = (1-x, -y)$$

$$(x,y) \rightarrow (z,u) = \begin{cases} (1,0) & \text{if } z > x, \\ (1, \min(0, u-y)) & \text{if } z = x, \\ (1 - x + z, u - y) & \text{otherwise}. \end{cases}$$

A formula φ is a tautology of S_n, i.e., $\varphi \in \mathrm{Taut}(S_n)$ or $\models_{S_n} \varphi$, if for all interpretations v in S_n, $v(\varphi) = 1$, and φ is a tautology of S_n^ω, i.e., $\varphi \in \mathrm{Taut}(S_n^\omega)$ or $\models_{S_n^\omega} \varphi$, if for all interpretations v in S_n^ω, $v(\varphi) = (1,0)$. That a formula φ is a theorem of the logic Ł is denoted by $\varphi \in \mathrm{Th}(\text{Ł})$ or $\vdash_{\text{Ł}} \varphi$.

McNaughton [7] establishes a connection between a class of real valued functions on $[0,1]^n$ and formulas of $\text{Ł}_{\aleph_0}$which will be used in the proofs that follow.

Definition 2.2 Let φ be a formula and f be an n-ary real function. φ represents f (or f is represented by φ) if and only if

(i) φ has exactly n propositional variables,
(ii) the field of f is $[0,1]$, and
(iii) $\forall x_1, \ldots, x_n$ and for every valuation v such that $v(p_i) = x_i$, for $1 \leq i \leq n$,

$$f(x_1, \ldots, x_k) = v(\varphi).$$

Theorem 2.3 (McNaughton) *The formula φ represents the real function f if and only if*

(i) *f is continuous over $[0,1]^n$ and $Range(f) \subseteq [0,1]$,*
(ii) *the domain, $[0,1]^n$, is partitioned into a finite number, j, of subdomains D_i, $1 \leq i \leq j$, where the partition is exhaustive and the interiors of the subdomains are mutually exclusive, and*
(iii) *there are j polynomials $\pi_1, \ldots, \pi_j$ each of the form*

$$\pi_i = b_i + m_{1,i}x_1 + \cdots + m_{n,i}x_n$$

with b_i, $m_{i',i}$ integers such that if $\langle x_1, \ldots, x_n \rangle \in D_i$ then

$$f(x_1, \ldots, x_n) = \pi_i(x_1, \ldots, x_n).$$

3 The lattice of genera of formulas of $Ł_{\aleph_0}$ In [11] Rose introduces the notion of the genus of a formula and proves Lemma 3.7 below preliminary to proving that the ordinal degree of completeness of $Ł_{\aleph_0}$ is ω. In this section Rose's lemma is generalized. It is perhaps useful to think of the genus of a formula as a measure of its power as an axiom, since by Theorem 3.8 below, the logic obtained by adding the formula φ to the axioms of $Ł_{\aleph_0}$, where φ is of genus G_φ, allows the derivation of a formula ψ of genus G_ψ if and only if $G_\varphi \leq G_\psi$. The lower the genus of φ in the lattice ordering, the more power φ has as an axiom. The definition of the genus of a formula is:

Definition 3.1 (Rose) Suppose that the formula $\varphi(p_1, \ldots, p_n)$ is not a theorem of $Ł_{\aleph_0}$ but is valid in the Łukasiewicz calculi with $b_1 + 1, \ldots, b_i + 1, c_1 + 1, \ldots, c_j + 1$ values but in no others. If for each b_k, $1 \leq k \leq i$, there is a real number $\epsilon > 0$ such that if v is a valuation then if $|v(p_h) - a_h/b_k| < \epsilon$, $1 \leq h \leq n$, for a_h an integer $0 \leq a_h \leq b_k$, then $v(\varphi(p_1, \ldots, p_n)) = 1$, but no such ϵ exists for any of the c's, then φ is of *genus* $\langle b_1 + 1, \ldots, b_i + 1; c_1 + 1, \ldots, c_j + 1 \rangle$.

Let G_φ denote the genus of φ. If $G_\varphi = \langle b_1 + 1, \ldots, b_i + 1; c_1 + 1, \ldots, c_j + 1 \rangle$, let B_φ denote $\{b_1 + 1, \ldots, b_i + 1\}$, C_φ denote $\{c_1 + 1, \ldots, c_j + 1\}$, and $\cup G_\varphi$ denote $B_\varphi \cup C_\varphi$. For any φ, $B_\varphi \cap C_\varphi = \emptyset$. Since for all n, $2 \leq n \leq \aleph_0$, every theorem of $Ł_n$ is a theorem of $Ł_2$, a formula φ has a genus if and only if φ is a theorem of the classical two-valued system and not a theorem of the $\aleph_0$-valued Łukasiewicz system. Thus the theorems of $Ł_2$ can be partitioned according to genus, i.e., $[\varphi] = [\psi]$ just in case $G_\varphi = G_\psi$. The structure of this partition is the key to determining the axiomatization of the extensions of $Ł_{\aleph_0}$. The operations of meet and join in the lattice of genera are defined as follows:

$$G_\varphi \vee G_\psi = \langle B_\varphi \cup B_\psi;\ C_\varphi \cup C_\psi - B_\varphi \cup B_\psi \rangle$$

$$G_\varphi \wedge G_\psi = \langle B_\varphi \cap B_\psi;\ ((B_\varphi \cup C_\varphi) \cap (B_\psi \cup C_\psi)) - B_\varphi \cap B_\psi \rangle.$$

The genera of the theorems of $Ł_2$, i.e., $\{G_\varphi \mid \varphi \in \mathrm{Th}(Ł_2\}$, form a distributive lattice which we denote by G. G is bounded below, but not above. The minimal element of G is $\langle;2\rangle$. The partial order on the lattice of genera, G, is such that for G_φ, $G_\psi \in G$, $G_\varphi \leq G_\psi$ if and only if $G_\varphi \wedge G_\psi = G_\varphi$. That is, $G_\varphi \leq G_\psi$ just in case if b occurs before the ; in G_φ then b occurs before the ; in G_ψ, and $\bigcup G_\varphi \subseteq \bigcup G_\psi$.

The definition of genus can be improved. The addition of a maximal element to the lattice of genera will simplify some arguments below. Therefore, let $G_\varphi = \omega$ if and only if φ is a theorem of $Ł_{\aleph_0}$, and thus add the maximal element ω to the lattice of genera G. Also, as a result of the subset relations that hold between the sets of theorems of the various normal extensions of $Ł_{\aleph_0}$, the genus sometimes contains more elements than is necessary to convey the status of the formula. For example, $\mathrm{Th}(Ł_7) \subset \mathrm{Th}(Ł_4)$ since $4 - 1$ divides $7 - 1$ (a result provided by Lindenbaum), so one might just as well say $G_\varphi = \langle;7\rangle$ as that $G_\varphi = \langle;4,7\rangle$ since no more information is conveyed by the latter expression. However $G_\varphi = \langle 4;7\rangle$ is not redundant and $G_\varphi = \langle 7;4\rangle$ is contradictory. We decide, however, not to eliminate the redundancy because of the complexity that it introduces in the definitions of meet and join and thus into the arguments given below.

The following lemmas are used in the proof of Theorem 3.8.

Lemma 3.2 $\quad G_\varphi \wedge G_\psi = G_{\varphi\wedge\psi}$.

Proof: Suppose that $c + 1$ and $b + 1$ are in $G_\varphi \wedge G_\psi$. Then $b + 1$ is in both B_φ and B_ψ and thus in $G_{\varphi\wedge\psi}$ before the ;. $c + 1$ is in both G_φ and G_ψ and thus in $G_{\varphi\wedge\psi}$. So $G_\varphi \wedge G_\psi \leq G_{\varphi\wedge\psi}$.

Suppose that $c + 1$ and $b + 1$ are in $G_{\varphi\wedge\psi}$ before and after the ; respectively. Then $b + 1$ is in both B_φ and B_ψ and thus in $G_\varphi \wedge G_\psi$ before the ;. And $c + 1$ is in both G_φ and G_ψ and thus in $G_\varphi \wedge G_\psi$. So $G_{\varphi\wedge\psi} \leq G_\varphi \wedge G_\psi$.

So $G_\varphi \wedge G_\psi = G_{\varphi\wedge\psi}$.

Lemma 3.3 $\quad G_\varphi \vee G_\psi \leq G_{\varphi\vee\psi}$.

Proof: Suppose that $c + 1$ and $b + 1$ are in $G_\varphi \vee G_\psi$ before and after the ; respectively. Then $b + 1$ is in at least one of B_φ and B_ψ before the ; and thus in $G_{\varphi\vee\psi}$ before the ;. $c + 1$ is in at least one of G_φ and G_ψ and thus in $G_{\varphi\vee\psi}$. So $G_\varphi \vee G_\psi \leq G_{\varphi\vee\psi}$.

Lemma 3.4 *Let $\Phi = \{\varphi_1, \varphi_2, \ldots,\}$ be a set of formulas. There is a finite subset Φ_{fin} of Φ such that $\bigwedge_{\varphi\in\Phi} G_\varphi = \bigwedge_{\varphi\in\Phi_{fin}} G_\varphi$.*

Proof: For each genus G_φ there is only a finite number of distinct G_ψ such that $G_\psi \leq G_\varphi$ and $G_\psi \neq G_\varphi$.

Lemma 3.5 *In $S_{\aleph_0}$ if whenever the formula φ takes the value 1 the formula ψ does so, then $G_{\varphi\wedge\psi} = G_\varphi$.*

Proof: Since whenever the formula φ takes the value 1 the formula ψ does so, $G_\varphi \leq G_\psi$ and so $G_\varphi = G_\varphi \wedge G_\psi$. But $G_\varphi \wedge G_\psi = G_{\varphi\wedge\psi}$. So $G_{\varphi\wedge\psi} = G_\varphi$.

Lemma 3.6 $\quad$ *$G_\varphi \leq G_{\varphi(\psi/p)}$, i.e., any substitution instance of a formula has a genus $\geq$ the genus of the formula.*

Proof: An induction on the complexity of formulas shows that for any formula $\psi(p_1, \ldots, p_n)$ if $v(p_i) = a_j/c$ for $1 \leq i \leq n$ and $0 \leq j \leq c$ then $v(\psi) = a/c$ for some integer a, $0 \leq a \leq c$.

Suppose $c + 1 \in G_\varphi$. Then for $v(p_i) = a_j/c$ with $1 \leq i \leq n$ and a_j an integer $0 \leq a_j \leq c$, $v(\psi) = a/c$ for some $0 \leq a \leq c$ and thus $v(\varphi(\psi/p_i)) = 1$.

Suppose $b + 1 \in G_\varphi$ before the ;. Let ϵ be the number which is associated with b for G_φ in the definition of genus. By McNaughton's theorem f is a continuous function on $[0,1]^n$ where f is the function associated with the formula ψ. So there is a positive number δ such that if for all p_i and all a_h, $|v(p_i) - a_{h_i}/b| < \delta$ then for all p_i and all a_h, $|v(\psi(p_1, \ldots, p_n)) - f(a_{h_1}/b, \ldots, a_{h_n}/b)| < \epsilon$. Let ϵ' be the lesser of δ and ϵ, then $|v(p_i) - a_{h_i}/b| < \epsilon'$ implies that $v(\varphi(\psi/p)) = 1$, i.e., $b + 1 \in G_\varphi(\psi/p)$ before the ;.

Lemma 3.7 *If we adjoin a formula φ of genus $\langle b_1, \ldots, b_i; c_1, \ldots, c_j\rangle$ to the axioms of the formalization of $Ł_{\aleph_0}$ then every formula of this genus becomes provable in the formulation.*

Proof: See Rose [11], pp. 181–4.

Let $Ł_{\aleph_0} + \varphi$ denote the extension of $Ł_{\aleph_0}$ obtained by adding the formula φ to the axioms of $Ł_{\aleph_0}$. The following theorem shows that $G_\varphi = G_\psi$ if and only if $\vdash_{Ł_{\aleph_0}+\varphi} \psi$ and $\vdash_{Ł_{\aleph_0}+\psi} \varphi$. The theorem thus identifies the partial order of the lattice G with extensions of $Ł_{\aleph_0}$.

Theorem 3.8 *Let $Ł_{\aleph_0+\varphi}$ be the logic obtained by adding the formula $\varphi(p_1, \ldots, p_n)$ to the axioms of $Ł_{\aleph_0}$ where φ is of genus G_φ and $\psi(q_1, \ldots, q_m)$ is of genus G_ψ. Then $G_\varphi \leq G_\psi$ if and only if ψ is a theorem of $Ł_{\aleph_0+\varphi}$.*

Proof: ($\Rightarrow$) Suppose $G_\varphi \leq G_\psi$. By Lemma 3.7 every formula of genus G_φ is a theorem of $Ł_{\aleph_0} + \varphi$.

By Lemma 3.2 $G_{\varphi\wedge\psi} = G_\varphi$ and by 3.13 of [12] $\vdash_{Ł_{\aleph_0}} (\varphi \wedge \psi) \supset \psi$ so $\vdash_{Ł_{\aleph_0}+\varphi} (\varphi \wedge \psi) \supset \psi$. By Lemma 3.7 $\vdash_{Ł_{\aleph_0}+\varphi} (\varphi \wedge \psi)$ and by modus ponens $\vdash_{Ł_{\aleph_0}+\varphi} \psi$.

($\Leftarrow$) Suppose $\vdash_{Ł_{\aleph_0}+\varphi} \psi$, that is, there is a deduction in $Ł_{\aleph_0} + \varphi$ of ψ. An induction on the steps of the deduction will show that $G_\varphi \leq G_\psi$. Let $B_1, \ldots, B_n$, be the steps of a deduction of ψ in $Ł_{\aleph_0} + \varphi$ with $B_n = \psi$. Suppose $b + 1$ is in G_φ before the ; and $c + 1$ is in G_φ after the ;.

Base case. Each of the axioms of $Ł_{\aleph_0}$, φ and by Lemma 3.6, substitution instances thereof take the value 1 in a neighborhood around a_i/b for $0 \leq a_i \leq b$, $1 \leq i \leq n$ and each takes the value 1 at a_i/c for $0 \leq a_i \leq c$, $1 \leq i \leq n$.

Inductive hypothesis. Suppose that each of the B_i for $i < j$ takes the value 1 in a neighborhood around a_i/b for $0 \leq a_i \leq b$, $1 \leq i \leq n$ and each takes the value 1 at a_i/c for $0 \leq a_i \leq c$, $1 \leq i \leq n$.

Inductive step. B_j is either an axiom or φ itself or follows from B_h and $B_i = B_h \supset B_j$ by modus ponens or B_j is a substitution instance of an axiom, φ, or a previous B_h. The case where B_j is either an axiom or φ itself or a substitution instance of a previous step is treated just as in the base case.

Consider modus ponens. By the inductive hypothesis there are numbers ϵ_1 and ϵ_2 such that if for all x_j, $|v(x_j) - a/b_k| < \epsilon_1, 1 \leq j \leq n$ then $v(B_h) = 1$ and if for all x_j, $|v(x_j) - a/b_k| < \epsilon_2, 1 \leq j \leq n$ then $v(B_i) = 1$. Let ϵ be the lesser of

ϵ_1 and ϵ_2. Since if $v(B_h) = 1$ and $v(B_h \supset B_j) = 1$ then $v(B_j) = 1$ for any argument, if for all x_j, $|v(x_j) - a/b_k| < \epsilon$, $1 \leq j \leq n$ then $B_j = 1$ as well.

By induction, each step in the deduction, and thus ψ, takes the value 1 in a neighborhood around a_i/b for $0 \leq a_i \leq b$, so $b + 1$ is in G_φ before the ;. A similar argument shows that $c + 1$ is in G_ψ after the ;.

4 Connections between Rose and Komori In this section we provide, for every axiomatic extension of $\text{Ł}_{\aleph_0}$, an algebra that is characteristic for theoremhood. Conversely, we show that for an appropriate algebra a single axiom can be added to those for $\text{Ł}_{\aleph_0}$ to give an axiomatization sound and complete for the algebra. The connection between genus and the two types of algebras S_n and S_n^ω makes this possible.

We will show that $\text{Th}(\text{Ł}_n^\omega) = \text{Taut}(S_n^\omega)$. Since the algebra S_n^ω is an extension of the algebra S_n, $\text{Taut}(S_n^\omega) \subset \text{Taut}(S_n)$. Also $\bigcap_{j \in J} \text{Taut}(S_j^\omega) \subseteq \text{Taut}(S_m^\omega)$ if and only if there is a $j \in J$ such that $m - 1$ divides $j - 1$. Furthermore $\text{Taut}(S_i) \subseteq \text{Taut}(S_2)$, and $\text{Taut}(S_j^\omega) \subseteq \text{Taut}(S_2^\omega) \subseteq \text{Taut}(S_2)$. A generalization of Lindenbaum's theorem follows from Theorem 3.8, namely, if I and J are sets of positive integers.

$$\bigcap_{i \in I} \text{Taut}(S_i) \cap \bigcap_{j \in J} \text{Taut}(S_j^\omega) \subseteq \text{Taut}(S_m)$$

if and only if there is an $n \in I \cup J$ such that $m - 1$ divides $n - 1$.

Let Ł be an extension of $\text{Ł}_{\aleph_0}$, then by Komori [4] there exist finite sets of integers I and J such that

$$\text{Th}(L) = \bigcap_{i \in I} \text{Taut}(S_i) \cap \bigcap_{j \in J} \text{Taut}(S_j^\omega).$$

Let us call $\prod_{i \in I} S_i \times \prod_{j \in J} S_j^\omega$ the Komori representation of L. We now provide lemmas required for the soundness and completeness result.

The intuitive notion behind the next lemma is that evaluation in $S_{\aleph_0}$, the standard model for $\text{Ł}_{\aleph_0}$, "preserves closeness."

Lemma 4.1 *If $\varphi(p_1, \ldots, p_n)$ contains k $\supset$'s and a valuation v in $S_{\aleph_0}$ assigns to each p_i a value which differs from some j_i/l by less than ϵ then $v(\varphi)$ differs from some j/l by less than $2^k \cdot \epsilon$.*

Proof: By induction on the number of $\supset$'s in φ.

Base case: $k = 0$. $\varphi = p$ or $\sim p$ so $v(\varphi)$ differs from some j/l by less than ϵ.

Inductive hypothesis: The lemma holds when $k < m$.

Inductive step: Suppose that $\varphi = \psi \supset \theta$ contains m $\supset$'s. Then by the inductive hypothesis for some integers j_ψ and j_θ

$$|v(\psi) - j_\psi/l| < 2^{m-1} \cdot \epsilon,$$

and

$$|v(\theta) - j_\theta/l| < 2^{m-1} \cdot \epsilon.$$

If $v(\psi) \leq v(\theta)$ then $v(\varphi) = 1$. For the case $v(\psi) \geq v(\theta)$, $|(1 - v(\psi) + v(\theta)) - (1 - j_\psi/l + j_\theta)/l)\ | < 2^m \cdot \epsilon$.

Lemma 4.2 *If $G_\varphi = \langle B; C\rangle$ then $k+1 \in B$ if and only if $\vDash_{S^\omega_{k+1}} \varphi$.*

Proof: ($\Rightarrow$) Suppose $G_\varphi = \langle B; C\rangle$ and $k+1 \in B$. So $\exists\epsilon$ such that if $\forall p_i\ |p_i - j/k| < \epsilon$, then $v(\varphi) = 1$ in $S_{\aleph_0}$.

Let v be a valuation in S^ω_{k+1}, and let $n = \max\{|y| : (x,y)$ is used in the valuation of $\varphi\}$.

Define $h: S^\omega_{k+1} \mapsto S_{\aleph_0}$ such that $h(x,y) = x + \epsilon\frac{y}{2n}$. Note that $x = j/k$ for some j. We require that $\epsilon < 1/2k$ in order to insure the one-oneness of h.

h is a homomorphism since:

$$h(\neg(x,y)) = h(1-x,-y) = 1 - x - \epsilon\frac{y}{2n} = \neg h(x,y).$$

Consider $h((x,y) \to (z,w))$.

Case 1: $x < z$. $h((x,y) \to (z,w)) = h(1,0) = 1 = h(x,y) \to h(z,w)$.

Case 2: $x = z$.

$$\begin{aligned} h((x,y) \to (z,w)) &= h(1, \min(0, w-y)) \\ &= 1 + \min(0, w-y)/2n \cdot \epsilon \\ &= h\left(x + \epsilon\frac{y}{2n}\right) \to h\left(z + \epsilon\frac{w}{2n}\right) \\ &= h(x,y) \to h(z,w). \end{aligned}$$

Case 3: $x > z$.

$$\begin{aligned} h((x,y) \to (z,w)) &= h(1 - x + z, w - y) \\ &= 1 - x + z - ((w-y)/2n) \cdot \epsilon \\ &= h\left(x + \epsilon\frac{y}{2n}\right) \to h\left(z + \epsilon\frac{w}{2n}\right) \\ &= h(x,y) \to h(z,w). \end{aligned}$$

h is one-one, so h is a monomorphism and thus h gives a valuation in $S_{\aleph_0}$ and since $k+1 \in B$, $h(v(\varphi)) = 1$. Because h is a monomorphism and $h(x,y) = 1$ then $(x,y) = (1,0)$. So if $k+1 \in B$ then $\vDash_{S^\omega_{k+1}} \varphi$.

($\Leftarrow$) Suppose $\vDash_{S^\omega_{k+1}} \varphi$ and $k+1 \notin B$.

Let n be the number of $\supset$'s in φ. Since $k+1 \notin B$, there is a $p \in v(\varphi)$ and $j, 0 \le j \le k$ such that $\forall\epsilon > 0$, $\exists x$, $0 < |j/k - x| < \epsilon$ such that if $v(p) = x$ then $v(\varphi) \neq 1$.

Since by McNaughton's theorem evaluations are continuous piecewise linear functions it follows that there is an ϵ such that when $v(p) = x$ either $\forall x j/k - \epsilon < x < j/k$ or $\forall x, j/k < x < j/k + \epsilon$, $v(\varphi(\ldots, p, \ldots)) \neq 1$.

Without loss of generality suppose $\forall x, j/k < x < j/k + \epsilon$, $v(\varphi(\ldots, p, \ldots)) \neq 1$ when $v(p) = x$. Let $x = j/k + 1/k'$ where $1/k' < \epsilon$, and $(2^{n+2})/k' < 1/k$, and k divides k'.

Let $h : S_{k'+1} \mapsto S^{\omega}_{k+1}$ be defined by

$$h(i/k') = \begin{cases} (j/k, 0) & \text{if } i/k' = j/k \\ (j/k, \pm m) & \text{if } i/k' = j/k \pm m/k'. \end{cases}$$

h is homomorphism since:

$$h(\neg x) = h(1 - i/k') = (1 - j/k, \pm m) = \neg h(x).$$

Consider $h(x \to y)$.

Case 1: $x \leq y$. $h(x \to y) = h(1) = (1,0) = h(x) \to h(y)$.

Case 2: $x > y$. $h(x \to y) = h(1 - x + y) = (1 - j_1/k + j_2/k, \pm m_1 \pm m_2) = h(x) \to h(y)$.

So h is a monomorphism and $v(\varphi(\ldots, p, \ldots)) \neq 1$ when $v(p) = x$. So $h(v(\varphi))$ is a valuation of φ in S^{ω}_{k+1} and $h(v(\varphi)) \neq 1$ when $v(p) = x$ which contradicts the assumption that $\models_{S^{\omega}_{k+1}} \varphi$, so $k + 1 \in B$.

Lemma 4.3 *If $G_\varphi = \langle B; C \rangle$ then $k + 1 \in B \cup C$ if and only if $\models_{S_{k+1}} \varphi$.*

Proof: By the definition of genus.

The following theorem gives a complete characterization of all normal extensions of $Ł_{\aleph_0}$.

Theorem 4.4 (Soundness and weak completeness) *Let $\Phi = \{\varphi_1, \varphi_2, \ldots,\}$ be a set of formulas and let $G_\Phi = \bigwedge_{\varphi \in \Phi} G_\varphi$. Suppose $G_\Phi = \langle B; C \rangle$, and $Ł_{\aleph_0 + \Phi}$ is the logic obtained by adding $\{\varphi \mid \varphi \in \Phi\}$ to the axioms of $Ł_{\aleph_0}$. Then*

$$Th(Ł_{\aleph_0+\Phi}) = \bigcap_{i \in C} Taut(S_i) \cap \bigcap_{i \in B} Taut(S^{\omega}_i),$$

that is, $\prod_{i \in C} S_i \times \prod_{i \in B} S^{\omega}_i$ is characteristic for theoremhood for $Ł_{\aleph_0+\Phi}$.

Proof: Suppose $\Phi = \{\varphi_1, \varphi_2, \ldots,\}$ is a set of formulas and $G_\Phi = \bigwedge_{\varphi \in \Phi} G_\varphi$. Suppose $G_\Phi = \langle B; C \rangle$, $\mathrm{Taut}(L) = \bigcap_{i \in C} \mathrm{Taut}(Ł_i) \cap \bigcap_{i \in B} \mathrm{Th}(Ł^{\omega}_i)$ and $Ł_{\aleph_0+\Phi}$ is the logic obtained by adding the set Φ of formulas to the axioms of $Ł_{\aleph_0}$.

$\vdash_{Ł_{\aleph_0+\Phi}} \psi$ if and only if $G_\Phi \leq G_\psi$ by Theorem 3.8. And by Lemmas 4.2 and 4.3, $G_\Phi \leq G_\psi$ if and only if $\psi \in \bigcap_{i \in C} \mathrm{Taut}(S_i) \cap \bigcap_{i \in B} \mathrm{Taut}(S^{\omega}_i)$.

Corollary 4.5 *For every extension of $Ł_{\aleph_0}$, L, there is a formula φ such that $Th(L) = Th(Ł_{\aleph_0+\varphi})$, i.e., φ is an axiom that distinguishes L from other normal extensions of $Ł_{\aleph_0}$.*

Proof: Let $\Phi = \{\varphi_1, \varphi_2, \ldots,\}$ be a set of formulas and let $G_\Phi = \bigwedge_{\varphi \in \Phi} G_\Phi$. There is some formula φ such that $G_\Phi = G_\varphi$, so $Ł_{\aleph_0+\Phi} = Ł_{\aleph_0+\varphi}$.

Corollary 4.6 $Ł_{\aleph_0+\psi}$ *is an extension of* $Ł_{\aleph_0+\varphi}$ *if and only if* $G_\psi \leq G_\varphi$.

In light of the above theorem we can associate a genus with an extension of $Ł_{\aleph_0}$, and in light of the last corollary we can see that the extensions of $Ł_{\aleph_0}$ are partially ordered by the lattice of genera.

The notions of *the cardinal degree and the ordinal degree of completeness* of an axiom set were introduced by Tarski. See Tarski [14] page 100. The *car-*

dinal degree of completeness of a logic characterized by the set of axioms Φ is the number of logics which contain Φ. We say that a set of axioms is absolutely consistent if there is some formula which is not a consequence of the axioms.

Definition 4.7 The *ordinal degree of completeness* of a set Φ of axioms is the smallest ordinal $\alpha \neq 0$ such that there is no increasing sequence of type α of absolutely consistent nonequivalent sets of axioms which begins with Φ.

According to Łukasiewicz and Tarski [5], Lindenbaum proved that the ordinal degree of completeness of $Ł_3$ is 3. Tarski then generalized this result to show that the ordinal degree of completeness of $Ł_n$ is 3 if $n - 1$ is prime. That is, if $n - 1$ is prime adding any formula which is a nontheorem of $Ł_n$ but is a theorem of $Ł_2$ to the axioms of $Ł_n$ yields $Ł_2$, and adding any formula which is a nontheorem of $Ł_2$ to the axioms of $Ł_n$ yields inconsistency. Then in May 1930 "the problem of the degree of completeness was solved for systems $Ł_n$ with an arbitrary natural n; this was the joint result of members of a proseminar conducted by Łukasiewicz and Tarski in the University of Warsaw." (See the footnote on p. 49 of [14].) Rose [10] provides a proof of this result, namely that for any $n \geq 2$ the ordinal degree of completeness of $Ł_n$ is $d(n-1)+1$ where $d(n)$ is the number of distinct divisors of n including 1 and n. Rose went on in [11] to show that the ordinal degree of completeness of $Ł_{\aleph_0}$ is ω. These results follow from Theorem 4.4. Both the cardinal and the ordinal degree of completeness of any extension of $Ł_{\aleph_0}$ are determined by the genus of the added axiom.

Every proper extension of $Ł_{\aleph_0}$ has a finite cardinal degree of completeness determined by the number of elements of the lattice G which are less than or equal to G_φ. The ordinal degree of completeness for a proper extension of $Ł_{\aleph_0}$ is determined by the longest path in the lattice from the genus of the extension to the genus $\langle ;2\rangle$. For every integer n there is an extension with ordinal degree of completeness greater than n. Thus:

Corollary 4.8 *The ordinal degree of completeness of* $Ł_{\aleph_0}$ *is* ω.

The following result, published in Tokarz [15], now follows as a corollary to Theorem 4.4.

Definition 4.9 Let $C = \langle a_1, \ldots, a_n\rangle$ be a sequence of natural numbers. Let $N_C(a_i)$, for $1 \leq i \leq n$, denote the number of subsequences D of C such that:

$a_i \in D$ and for every $b \in D$, $a_i \geq b$,
if $j \neq k$ and $a_j, a_k \in D$ then $a_j - 1$ is not a divisor of $a_k - 1$,

Let $c(n) = \langle a_1, \ldots, a_k\rangle$ be the sequence such that:

$a_1 = n$,
$a_1 > \cdots > a_k > 1$,
for every i, $1 \leq i \leq k$, $a_i - 1$ is a divisor of $n - 1$.

Corollary 4.10 *For finite n, the cardinal degree of completeness of* $Ł_n$ *is*

$$\left(\sum_{a_i \in C(n)} N_{c(n)}(a_i)\right) + 1.$$

Scroggs [13] shows that the modal logic S5 is pretabular, that is, S5 has no finite characteristic matrix but every proper normal extension does. Dunn and

Meyer [3] proves that Dummett's *LC* is pretabular and Maksimova [6] shows that there are exactly three pretabular extensions of intuitionist propositional logic (one of which is *LC*). Another corollary to Theorem 4.4 is that there is only one pretabular extension of $\mathbf{Ł}_{\aleph_0}$, namely $\mathbf{Ł}_{\aleph_0+\varphi}$ where $G_\varphi = \langle 2;\rangle$, that is, the logic for which Chang's C, equivalently, for which Komori's S_2^ω is characteristic.

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