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Stable Groups, Mostly of Finite Exponent

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Abstract We prove certain properties of stable groups of finite exponent. In particular, an $\mathfrak{R}$-group of finite exponent has normal-by-finite 2–Sylow subgroups; if it has exponent $3 \cdot 2^n$ for some $n < \omega$, then it is nilpotent-by-finite. We give an easy proof of the fact that a locally finite subgroup of a stable group of finite exponent is nilpotent-by-finite. For groups of infinite exponent, we prove the definability of an algebraically closed field of characteristic 2 under certain circumstances. Finally, we prove two general propositions about normal subgroups of stable groups.

In this paper we shall be concerned with an arbitrary subgroup of a stable group (in short: a *substable* group). Recall that a subgroup H of a group G is definable if there is some formula $\varphi(x)$ with $H = \varphi(G)$; a subgroup is type-definable if it is the intersection of definable subgroups in a saturated model. If $H < G$ is any subgroup and $K \leq H$ is such that there is a formula φ with $K = \varphi(H)$, then K is relatively definable (with respect to H); the definition for relative type-definability is analogous. Note that if H is substable and $K < H$ relatively definable, then also H/K is substable: If $\varphi(H) = K$, then H/K may be viewed as subgroup of $G/\varphi(G)$; if $K \lhd H$ we may prefer to replace $\varphi(x)$ by $\psi(x) = \bigwedge_{h \in H} \psi(x^h)$, then $N_G(\psi(G)) \geq H$ and we may view H/K as subgroup of $N_G(\psi(G))/\psi(G)$. The connected component H^0 of a substable group H is the intersection with H of all definable subgroups K such that the index $|H : H \cap K|$ is finite; H^0 is normal in H and itself connected. If H is definable, we need only consider definable subgroups $K \leq H$ of finite index. So the index $|H : H^0|$ is at most $2^{|T|}$ and a saturated model and for a type-definable H the connected component has comparable size. But in the absence of saturation or if H is just substable, H^0 may even be reduced to the identity! For a relatively definable H this may be remedied in some cases by considering the locally connected component H^c, which is the intersection of all conjugates H^g such that the index $|H : H \cap H^g|$ is finite. By Baldwin-Saxl, this is again a relatively definable subgroup of finite index. Finally, a group is small if its theory has only countably many pure types.

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One of the guiding themes behind the study of stable groups is the idea that they should behave, at their best, somewhat like algebraic groups or like finite groups. One might try to prove then that a substable group of finite exponent is nilpotent-by-finite and in particular that its Sylow subgroups are normal-by-finite. Of course this might not be true as such: it might well be possible to construct a stable Tarski monster (or one of those constructed so far might turn out to be stable). However, there is a beginning of a Sylow theory: a substable 2-group is nilpotent-by-finite, and the maximal 2-subgroups (2-Sylow subgroups) of a substable group of finite exponent are nilpotent and all conjugate. Furthermore, conjugacy of the 2-Sylows also holds in a small stable group (cf. Poizat and Wagner [8], Théorème 11, Proposition 12, and Théorème 14).

First we note that for substable groups the Frattini-argument is valid:

Remark 1 Let G be a substable group of finite exponent. Then any subgroup $H \leq G$ containing the normalizer of a 2-Sylow S of G is self-normalizing; if N is normal in G and P is a 2-Sylow of N, then $G = N \cdot N_G(P)$. These results also hold if G is small stable, provided H and N are definable.

Proof: Suppose $n \in N_G(H)$. Then S^n is another maximal 2-group of H, so there is $h \in H$ with $S^{nh} = S$, that is $nh \in N_G(S) \leq H$. Therefore $n \in H$ as well.

For the second part, consider arbitrary $g \in G$. Then P^g is a 2-Sylow of $N^g = N$, so there is $n \in N$ with $P^g = P^n$, that is $gn^{-1} \in N_G(P)$.

Theorem 2 *Let S and T be 2-Sylow subgroups of some substable group G of finite exponent, such that their intersection I is maximal subject to $S^c \neq T^c$. Then $N = N_G(S^c) \cap N_G(T^c) \cap N_G(I)$ acts transitively on the infinite group $A = (N_S(I)/I)[2]$ (and also on $(N_T(I)/I)[2]$). Furthermore, N/I has odd exponent and any abelian subgroup of $N/C_N(A)$ is finite.*

Proof: First note that both S and T are relatively definable as maximal nilpotent 2-groups of finite exponent. Secondly, as $S^c \neq T^c$, clearly I must have infinite index in S or in T. But S^c and T^c are also conjugate, so if I had finite index in S, then $S^c \leq I$, whence $S^c < T^c$ and we could conjugate T^c to a proper subgroup, contradicting stability. Thus I has infinite index in both S and T.

As I has infinite index in T, there is a minimal k such that $|Z_k(T) : I \cap Z_k(T)|$ is infinite. Then $|Z_{k-1}(T) : I \cap Z_{k-1}(T)|$ is finite, so a subgroup X of finite index in $Z_k(T)$ centralizes T modulo I and in particular normalizes I. Hence $N_T(I)/I$ is infinite, and so is $N_{T^c}(I)/I$. We note for later use that X/I is an infinite abelian 2-group of finite exponent, which must therefore contain infinitely many involutions.

Fix an involution $j \in N_{T^c}(I)/I$ and consider any involution $i \in N_S(I)/I$. If the order of ij were even, both i and j would commute with a common third involution $k \in N_G(I)/I$ and there were maximal 2-subgroups $S' \supset I \cup \{i,k\}$ and $T' \supset I \cup \{k,j\}$, with $S^c = S'^c = T'^c = T^c$ due to the maximality of I, a contradiction. Hence the order is odd and there is an involution $k_i = ij\ldots i = ji\ldots j \in N_G(I)/I$ ($\mathrm{o}(ij)$ factors) with $i^{k_i} = j$ and $j^{k_i} = i$ (mod I). Since I is maximal, we must have $(S^c)^{k_i} = T^c$ and $(T^c)^{k_i} = S^c$; in particular $j \in T^c$ implies $i \in S^c$. But then, if $i' \in N_S(I)/I$ is another involution, $k_i k_{i'}^{-1} \in N_G(S^c) \cap N_G(T^c) \cap N_G(I) = N$ maps i to i'.

As above $N_{S^c}(I)/I$ is infinite and as a nilpotent 2-group contains a central involution. But all other involutions in $N_S(I)/I$ are conjugate and thus central as well: They form a central subgroup, which must be infinite, as there are infinitely many involutions in $N_S(I)/I$.

Suppose now that there were an involution $n \in N/I$. Then there were 2-Sylow subgroups $S' \supset S^0 \cup \{n\}$ and $T' \supset T^0 \cup \{n\}$, and $S' \cap T' > I$. Hence $S^0 = S'^0 = T'^0 = T^0$, contradiction.

As for the last statement, this follows from the next Lemma.

Lemma 3 *Let A and B be substable abelian groups of finite, coprime exponent, such that B acts on A. Then a subgroup of B of finite index stabilizes A.*

Proof: We may suppose that A has exponent p^m and B has exponent q^n, for two different primes p and q. For if B_p^q is a subgroup of finite index of B_p stabilizing A_q, then $\bigcap_q B_p^q$ is a subgroup of finite index in B_p stabilizing A, and $\bigoplus_p \bigcap_q B_p^q$ is a subgroup of finite index in B stabilizing the whole of A.

Let $C \leq A$ be a minimal nontrivial intersection of kernels of endomorphisms, every one of which is generated by at most $q^n + 1$ elements from B. There are only finitely many ways in which such an endomorphism may be generated, hence C is definable by Baldwin-Saxl. As B is abelian, C is B-invariant, and for any $q^n + 1$ elements $\bar{b} \in B$ any endomorphism of C generated by $\bar{b}$ is either zero or an automorphism. Hence these $\bar{b}$-generated endomorphisms form a commutative field F. But F has at most q^n elements of order q^n, so two of the $\bar{b}$ must have the same action on C. As $\bar{b}$ was arbitrary, there are at most q^n different actions by elements of B on C and a subgroup B_1 of finite index fixes C.

Now consider $C_2 \leq A$, a maximal centralizer of some subgroup $B_2 \leq B$ of finite index, and suppose $C_2 < A$. Again a subgroup B_3 of finite index stabilizes some set $C_3/C_2 \leq A/C_2$. But then for $d \in C_3$ and $a \in B_3$ we get $(B_3 - 1)^2 C_3 = 0$, whence $C_3 B_3$ is 3-nilpotent. But the order of a is coprime to p, so $ad = d$ for all $a \in B_3$, contradicting the maximality of C_2.

Note that any two involutions in $N_S(I)/I$ and $N_T(I)/I$ are conjugate in $N(I)/I$ by some involution k interchanging S^c and T^c. Hence k normalizes N. If it had only finitely many fixed points, by a result of Hartley [4] k would invert an abelian subgroup of finite index. But then a subgroup of finite index would stabilize A, in contradiction to the transitivity of the action of N/I on the infinite group A. So $C_N(k/I)$ is infinite.

Definition 4 An $\mathfrak{R}$-group is a group such that for any definable transitive group action, if a generic point x is algebraic over some point y, then y is generic as well.

Information on $\mathfrak{R}$-groups may be found in Wagner [9],[10]. Important examples are superstable groups and small stable groups.

Corollary 5 *Let G be an $\mathfrak{R}$-group of finite exponent. Then there is a normal nilpotent 2-group I such that G/I has finite 2-Sylow subgroups.*

Proof: Consider a 2-Sylow subgroup S and suppose that S^c is not normal. Then there is a conjugate T of S with $S^c \neq T^c$ and maximal intersection I, and $N = N_G(S^c) \cap N_G(T^c) \cap N_G(I)$ acts transitively on $A = (N_S(I)/I)$ [2]. But since G is $\mathfrak{R}$, $N/C_N(A)$ contains an infinite abelian subgroup, contradicting Theorem 2.

We can also say something similar in the case of small stable groups of possibly infinite exponent. First a general Lemma and Proposition:

Lemma 6 *Let N be a definable transitive group of automorphisms of the definable abelian group A of infinite exponent. If $A \rtimes N$ is stable, then a division ring is definable.*

Proof: We choose a minimal type-definable subgroup $A_0 \leq A$. As the intersection of A_0 with any conjugate is either trivial or the whole of A_0, $N_N(A_0)$ acts transitively on $A_0 - \{0\}$. Note that by minimality any definable endomorphism of A_0 is either zero or an automorphism. We are going to interpret the division ring R of definable endomorphisms of A_0 in $M = N_N(A_0)/C(A_0)$. Fix $0 \neq a \in A_0$. For $0 \neq r \in R$ there is $n_r \in N_N(A_0)$ such that $ra = n_r a$, that is $\ker(r - n_r)$ is nontrivial and thus equals A_0. So $r \mapsto n_r$ is the required isomorphism between $R^\times$ and M. Of course addition is now definable on M, by putting $g + h = n$ iff $ga + ha = na$.

If the theory is $\mathfrak{R}$, then by [9], Theorem 2.7, the division ring is an algebraically closed field.

Proposition 7 *Let N be a definable group of automorphisms of the definable abelian group A of exponent p such that A is N-analyzable. If $A \rtimes N$ is $\mathfrak{R}$ and definable abelian subgroups of N have an exp(A)-divisible connected component, then an algebraically closed field is definable.*

Proof: By [9], Theorem 3.1, there are finitely many definable abelian subgroups N_i, $i < k$, of N such that N is $\{N_i : i < k\}$-analyzable, and so is A. By taking k minimal and replacing successively N_i by the intersection of all definable $N' \leq N_i$ with N_i/N' analyzable in $\{N_j : j > i\}$, we may assume that the generic types of the N_i are pairwise foreign. Note that the new N_i are possibly only type-definable; in any case they are connected.

Let M be the intersection of all $N' \leq N_0$ such that the quotient N_0/N' is analyzable in a formula φ with the following property (*): whenever N_0 is analyzable in a set Σ of formulas, it is also analyzable in $\Sigma - \{\varphi\}$. (This is a kind of strongly connected component, and called the Frattini-free component in [10].) We note that $\text{gen}(M)$ is foreign to all φ with property (*).

Let $Z = C_A(M)$, a proper subgroup of A. Then if $a \in A$ is centralized by some $n \in M$ modulo Z, we have $(n-1)^2 a = 0$, whence $(n^p - 1)a = (n-1)^p a = 0$ and a is centralized by n^p. However, $M^p = M$, so no point in A/Z is stabilized by the whole of M. Now choose an N-minimal subgroup B of A/Z.

We claim that $\text{gen}(B)$ is not foreign to N_0. Indeed, as A is analyzable in $\{N_i : i < n\}$, so is B. So if $\text{gen}(B)$ were foreign to N_0, by minimality it were almost N_i-internal for some $0 \neq i$. On the other hand $N_0/C_{N_0}(B)$ is an infinite B-internal quotient, as already $M/C_M(B)$ is infinite; together this implies that $\text{gen}(N_0)$ cannot be foreign to N_i, contradiction.

Next we claim that $\text{gen}(B)$ is not foreign to M. For otherwise, as it is not foreign to N_0, there is a formula φ with property (*), such that $\text{gen}(B)$ is not foreign to φ. By minimality B is almost φ-internal; as $M/C_M(B)$ is infinite and B-internal, gen(M) cannot be foreign to φ, contradiction. Hence B is M-analyzable.

Let M_0 be a definable abelian supergroup of M with $C_A(M_0) = C_A(M)$, and let B_0 be an M_0-minimal subgroup of B. We finally show that gen(B_0) is not foreign to $\varphi(x) =$ "$x \in M_0/C_{M_0}(B_0)$". So suppose otherwise. We claim that φ has property (*). Indeed, if Σ is a set of formulas such that N_0 is $\Sigma \cup \{\varphi\}$-analyzable, then, as B is M-analyzable, also B_0 is $\Sigma \cup \{\varphi\}$-analyzable. So gen(B_0) cannot be foreign to Σ; by minimality B_0 is almost Σ-internal. Clearly φ is B_0-internal, whence almost Σ-internal, and N_0 must be Σ-analyzable. So φ has property (*). But this implies $C_M(B_0) \geq M$, contradiction.

Therefore gen(B_0) is not foreign to φ, so by [9], Theorem 4.2, there is a definable algebraically closed field K with $B_0 \cong K^+$ and $M_0/C_{M_0}(B_0) \hookrightarrow K^\times$.

Note that the divisibility condition is in particular satisfied if N has only finitely many elements of order exp(A). Furthermore, if N acts transitively on A, then gen(A) is even N-internal. So Lemma 6 and Proposition 7 together prove the definability of an algebraically closed field for definable transitive group actions under $\mathfrak{R}$ plus the divisibility condition.

Theorem 8 *Let G be a small stable group, and S and T be two infinite 2-Sylows such that $I = S \cap T$ is maximal subject to having infinite index in both. Then there are relatively definable subgroups $S^c \leq S$ and $T^c \leq T$ of finite index such that $N = N_G(I) \cap N_G(S^c) \cap N_G(T^c)$ acts transitively on the group $A = (N_S(I)/I)[2]$. N/I does not contain involutions. If A is infinite, then an algebraically closed field of characteristic 2 is definable.*

Proof: First we have some problems about definability, as the 2-Sylows need not be relatively definable any longer. But for any 2-Sylow S there is a finite 2-extension of a definable nilpotent group $\bar{S}$ with S as 2-Sylow, and replacing $\bar{S}$ by $\bigcap_{g \in N_G(S)} \bar{S}^g$, we may assume $N_G(S) \leq N_G(\bar{S})$. As the 2-Sylows are all conjugate, we get a conjugate family of such supergroups, and a maximal intersection I must still exist (see [8], Section 1). There also is some kind of locally connected component: The intersection with S of all $\bar{S}'$ such that $|S : S \cap \bar{S}'|$ is finite forms a subgroup S^c of finite index contained in all intersections $S \cap S'$ of finite index in S, and the normalizer of S^c is maximal for a subgroup of finite index. Also these locally connected components are all conjugate, and it follows again from stability that if $S \cap S'$ has finite index in S, it also has finite index in S'. Furthermore, if we put $\bar{I} = \bigcap_{g \in N_G(I)} (\bar{S} \cap \bar{T})^g$, then we may work in $N_G(\bar{I})/\bar{I}$ instead of $N_G(I)/I$, as $N_G(I) \leq N_G(\bar{I})$ and $S \cap \bar{I} = I$. S/I is substable, so local nilpotency implies $N_S(I) > I$. Finally there is an increasing sequence I_k of definable 2-groups with $\bigcup_{k \in \omega} I_k = I$: I has a nilpotent normal subgroup J of finite index, and for big k the sets $J[2^k]$ form characteristic definable subgroups of J ([8], Lemme 16). So if $\bar{\imath}$ is a system of representatives of I/J, then $I_k := \langle \bar{\imath} \rangle J[2^k]$ will do.

Again we fix an involution $j \in N_T(I)/I$ and consider any involution $i \in N_S(I)/I$. There is $k < \omega$ with $i^2, j^2 \in I_k$. Consider a definable abelian group containing ij (modulo I_k), which is inverted by i and by j and normalizes I. By [8], Lemme 13 it is divisible-plus-bounded, and so it remains modulo I. So modulo I it either contains an involution k (which is impossible, since k would commute both with i and j), or there is r (inverted by i and by j) with $r^2 = ij$. Then $i^r = r^{-1}ir = irr = i(ij) = j$ and $j^r = r^{-1}jr = jr^2 = jij = i^j$. So there is $n_i = rj \in N_G(I)/I$ with $i^{n_i} = j$ and $j^{n_i} = i$, whence by maximality of I we have $(S^{n_i})^c = T^c$ and $(T^{n_i})^c = S^c$. If i' is another involution, then $n = n_i n_{i'}^{-1}$ takes i to i' and normalizes S^c and T^c. As there is a central involution in S/I, all involutions are central and $(N_S(I)/I)[2]$ forms a group A. Note that $\bar{A} := A \cdot \bar{I}/\bar{I}$ is isomorphic to A and definable as the orbit of any of its (nonzero) elements under $N \cup \{0\}$; if it is infinite, it is connected, as all involutions are conjugate to one in the connected component. The fact that N/I contains no involutions follows as in Theorem 2. Finally, if A is infinite, we can apply Proposition 7 to $\bar{A} \rtimes N/C_N(\bar{A})$. We need to check that any definable abelian subgroup $H/C_N(\bar{A})$ is 2-divisible. So let $h \in H$. Then $Z = Z(C_H(h))$ is an abelian subgroup of H. By [8], Lemme 13 Z is the sum of a divisible group and one of bounded exponent, and so is $Z/Z \cap I$. But this quotient has no involutions and thus is 2-divisible. As $I \leq C_N(\bar{A})$, also $Z/C_Z(\bar{A})$ is 2-divisible, whence $hC_N(\bar{A})$ has a square root. Obviously, the resulting field will have characteristic 2.

The next proposition may be useful in a variety of circumstances:

Proposition 9 *Let $\mathfrak{N} = \{N_i : i \in I\}$ be a family of pairwise normalizing k-nilpotent substable groups. Then $\mathfrak{N}$ generates a nilpotent subgroup. The result also holds with soluble instead of nilpotent.*

Proof: As any k-nilpotent subgroup is contained in a uniformly (in k) definable k-nilpotent subgroup, whose normalizers are in turn uniformly definable, we may assume that the N_i are actually uniformly definable and normal in some G. (This is also true for soluble instead of nilpotent.)

We first treat the nilpotent case. We put $N_i^j = Z_{k-j+1}(N_i)$, and $N_i^0 = G$. Then N_i^j is $(k-j)$-nilpotent, normal and uniformly definable. Let n be a bound for the length of a descending chain of intersections of the N_i^j, and consider a sequence $a_s \in N_{i(s)} \in \mathfrak{N}$. Let $b_0 = a_0$ and $b_{s+1} = [b_s, a_{s+1}]$.

If t is such that $b_s \in N_{i(s+1)}^t - N_{i(s+1)}^{t+1}$, then $b_{s+1} \in N_{i(s+1)}^{t+1}$, and because of normality b_{s+1} lies in all N_i^j's which contain b_s: $\bigcap_{b_s \in N_i^j} N_i^j > \bigcap_{b_{s+1} \in N_i^j} N_i^j$. But the sequence can descend at most n times, whence $b_n \in Z(N_{i(n+1)})$ and b_{n+1} must be trivial. It follows that $\langle \mathfrak{N} \rangle$ is nilpotent of class $n+1$.

In the soluble case, we use induction on the solubility class, the case of abelian groups having just been dealt with. So we may assume that the derived subgroups $\{N_i' : i \in I\}$ generate a soluble normal subgroup N, which we may assume to be definable. But $\{N_i N/N : i \in I\}$ is a family of normal abelian subgroups of G/N and generates a nilpotent group; the result follows.

In particular, a substable group G contains a normal relatively definable soluble subgroup $R_k(G)$ containing all normal k-soluble subgroups. So the existence of the soluble radical of G means that the increasing sequence of R_k's becomes stationary. Similarly there is a relatively definable normal nilpotent subgroup $F_k(G)$ containing all normal k-nilpotent subgroups, and the Fitting subgroup of G exists iff the F_k's become stationary.

The following Proposition has a similar flavor and generalizes a result of Borovik and Thomas [1] from the finite Morley rank to the general stable case:

Proposition 10 *Let $H(a)$ be an a-definable normal substable group and suppose that for any independent family $\{a_i \models tp(a) : i < \omega\}$ the intersection $\bigcap_{i<\omega} H(a_i)$ is empty. Then $H(a)$ is nilpotent (and so is $\langle H(a_i) : i < \omega\rangle$ by Proposition 9).*

Proof: Suppose otherwise. We clearly may assume that $\mathrm{tp}(a)$ is stationary, as the hypothesis holds for any stationarization of it (and the nilpotency class of $H(a)$ then forms part of $\mathrm{tp}(a)$). Furthermore, we work in a saturated model.

By Baldwin-Saxl there is a minimal intersection N of conjugates of $H(a)$ not contained in the hypercenter Z of $H = \langle H(a') : a' \models \mathrm{tp}(a)\rangle$. (Here the parameters of the different $H(a)$'s in the definition of N are not necessarily independent. Note that H is in general not at all definable, but its iterated centers are relatively definable as in any substable group.) If $N \leq H(a')$ for some a', then a' must fork with the parameters needed for the definition of N (otherwise N-conjugates of $H(a')$ independent over N would be independent over $\emptyset$, with nontrivial intersection). Hence $[N, H(a)] \leq Z$ for generic a by minimality, and so it is already contained in some iterated center $Z_i(H)$. But the centralizer modulo $Z_i(H)$ of some $H(a')$, $a' \models \mathrm{tp}(a)$, contains the centralizer modulo $Z_i(H)$ of some infinite independent set $\{H(a_i')\}_{i<\omega}$, as two independent series have the same centralizer modulo $Z_i(H)$. Hence the centralizer modulo $Z_i(H)$ of $\{H(a') : a' \models \mathrm{tp}(a)\}$ equals the centralizer modulo $Z_i(H)$ of a generic subfamily, and must thus contain N, contradicting the choice of N.

If G were superstable and φ a formula such that for any subformula $\psi \subseteq \varphi$ of the same (Shelah) rank the intersection $\bigcap\{H(a) : a \models \psi\}$ were trivial, then any completion of φ to a type of the same rank would satisfy the hypotheses of the proposition, so groups $H(a')$ with a' of maximal rank are nilpotent. But now by compactness there is a bound n on the nilpotency class (the set of $a \models \varphi$ of maximal rank is closed, and the set of a with $H(a)$ nilpotent is open), and there exists a formula ψ of lower rank such that for any $a' \models \varphi \wedge \neg\psi$, $H(a')$ is n-nilpotent.

A result of Kegel [5] states that a locally finite group of finite exponent with the chain condition on centralizers is nilpotent-by-finite; we are now aiming for an easy proof of this result in the more restricted case of a locally finite substable group of finite exponent.

Fact 11 [3] A periodic soluble group with chain condition on centralizers is nilpotent-by-abelian-by-finite. If the exponent is finite, then the group is nilpotent-by-finite.

Fact 12 [2] A locally nilpotent group with chain condition on centralizers is soluble; if its chains of centralizers have their length bounded by k, it is k-soluble.

Fact 13 [2] A periodic locally nilpotent group is nilpotent-by-finite; if it has finite exponent, it is nilpotent.

Remark 14 A periodic substable group G has a Fitting subgroup $F(G)$ and a soluble radical $R(G)$; if the exponent is finite, then the index $|R(G) : F(G)|$ is finite.

Proof: The maximal normal locally nilpotent subgroup of G is nilpotent-by-finite and hence has a maximal normal nilpotent subgroup F. As F is characteristic, it must be the Fitting subgroup.

Let k be a bound for the length of a chain of centralizers. Then by Facts 11 and 12 the soluble subgroups of G are $(k+1)$-soluble-by-finite. So we may divide out by $R_{k+1}(G)$ and assume that all normal soluble subgroups of G are finite. But then they must be centralized by the centralizer-connected component of G (i.e., the minimal centralizer of finite index in G). But this subgroup must have finite center, whence the group generated by all normal soluble subgroups of G must be finite as well.

Finally, the last part follows from Fact 11.

Theorem 15 *A locally finite substable group G of finite exponent is nilpotent-by-finite.*

Proof: We may divide out by the radical and thus assume that G is semisimple. By the chain condition on centralizers, we may assume that any proper centralizer has infinite index. We aim to prove that G is trivial, so we may replace G by a countable elementary restriction. Therefore there is a sequence $\{G_i : i < \omega\}$ of finite subgroups with $\bigcup_{i<\omega} G_i = G$. The classification of finite simple groups tells us that there are only finitely many simple groups of given exponent, so we may assume that the G_i have nontrivial normal subgroups A_i.

Case 1: The A_i may be chosen abelian. We replace A_i by some abelian nontrivial intersection of centralizers normalized by G_i, which intersects G nontrivially. But now $\bigcap_{i>j} N(A_i)$ is increasing and uniformly definable and hence eventually stationary from some j_0 onwards: $A_{j_0} \cap G$ is a normal abelian subgroup of G, contradiction.

Case 2: Almost all G_i are semisimple. If we choose the A_i to be minimal normal subgroups of the G_i, then they must be the direct product of simple nonabelian subgroups. However, their number is bounded by the chain condition on centralizers, so that the A_i and thus also the $N(A_i)$ are uniformly definable, and there must be an i_0 such that $\bigcap_{j\geq i_0} N(A_j)$ is maximal (and hence the whole of G): A_{i_0} is a nontrivial normal subgroup. However, as G has no proper centralizer of finite index, any finite normal subgroup must be central and hence abelian, contradiction.

This finishes the proof.

Proposition 16 *A periodic locally soluble substable group G is soluble.*

Proof: It is sufficient to show that G is soluble-by-finite; proceed as in Case 1 of the last Theorem.

In Poizat [7] it was proven that a group of finite Morley rank and exponent $3 \cdot 2^n$ is nilpotent-by-finite. We now generalize this to $\mathfrak{R}$-groups. The case of an arbitrary stable group of exponent $3 \cdot 2^n$ is still open.

Theorem 17 *Let G be an $\mathfrak{R}$-group of exponent $3 \cdot 2^n$. Then G is nilpotent-by-finite.*

Proof: We note first that a stable group with a generic element of order 3 is nilpotent-by-finite by [7] and Wagner [11]. We suppose, by way of contradiction, that G is not nilpotent-by-finite, and work in a saturated model. By Corollary 5 we may assume that the 2–Sylows are finite.

By Facts 11 and 12 and by compactness there is a bound k on the nilpotency class of the connected component of a locally finite subgroup of G. Now let S be a maximal locally connected k-nilpotent subgroup of G. If there is such S with $N_G(S)/S$ infinite, we replace G by $N_G(S)/S$ and obtain a group where all locally finite subgroups are finite. By the chain condition on centralizers we may assume that every proper centralizer is nilpotent-by-finite, whence finite. However since G is not abelian-by-finite, by [11], Theorem 11, an involution must have an infinite centralizer, so it must be central. Hence $G/Z_n(G)$ is a group of exponent 3, whence nilpotent-by-finite, contradiction.

Therefore $N_G(S)/S$ must be finite and $N_G(S)$ is a maximal locally finite subgroup of G. As the different maximal locally connected k-nilpotent S' are uniformly definable, so are the maximal locally finite subgroups and their subgroups containing S', and we choose two of them, S_1 and S_2, with maximal intersection I subject to $N_G(I)/I$ being infinite (e.g., S' and a true conjugate will satisfy this last requirement). We can replace G by $N_G(I)/I$; then G is not nilpotent-by-finite, but any two distinct maximal locally connected nilpotent subgroups S and T of G are disjoint: If $I' = S \cap T$, then I' has infinite index in both; as I' has infinite index in $N_S(I')$ and $N_T(I')$, $N_G(I')/I'$ is infinite. By the maximality of I, I' must be trivial.

By the chain condition on centralizers we may further assume that every proper centralizer is locally finite and therefore nilpotent-by-finite. Note that maximal locally finite subgroups remain uniformly definable and are self-normalizing. Furthermore G is centerless.

Again by [11], Theorem 11, an involution i in G has infinite centralizer. Hence there is a unique maximal locally connected nilpotent 3-subgroup $B(i) \geq C_G(i)^0$. If $j \notin N(B(i))$ is an involution, then $B(i)j$ contains exactly one involution (namely j): if k were a second one, then $jk \in B(i)$, so there is some $b \in B(i)$ with $jk = b$, that is $b^j = b^{-1}$. Hence $b \in B(i) \cap B(i^j)$, so $B(i) = B(i^j)$, contradicting $j \notin N(B(i))$.

We claim that $(C_G(i) \cap B(i)) \cdot i^G$ is generic over i. Indeed, if $g \in G$ is principal generic, then $i^g \notin N(B(i))$ (otherwise $i^G \subset \bigcap_{g \in G^0} N(B(i))^g$, so this generates an infinite normal subgroup whose locally connected component lies in all conjugates of $B(i)$). So $B(i)i^g$ contains a unique involution i^g and $(C_G(i) \cap B(i))g$ is algebraic over $B(i)i^g$. Hence $(C_G(i) \cap B(i))g$ is algebraic over $(C_G(i) \cap B(i))i^g$, and as the former is a generic element of $(C_G(i) \cap B(i)) \backslash G$, so is the latter by $\mathfrak{R}$.

Hence for principal generic g over i the set $B(i)g$ contains a unique involution k, which in addition is conjugate to i. Thus there are unique $k = i^h \in i^G$ and $b \in B(i)$ with $g = bk$, and $b \in C_G(i)$. But if there were $c \in B(i)^0 - C_G(i)$ (independent from g), then also $c^{-1}g$ were principal generic and hence $g \in cC_G(i) \cdot i^G$, contradicting the uniqueness of k and b. Hence $B(i)^0 \leq C_G(i)$. But now i is algebraic over $B(i)$ (that is, the canonical parameter needed for its definition): If C is the intersection with $B(i)$ of all centralizers of involutions containing $B(i)^0$ and i, j centralize C with $Ci = Cj$, then i inverts and fixes $ij \in C \leq B(i)$: $i = j$. Furthermore i and j normalize $B(i)$, as C has finite index in $N_G(B(i))$, there are only finitely many possibilities for i.

So if g is a generic over i with $g = bk$ as above, it follows from $\mathfrak{R}$ that b^k is generic: First i^k is algebraic over b^k as one of the finitely many involutions $j \in i^G$ with $b^k \in B(j)$. Then $B(i)k$ is algebraic over $C(i)k$, the set of elements conjugating i to i^k. Now k is the unique involution in $B(i)k$. Hence b is algebraic over b^k, and so is the generic $g = bk$.

Thus there is a generic element of order 3, and G is nilpotent-by-finite.

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