Notre Dame Journal of Formal Logic
Volume 34, Number 2, Spring 1993

Remarks on Strong Nonstructure Theorems

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Abstract In this paper we continue the work started in Hyttinen and Tuuri [4]. We study the existence of universal equivalence trees and the existence of strongly bistationary sets.

1 Introduction In this paper we answer some questions left open in [4]. The starting point in [4] was Shelah's nonstructure theorem for unstable (and DOP and OTOP) theories. In [4] we looked how strong nonstructure theorems can be proved in terms of Ehrenfeucht-Fraisse games (see below). In many cases we were able to prove maximal results by using rather strong cardinal assumptions, but also many questions were left unanswered. In this paper we answer two of those questions.

One case in which we studied strong nonstructure theorems in [4] was the case in which we assumed about the theory only that it is unsuperstable. In this case we cannot prove maximal results, as shown in [4]. The theorems we were able to prove depend on the existence of so-called strongly bistationary sets. In the first part of this paper we continue the studies on the existence of these sets. In the main result of this part we show that if $\lambda = \kappa^+$, $\kappa > \xi \geq \omega$, $cf(\kappa) < \kappa$ and $A \subseteq \{\alpha < \lambda \mid cf(\alpha) = \xi\}$ is stationary then there is strongly bistationary $B \subseteq A$.

In the second part of this paper we study the existence of universal equivalence trees (see Definition 2.3).

In Hyttinen and Shelah [2] and [3] we will continue the studies of strong nonstructure theorems in the case the theory is unsuperstable.

2 Preliminaries In this chapter we give the most important definitions and a theorem from [4] needed in this paper. The proof of the theorem in [4] is based on the construction of Shelah in [9].

Definition 2.1 Let λ be a cardinal and α an ordinal. Let t be a tree (i.e., for all $x \in t$, the set $\{y \in t \mid y < x\}$ is well-ordered by the ordering of t). If $x, y \in t$ and $\{z \in t \mid z < x\} = \{z \in t \mid z < y\}$, then we denote $x \sim y$, and the equivalence class of x for $\sim$ we denote $[x]$. By a λ,α-tree t we mean a tree which satisfies:

Received February 12, 1992; revised February 25, 1993

(i) $|[x]| < \lambda$ for every $x \in t$;
(ii) there are no branches of length $\geq \alpha$ in t;
(iii) t has a unique root;
(iv) if $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

Definition 2.2 Let t and t' be trees.

(i) The domain of the sum $t'' = t + t'$ is $t \cup \{(b,y) \mid b \text{ a branch of } t, y \in t'\}$. The ordering is defined in the natural way, i.e., t'' is obtained from t by putting a copy of t' on top of each branch in t.

(ii) The domain of the product $t'' = t \times t'$ is $\{(x,f,y) \mid x \in t', f \text{ a function from } \{y \in t' \mid y < x\} \text{ to the branches of } t, y \in t\}$. Here $(x,f,y) \leq (x',f',y')$ iff
(a) either $x = x'$, $f = f'$ and $y \leq y'$,
(b) or $x < x'$, $f \subseteq f'$ and $y \in f'(x)$.

Next we define Ehrenfeucht-Fraisse games $G^t_\kappa(\mathcal{A},\mathcal{B})$. In Section 4 only the special case $G^t_2(\mathcal{A},\mathcal{B})$ is needed. In Section 3 if we are only interested in strongly bistationary sets then also only the special case $G^t_2(\mathcal{A},\mathcal{B})$ is needed. But in order to understand the consequences of the existence of α-bistationary sets the full definition is needed (see Theorem 3.6).

Definition 2.3 Let t be a tree and κ a cardinal. The Ehrenfeucht-Fraisse game approximated by t between models $\mathcal{A}$ and $\mathcal{B}$, $G^t_\alpha(\mathcal{A},\mathcal{B})$, is the following. At each move α:
(i) player $\forall$ chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$ and either $a^\beta_\alpha \in \mathcal{A}$, $\beta < \kappa_\alpha$ or $b^\beta_\alpha \in \mathcal{B}$, $\beta < \kappa_\alpha$;
(ii) if $\forall$ chose from $\mathcal{A}$ then $\exists$ chooses $b^\beta_\alpha \in \mathcal{B}$, $\beta < \kappa_\alpha$, else $\exists$ chooses $a^\beta_\alpha \in \mathcal{A}$, $\beta < \kappa_\alpha$.

$\forall$ must move so that $(x_\alpha)_{\beta \leq \alpha}$ form a strictly increasing sequence in t. $\exists$ must move so that $\{(a^\beta_\gamma, b^\beta_\gamma) \mid \gamma \leq \alpha, \beta \leq \kappa_\gamma\}$ is a partial isomorphism $\mathcal{A} \to \mathcal{B}$. The player who first has to break the rules loses. We write $G^t(\mathcal{A},\mathcal{B})$ for $G^t_2(\mathcal{A},\mathcal{B})$.

If $\exists$ has a winning strategy for $G^t(\mathcal{A},\mathcal{B})$, then we write $\mathcal{A} \equiv^t \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are t-equivalent and t is an equivalence tree of $\mathcal{A}$ and $\mathcal{B}$. If $|\mathcal{A}| = \kappa$ and t is a tree such that for all $\mathcal{B}$ of power κ, $\mathcal{A} \equiv^t \mathcal{B}$ implies $\mathcal{A} \cong \mathcal{B}$, then we say that t is a universal equivalence tree of $\mathcal{A}$.

Definition 2.4 Hyttinen and Väänänen [5] If t, t' are trees, then the comparison game $G_\leq(t,t')$ is the following. At each move α:
(i) player $\forall$ chooses some $x_\alpha \in t$;
(ii) player $\exists$ chooses some $y_\alpha \in t'$.

$\forall$ must choose x_α so that his moves form a strictly increasing sequence in t. $\exists$ must choose y_α so that his moves form a strictly increasing sequence in t'. The player who first runs out of the tree loses. If $\exists$ has a winning strategy, then we write $t \leq t'$, if $\forall$ has then we write $t \gg t'$.

Definition 2.5 Let $\lambda > \omega$ be regular, $A \subseteq \lambda$ and t a tree. Then the cub-game $G^t(A)$ is the following. At each move α:
(i) first player $\forall$ moves $x_\alpha \in t$,
(ii) then player $\exists$ moves $b_\alpha \in A$,
(iii) and finally player $\forall$ moves $a_\alpha \in \lambda$.

$\forall$ must choose x_α and a_α so that $x_\alpha > x_\beta$ for all $\beta < \alpha$, and $a_\alpha > b_\alpha$. $\exists$ must choose b_α so that $b_\alpha > a_\beta$ for all $\beta < \alpha$, and if α is a limit, $\exists$ must choose $b_\alpha = \sup\{b_\beta \mid \beta < \alpha\}$ ($\exists$ can do this only if the limit is in A). The player who first has to break the rules loses the game.

Definition 2.6 Let $\lambda > \kappa > \omega$ be regular cardinals.

(i) A set $A \subseteq \lambda$ is bistationary if A and $\lambda - A$ are stationary.

(ii) A set $A \subseteq \lambda$ is κ-cub if it is unbounded and closed under supremums of increasing sequences of length α for all α which are $<\lambda$ and of cofinality $\geq\kappa$.

(iii) A set $A \subseteq \lambda$ is κ-stationary if it intersects every κ-cub set.

(iv) A set $A \subseteq \{\alpha \in \lambda \mid cf(\alpha) = \kappa\}$ is strongly bistationary if A is stationary and for all $\alpha < \lambda$, $\forall$ does not have a winning strategy for $G^\alpha(\lambda - A)$.

(v) By E^ω_λ we denote a stationary subset of $\{\alpha < \lambda \mid cf(\alpha) = \omega\}$, such that $E^\omega_\lambda \cap \alpha$ is nonstationary in α for every $\alpha < \lambda$. The existence of such a set follows, e.g., from $\Box_\kappa$, if $\lambda = \kappa^+$. (See Kanamori and Magidor [6].)

(vi) An increasing sequence of ordinals $s = (\alpha_\beta)_{\beta<\gamma}$ is closed if γ is a successor and $\alpha_\delta = \sup\{\alpha_\beta \mid \beta < \delta\}$ for all limit $\delta < \gamma$.

(vii) If $A \subseteq \lambda$, we write $t(A)$ for the tree of all closed increasing sequences of elements of A ordered by the initial segment relation.

Theorem 2.7 *Let T be a complete unsuperstable theory. Let $\lambda > |T| + \omega$ be regular and let $A \subseteq \{\alpha \in \lambda \mid cf(\alpha) = \omega\}$ be strongly bistationary. Then there are non-isomorphic models $EM(J^0_A,\Phi)$ and $EM(J^1_A,\Phi)$ (as they were denoted in* [4]*) of T of cardinality λ such that if $\forall$ has a winning strategy for $G^t_\lambda(EM(J^0_A,\Phi), EM(J^1_A,\Phi))$, then $t \gg t(\lambda - A)$. Especially then $t \gg \alpha$ for all $\alpha < \lambda$.*

In [4], Theorem 2.7 was stated in slightly weaker form because it made no difference in that context. In the context of this paper it will make a difference. Anyway, from the proof in [4] it is clear that Theorem 2.7, also in this form, is true.

3 On the existence of strongly bistationary sets

Theorem 2.7 is of no use if there are no strongly bistationary sets. In [4] we were able to prove the existence in the following cases.

Theorem 3.1

(i) *Assume either*

(a) $\lambda = \omega_1$, *or*

(b) $\lambda = \kappa^+$, *κ regular, $\kappa^{<\kappa} = \kappa$ and λ is not weakly compact in L.*

Let $B \subseteq \{\alpha \in \lambda \mid cf(\alpha) = \omega\}$ be stationary. Then there is $A \subseteq B$ strongly bistationary.

(ii) *If $\lambda = \kappa^+ > \omega_1$, $\kappa^{<\kappa} = \kappa$, then E^ω_λ is strongly bistationary.*

The next lemma is essentially Claim 2 in the proof of Lemma 9.10 in [4].

Lemma 3.2 *Let $\lambda > \xi \geq \omega$ be regular cardinals and let $A \subseteq \{\alpha \in \lambda \mid cf(\alpha) = \xi\}$. If α is the least ordinal for which $\forall$ has a winning strategy for $G^\alpha(\lambda - A)$ then α is a regular cardinal or $\alpha = \xi + 1$.*

Proof: Assume that the lemma is false. If α is a limit ordinal then it is easy to see that $\forall$ has a winning strategy already for $G^{cf(\alpha)}(\lambda - A)$, a contradiction. If

α is a successor then it must be of the form $\beta + 1$ where $cf(\beta) = \xi$. But then $\forall$ has a winning strategy already for $G^{\xi+1}(\lambda - A)$, a contradiction.

Definition 3.3 Let κ be an infinite cardinal and $E \subseteq \kappa^+$. We write $\square_\kappa(E)$ if there is a sequence $(C_\alpha \mid \alpha < \kappa^+ \text{ and } \alpha \text{ limit})$ such that
(i) C_α is cub in α,
(ii) if $cf(\alpha) < \kappa$ then the order type of C_α is less than κ,
(iii) if $\beta < \alpha$ is a limit point of C_α then $\beta \in E$ and $C_\beta = C_\alpha \cap \beta$.

Notice that if $\square_\kappa$ holds and $A \subseteq \{\alpha < \kappa^+ \mid cf(\alpha) = \omega\}$ is stationary then there is a stationary set $E \subseteq A$ such that $\square_\kappa(E)$ holds (see Devlin [1] Lemma IV 2.10).

The next theorem is an improvement of case (ii) in Theorem 3.1.

Theorem 3.4 *If $\lambda = \kappa^+$, $\kappa > \omega$ regular, $E \subseteq \{\alpha < \lambda \mid cf(\alpha) = \omega\}$ stationary and $\square_\kappa(E)$ then E is strongly bistationary.*

Proof: By Lemma 3.2 it is enough to show that $\forall$ does not have a winning strategy for $G^\kappa(\lambda - E)$. For a contradiction assume that τ is a winning strategy of $\forall$ for $G^\kappa(\lambda - E)$. Let $C \subseteq \lambda$ be the set of all those α such that for all $\beta < \alpha$ and for all $\gamma < \kappa$ holds $x^\beta_\gamma < \alpha$ if the following sequence exists

$$(c^\beta_\delta, x^\beta_\delta)_{\delta \le \gamma}$$

where $x^\beta_\delta = \tau(c^\beta_\xi)_{\xi \le \delta}$ and c^β_δ is the least c in C_β (from the definition of $\square_\kappa(E)$) greater than x^β_ξ, for all $\xi < \delta$.

It is easy to see that C is unbounded in λ and closed under sequences of length κ. Choose any $\alpha \in C$ of cofinality κ. If $\forall$ plays according to τ and at each move $\beta < \kappa$ $\exists$ chooses c^α_β then $\exists$ wins, a contradiction.

Let us now look at when we can prove the existence of strongly bistationary sets in ZFC. By collapsing a weakly compact cardinal to ω_2 we get a model where if $A \subseteq \{\alpha < \omega_2 \mid cf(\alpha) = \omega\}$ is a stationary set then the set

$$\{\alpha < \omega_2 \mid cf(\alpha) = \omega_1,\ A \cap \alpha \text{ is stationary in } \alpha\}$$

is an ω_1-cub set (see Magidor [7]). So in this model A cannot be strongly bistationary. However this cannot be done if instead of ω_2 we have a successor of singular. In fact we can then prove the existence of a strongly bistationary set. Before doing this let us study a weaker notion.

Definition 3.5 Let $\lambda > \xi \ge \omega$ be regular cardinals, $A \subseteq \{\alpha < \lambda \mid cf(\alpha) = \xi\}$ and $\alpha < \lambda$. We say that A is α-bistationary if A is stationary and $\forall$ does not have a winning strategy for $G^\alpha(\lambda - A)$.

From the proof of Theorem 2.7 one can immediately see:

Theorem 3.6 *Let T be a complete unsuperstable theory. Let $\lambda > |T| + \omega$ be regular, $\kappa < \lambda$ and let $A \subseteq \{\alpha \in \lambda \mid cf(\alpha) = \omega\}$ be κ-bistationary. Then there are models $EM(J^0_A, \Phi)$ and $EM(J^1_A, \Phi)$ (as they were denoted in* [4]*) of T of cardinality λ such that for all $\alpha < \kappa^+$ $\forall$ does not have a winning strategy for $G^\alpha_\lambda(EM(J^0_A, \Phi), EM(J^1_A, \Phi))$.*

Theorem 3.7 *Let $\lambda > \kappa > \xi \ge \omega$ be regular cardinals and $A \subseteq \{\beta < \lambda \mid cf(\beta) = \xi\}$ stationary. Assume $A = \bigcup_{i<\kappa} A_i$, where each A_i is stationary and $A_i \cap A_j = \emptyset$ if $i \ne j$. Then for all $\alpha < \kappa$ there is $j_\alpha < \kappa$ such that $A^{j_\alpha} = \bigcup_{j_\alpha < i < \kappa} A_i$ is α-bistationary.*

Proof: Let $\alpha < \kappa$. For a contradiction assume that for all $j < \kappa$ there is τ_j a winning strategy of $\forall$ for $G^\alpha(\lambda - \bigcup_{j<i<\kappa} A_i)$. By induction on $\beta < \alpha$ choose γ_j^β, $j < \kappa$, and ϵ^β so that

(i) $\gamma_j^\beta = \tau_j(\epsilon^{\beta'})_{\beta'<\beta}$

(ii) ϵ^β is larger than any $\gamma_j^{\beta'}$, $j < \kappa$, $\beta' < \beta$.

Because τ_j is a winning strategy, there are $i(j)$ and $\beta(j)$ such that $\beta(j)$ is the least limit β such that

$$\bigcup_{\beta'<\beta} \epsilon^{\beta'} \in \bigcup_{j<i<\kappa} A_i$$

and $i(j)$ is the unique i such that

$$\bigcup_{\beta'<\beta(j)} \epsilon^{\beta'} \in A_i.$$

Then $i(j) > j$. Let

$$\epsilon(j) = \bigcup_{\beta'<\beta(j)} \epsilon^{\beta'}.$$

Because $\alpha < \kappa$ there are less than κ possible values of $\epsilon(j)$. So for some ϵ there is an unbounded set of j in κ for which $\epsilon(j) = \epsilon$. If i is such that $\epsilon \in A_i$ then there is an unbounded set of j in κ such that $i(j) = i$. Especially there is some $j > i$ such that $i(j) = i$, a contradiction.

Theorem 3.8 *Assume $\lambda = \kappa^+$, $\kappa > \xi \geq \omega$, $cf(\kappa) < \kappa$ and $A \subseteq \{\alpha < \lambda \mid cf(\alpha) = \xi\}$ stationary. Then there is $B \subseteq A$ strongly bistationary.*

Proof: Let $A = \bigcup_{i<\lambda} A_i$, A_i stationary and $A_i \cap A_j = \emptyset$ if $i \neq j$. By Lemma 3.2 it is enough to find $B \subseteq A$ such that for each regular cardinal $\xi < \kappa$, B is ξ-bistationary.

Let κ_α, $\alpha < \gamma = cf(\kappa) < \kappa$, be a cofinal sequence of regular cardinals below κ. For each $\alpha < \gamma$ and each $j < \lambda$ of cofinality κ_α there is, by Theorem 3.7, $m(\alpha,j) < j$ such that $\bigcup_{m(\alpha,j)<i<j} A_i$ is κ_α-bistationary. By Fodor's Lemma there is $m(\alpha)$ such that $m(\alpha,j) = m(\alpha)$ in an unbounded set of j. Let $m = \bigcup_{\alpha<\gamma} m(\alpha) + 1$. Let $B = A_m$. Clearly this is as required.

4 On the existence of universal equivalence trees Universal equivalence trees could be called generalized Scott ranks for uncountable models. In [4] we showed that if CH holds, then there is a linear order of power ω_1 with no universal equivalence ω_2, ω_1-tree. On the other hand, by a remark of Hella [4], if $2^\omega = 2^{\omega_1}$, then trivially every model of power ω_1 has a universal equivalence ω_2, ω_1-tree. This is because every $(2^{\omega_1})^+$, ω_1-tree can then be majored by an ω_2, ω_1-tree.

Next we are going to prove another positive result about the existence of universal equivalence trees. We will show that in a model of ZFC constructed by Mitchell [8], every linear order of power ω_1 has a universal equivalence ω_2, ω_1-tree t which is small in the following sense: there is a tree t' of power ω_1 such that $t = t' + 1$.

Definition 4.1 Let $A'(\kappa, \kappa^+)$ denote the following assumption: "There is a tree of power κ with at least κ^+ branches of length κ."

Thus $A'(\omega_1,\omega_2)$ states the existence of a generalized Kurepa tree whose levels are allowed to be of power ω_1 instead of power ω. Using an inaccessible cardinal, Mitchell constructs a model of ZFC where $A'(\kappa,\kappa^+)$ fails.

Theorem 4.2 (Mitchell [8]) *Suppose that in $\mathfrak{M}$, θ and κ are regular, λ is inaccessible, and $\theta < \kappa < \lambda$. Then there is an extension $\mathfrak{N}$ of $\mathfrak{M}$ in which $2^\theta = 2^\kappa = \kappa^+ = \lambda$, all cardinals δ such that $\delta \leq \kappa$ or $\delta \geq \lambda$ are preserved, and $A'(\kappa,\kappa^+)$ fails. In particular,* Con(ZFC *and* $\neg A'(\omega_1,\omega_2)$) *is equivalent to* Con(ZFC *and* $\exists\kappa$(κ *is inaccessible*)).

In Theorem 4.2 above we are interested in the case where $\theta = \omega$ and $\kappa = \omega_1$. Note that the failure of $A'(\omega_1,\omega_2)$ trivially implies the failure of CH.

Definition 4.3

(i) Let η be a linear order. An *interval* of η is a subset $I \subseteq \eta$ such that if $x,y \in I$ and $x < z < y$, then $z \in I$. I inherits the ordering of η.

Let I and I' be nonempty intervals of η. We denote $I <_0 I'$ if $I \cap I' = \emptyset$ and for all $a \in I$ and $b \in I'$, $a < b$.

(ii) A *Dedekind cut* of η is a pair of intervals (possibly empty) (I,I') such that I is an initial segment of η, I' is an end segment of η, $I \cap I' = \emptyset$ and $\eta = I \cup I'$.

(iii) If $a \in \eta$ then $\eta^{<a}$ denotes the interval $\eta \restriction \{x \in \eta \mid x < a\}$. We define $\eta^{\leq a}$, $\eta^{>a}$, and $\eta^{\geq a}$ similarly.

(iv) By $\mathrm{cf}(\eta)$ we denote the cofinality of a linear order η and by $\mathrm{ci}(\eta)$ the *coinitiality* of η, that is, $\mathrm{ci}(\eta) = \mathrm{cf}(\eta^*)$, where η^* denotes η with the ordering reversed.

(v) The *type* of a Dedekind cut (I,I') is $(\mathrm{cf}(I),\mathrm{ci}(I'))$ and the *depth* of (I,I') is $\max\{\mathrm{cf}(I),\mathrm{ci}(I')\}$.

Definition 4.4 Let $\mathfrak{A}$ and $\mathfrak{B}$ be models and t a tree. Let

$$P = ((a_\alpha, b_\alpha, x_\alpha))_{\alpha<\beta}$$

be a play (that is, sequence of moves) in $G^t(\mathfrak{A},\mathfrak{B})$, where for all $\alpha < \beta$, $a_\alpha \in \mathfrak{A}$ and $b_\alpha \in \mathfrak{B}$. Suppose $\exists$ has not lost in P. Then we denote the partial mapping $\{(a_\alpha,b_\alpha) \mid \alpha < \beta\}$ by $\mathrm{pm}(P)$.

Lemma 4.5 *Assume* $\neg$CH. *Let η and η' be linear orders of power ω_1 and let (I,I') be a Dedekind cut of η of depth ω_1. Suppose σ is a winning strategy of $\exists$ in the game $G^{\omega\cdot\omega+1}(\eta,\eta')$. We denote $G = G^{\omega\cdot\omega}(\eta,\eta')$. Then there is a play P of G where $\exists$ has followed σ and $\forall$ still has moves left, and a Dedekind cut (J,J') of η', such that no matter how $\forall$ continues the play P, at each move, assuming $\exists$ follows σ, if $\forall$ chooses from $I \cup J$ then $\exists$ chooses from $I \cup J$ and if $\forall$ chooses from $I' \cup J'$ then $\exists$ chooses from $I' \cup J'$.*

Proof: If I or I' is empty, then the claim is trivial. Suppose then I and I' are not empty, and, for instance, $\mathrm{cf}(I) = \omega_1$.

Assume for a contradiction that the claim does not hold. Let B be a full binary tree of height ω. For each $b \in B$ we construct an interval J_b of η' and a play P_b of G, such that:

(i) if b' and b'' are the immediate successors of b, then $P_b \subseteq P_{b'}$, $P_b \subseteq P_{b''}$, $J_b = J_{b'} \cup J_{b''}$ and $J_{b'} \cap J_{b''} = \emptyset$;
(ii) $\mathrm{ran}(\mathrm{pm}(P_b) \restriction I) \cap J_b \neq \emptyset$ and $\mathrm{ran}(\mathrm{pm}(P_b) \restriction I') \cap J_b \neq \emptyset$.

Using our assumption it is easy to do the construction.

Now, let $(b_n)_{n<\omega}$ be an arbitrary branch in B. Let $P = \bigcup_{n<\omega} P_{b_n}$ and $K = \bigcap_{n<\omega} J_{b_n}$. As ∃ has in P followed σ, ∃ can still make one move without losing. Let ∀ continue P be moving an element of I which is greater than any in $\mathrm{dom}(\mathrm{pm}(P)) \cap I$ (here we use $\mathrm{cf}(I) = \omega_1$). By (ii), ∃ must respond by an element in K, which shows that K is nonempty. Thus we see that η' is at least of cardinality 2^ω, a contradiction.

Next we construct trees $T(\eta)$ for linear orders η of power ω_1 with at most ω_1 Dedekind cuts of depth ω_1. We show that it is consistent relative to the existence of an inaccessible cardinal that $(\omega \cdot \omega) \times T(\eta) + 1$ is a universal equivalence tree for an η.

Construction 4.6 Let η be a linear order of power ω_1 with at most ω_1 Dedekind cuts of depth ω_1. Let $\{a_\alpha \mid \alpha < \omega_1 \text{ and } \alpha \text{ even}\}$ enumerate η and let $\{(D_\alpha, D'_\alpha) \mid \alpha < \omega_1 \text{ and } \alpha \text{ odd}\}$ enumerate the Dedekind cuts of η of depth ω_1. We construct a tree $T(\eta)$.

We construct by induction trees T_α, $\alpha < \omega_1$, such that:

(i) T_α is of height $\leq \alpha + 1$ and $T_\alpha \subseteq T_\beta$ if $\alpha < \beta$.
(ii) The nodes of T_α are nonempty intervals I of η, and the root of T_α is η.
(iii) If $I \in T_\alpha$, and I_δ, $\delta < \epsilon$, are the immediate successors of I, then $I_\delta \cap I_{\delta'} = \emptyset$ for all $\delta < \delta' < \epsilon$, and $I = \bigcup_{\delta<\epsilon} I_\delta$.
(iv) Suppose $(I_\delta)_{\delta<\epsilon}$ is an initial segment of a branch in T_α, ϵ is a limit ordinal, and $I = \bigcap_{\delta<\epsilon} I_\delta$. If I is nonempty, then I is the unique supremum of $(I_\delta)_{\delta<\epsilon}$ in T_α. If I is empty, then $(I_\delta)_{\delta<\epsilon}$ has no supremum in T_α.

Conditions (i)–(iv) imply the following properties:

(v) if $I < I'$ in T_α, then $I' \subseteq I$;
(vi) if I and I' are nodes of T_α, and I and I' are not comparable in the ordering of the tree T_α, then $I \cap I' = \emptyset$;
(vii) $\bigcup\{I \mid I \text{ is a leaf of } T_\alpha\} = \eta$.

(In the proofs below we will also use some other properties of our construction which are not listed in (i)–(vii).)

Let T_0 be a tree whose only node is η. Suppose then that we have defined T_β, $\beta < \alpha$.

Case 1. If α is a limit, let $T'_\alpha = \bigcup_{\beta<\alpha} T_\beta$. We extend T'_α to T_α by the following process: at the tip of each branch $(I_\delta)_{\delta<\epsilon}$ of a limit ordinal length in T'_α we add the node $I = \bigcap_{\delta<\epsilon} I_\delta$, if I is nonempty.

Case 2. Suppose $\alpha = \gamma + 1$ and γ is even. Let I be the unique leaf in T_γ such that $a_\gamma \in I$. We build T_α from T_γ by adding the node $I' = I^{\leq a_\gamma}$ on top of I, and also the node $I'' = I^{>a_\gamma}$, if I'' is nonempty.

Case 3. Suppose $\alpha = \gamma + 1$ and γ is odd. Suppose that $(\mathrm{cf}(D_\gamma), \mathrm{ci}(D'_\gamma)) = (\kappa, \lambda)$. (Note that κ or λ may be 0, if either interval is empty.) We build the tree

T_α from T_γ by the following procedure which eliminates the deep Dedekind cut from the leaves of T_α.

(a) Suppose there is a leaf I in T_α such that $I \cap D_\gamma$ and $I \cap D'_\gamma$ are nonempty. Let $(b_\delta)_{\delta<\kappa}$ be a cofinal sequence of points in $I \cap D_\gamma$ and $(c_\delta)_{\delta<\lambda}$ a coinitial sequence of points in $I \cap D'_\gamma$. Then for all $\delta < \kappa$ and $\delta' < \lambda$, we add the intervals $I^{\leq b_\delta} - \bigcup_{\epsilon<\delta} I^{\leq b_\epsilon}$ and $I^{\geq c_{\delta'}} - \bigcup_{\epsilon<\delta'} I^{\geq c_\epsilon}$ as new nodes on top of I.
(b) Suppose there is a leaf I in T_α which is an end segment of D_γ. Let $(b_\delta)_{\delta<\kappa}$ be a cofinal sequence of points in I. Then for all $\delta < \kappa$, we add the intervals $I^{\leq b_\delta} - \bigcup_{\epsilon<\delta} I^{\leq b_\epsilon}$ on top of I.
(c) Suppose there is a leaf I in T_α which is an initial segment of D'_γ. Let $(c_\delta)_{\delta<\kappa}$ be a coinitial sequence of points in I. Then for all $\delta < \lambda$, we add the intervals $I^{\geq c_\delta} - \bigcup_{\epsilon<\delta} I^{\geq c_\epsilon}$ on top of I.

This ends the induction. Finally, we let $T(\eta) = \bigcup_{\alpha<\omega_1} T_\alpha$.

Lemma 4.7 *Let η be as in Construction 4.6. Then $T(\eta)$ is an ω_2, ω_1-tree of power ω_1. In addition, each leaf of $T(\eta)$ is an interval containing just one element and each element of η occurs exactly in one leaf of $T(\eta)$.*

Proof: Since each level of $T(\eta)$ consists of disjoint intervals of η, each level is of power $\leq\omega_1$.

We show that there is no ω_1-branch in $T(\eta)$. Assume for a contradiction that $(I_\delta)_{\delta<\omega_1}$ is an ω_1-branch in $T(\eta)$. From the construction it is obvious that we can find a sequence $s = (d_\delta)_{\delta<\omega_1}$ of elements of η, such that s is either strictly descending or strictly ascending in η and for all $\delta < \omega_1$, $d_\delta \in I_\delta$. Suppose, for instance, that s is strictly ascending. Let $J = \bigcup_{\delta<\omega_1} \eta^{\leq d_\delta}$ and $J' = \eta - J$. Now each I_δ contains a nonempty end segment of J. Let γ be such that $(D_\gamma, D'_\gamma) = (J, J')$. From (iv) of Construction 4.6 it follows that there is a leaf I of $T_{\gamma+1}$ which belongs to s. We have a contradiction, because in $T_{\gamma+1}$ there is no leaf which would contain a nonempty end segment of J.

We prove then the last claim. Suppose I is a leaf of $T(\eta)$ containing elements $a < b \in \eta$. By Case 2 in Construction 4.6, there is a node in $T(\eta)$ whose greatest element is a. But now we have a contradiction to (v) and (vi) of Construction 4.6. Suppose $a \in \eta$ is arbitrary. Let A be the set of those nodes of $T(\eta)$ which contain a. From (iii) and (iv) of Construction 4.6 it follows that A is a branch of $T(\eta)$. As A is of length $< \omega_1$, it follows from (iv) that A has to be of a successor length; that is, the topmost element of A is a leaf.

Theorem 4.8 *Assume* $\neg$CH. *Let η and η' be linear orders of power ω_1 with at most ω_1 Dedekind cuts of depth ω_1. Let $t = (\omega\cdot\omega) \times T(\eta) + 1$. If $\eta \cong^t \eta'$, then $\eta \cong \eta'$.*

Proof: Let σ be a winning strategy of $\exists$ in $G^t(\eta, \eta')$. If $x \in t' = (\omega\cdot\omega) \times T(\eta)$, then we denote by $p(x)$ the projection of x in $T(\eta)$.

For each $I \in T(\eta)$ we will construct an interval $f(I)$ of η' and a play $P(I)$ in $G^{t'}(\eta, \eta')$. We order the intervals $f(I)$, $I \in T(\eta)$, into a tree T' in the following way:

$$f(I) \leq f(I') \text{ in } T' \text{ iff } I \leq I' \text{ in } T(\eta).$$

Let us denote $T'_\alpha = T' \restriction \{f(I) \mid I \in T_\alpha\}$. We do the construction so that the following hold:

(i) T'_α satisfies (i)–(iv) of Construction 4.6, where T_α is replaced by T'_α and η by η'.
(ii) If I is a node in $T(\eta)$ and I' and I'' are its immediate successors in $T(\eta)$, then $I' <_0 I''$ iff $f(I') <_0 f(I'')$.
(iii) $P(I)$ is a play where $\exists$ has followed σ, $\forall$ has moved only elements x in t' such that $p(x) < I$ in the ordering of $T(\eta)$, and for all continuations P' (in $G^t(\eta, \eta')$) of $P(I)$, $I \cup f(I)$ is closed under σ.
(iv) If $I \subset I'$, then $P(I) \subseteq P(I')$.

We define $f(\eta) = \eta'$ and $P(\eta) = (\ )$. Suppose $f(I)$ and $P(I)$ have been constructed for all $I \in \bigcup_{\beta<\alpha} T_\beta$. We are going to define $f(I)$ and $P(I)$ for $I \in T_\alpha$.

Consider first the case where α is a limit. Then if $I = \bigcap_{\delta<\epsilon} I_\delta$ is a new added node in T_α (see Case 1 in Construction 4.6), then we define $f(I) = \bigcap_{\delta<\epsilon} f(I_\delta)$ and $P(I) = \bigcup_{\delta<\epsilon} P(I_\delta)$.

Consider then the case where $\alpha = \gamma + 1$ and γ is odd (Case 3 in Construction 4.6). Let the notation be as in Case 3. We treat as an example Case 3(a), where $\lambda = \kappa = \omega_1$, that is, $(\mathrm{cf}(D_\gamma), \mathrm{ci}(D'_\gamma)) = (\omega_1, \omega_1)$. (Case 2 and other subcases of Case 3 are similar and easier.) The problem is to split $f(I)$ into pieces like I was split, and define the corresponding plays.

In the following construction for Case 3(a) we only let $\forall$ continue $P(I)$ so that his moves x in the approximating tree t' are such that the projection in $T(\eta)$ is $p(x) = I$. First we apply Lemma 4.5 to split $f(I)$ into two pieces J and J' and to find a continuation P' of $P(I)$, such that in the next $\omega \cdot \omega$ moves when P' is continued, the winning strategy σ of $\exists$ maps every point chosen by $\forall$ in $D_\gamma \cap I$ and $D'_\gamma \cap I$ to J and J' respectively, and vice versa. As σ is a winning strategy of $\exists$, $\mathrm{cf}(J) = \omega_1$ and $\mathrm{ci}(J') = \omega_1$. Let $(d_\delta)_{\delta<\omega_1}$ be a cofinal sequence in J and $(e_\delta)_{\delta<\omega_1}$ a coinitial sequence in J'.

We describe then how to split J into ω_1 pieces. (The procedure for J' is symmetric.) It is straightforward to show that there is a cub set $C \subseteq \omega_1$ whose elements $\epsilon \in C$ have the property: If $\forall$ continues P' by a finite number of moves moving only elements from the set $\{b_\delta \mid \delta < \epsilon\} \cup \{d_\delta \mid \delta < \epsilon\}$, then $\exists$'s responses, using σ, are in the union of $\bigcup_{\delta<\epsilon} I^{\le b_\delta}$ and $\bigcup_{\delta<\epsilon} J^{\le d_\delta}$. (We say then that ϵ is *closed under finite continuations of P'*.)

We will define by induction ordinals

$$\epsilon_0 < \cdots < \epsilon_{\zeta<\omega_1} < \cdots$$

which are closed under finite continuations of P', and elements

$$r_\zeta \in J, \qquad \zeta < \omega_1.$$

Suppose we have for some $\zeta < \omega_1$ defined $\epsilon_0 < \cdots < \epsilon_{\delta<\zeta} < \cdots$ and the elements r_ξ, where $\xi < \epsilon_\delta$ for some $\delta < \zeta$. As C is cub, $\epsilon' = \sup\{\epsilon_\delta \mid \delta < \zeta\} \in C$. Let $\forall$ play so that he continues P' and he moves in the first ω moves b_ξ and d_ξ for all $\xi < \epsilon'$, yielding a play $P'' \supseteq P'$. Let C' be the set of those ϵ closed under finite continuations of P''. Since C' is cub, there is $\epsilon_\zeta \in C \cap C'$, $\epsilon_\zeta > \epsilon'$. Now $\forall$ continues P'' and moves in the next ω moves b_ξ and d_ξ for all $\epsilon' \le \xi < \epsilon_\zeta$, yielding a play P'''. Let $r_\xi = (\mathrm{pm}(P'''))(b_\xi)$ for all $\epsilon' \leq \xi < \epsilon_\zeta$. This completes the description of the induction.

Let

$$I_\delta = I^{\leq b_\delta} - \bigcup_{\epsilon<\delta} I^{\leq b_\epsilon}.$$

Now we define

$$f(I_\delta) = J^{\leq r_\delta} - \bigcup_{\epsilon<\delta} J^{\leq r_\epsilon}$$

for all $\delta < \omega_1$. The corresponding play $P(I_\delta)$ is P''' above from the phase of induction where r_δ was defined.

This ends the construction of $f(I)$ and $P(I)$, $I \in T(\eta)$. Let us now define the isomorphism $g : \eta \to \eta'$. If $a \in \eta$, then by Lemma 4.7, there is exactly one leaf I in $T(\eta)$ which contains a, and I is actually the singleton of a. As $\exists$ can continue $P(I)$ still $\omega \cdot \omega + 1$ moves without losing. Also, $f(I)$ has to be of the form $\{a'\}$, where $a' \in \eta'$. We let $g(a) = a'$.

We have to prove that if $a < b$, then $g(a) < g(b)$. Let $A = \{I \in T(\eta) \mid a \in I, b \in I\}$. Obviously, the elements of A form an initial segment of a branch in $T(\eta)$. By (iv) of Construction 4.6, this segment is of a successor length. Let I be the topmost element in A. Now it follows from condition (ii) above and from (v) of Construction 4.6, applied to appropriate T'_α, that $g(a) < g(b)$.

Finally, we show that g is onto. Let b be an arbitrary element in η'. Let $A = \{I \in T(\eta) \mid b \in f(I)\}$. Using properties (iii) and (iv) of Construction 4.6, applied to appropriate T'_α, it is easy to see that the elements of A form a branch $s = (I_\alpha)_{\alpha<\beta}$ in $T(\eta)$. Let $P = \bigcup_{\alpha<\beta} P(I_\alpha)$, $I = \bigcap_{\alpha<\beta} I_\alpha$ and $I' = \bigcap_{\alpha<\beta} f(I_\alpha)$.

Suppose for a contradiction that s is of a limit length. Then $\forall$ has still one move left and he can continue P by moving an element in I', and $\exists$ loses because he must move in I, which is empty. This is the place where we use the "+1" part in the definition of t. This contradicts the fact that σ is a winning strategy of $\exists$ in $G^t(\eta, \eta')$.

Thus s must be of a successor length. Let I be the topmost node of s. We know that I is the singleton of some element $a \in \eta$. As above we see that $f(I) = \{b\}$, and thus $g(a) = b$.

Next we prove that in the model of Mitchell every linear order of power ω_1 has at most ω_1 Dedekind cuts of depth ω_1.

Let $U(\eta)$ be constructed like $T(\eta)$, except that if $\alpha = \gamma + 1$ and γ is odd we do nothing and let $T_\alpha = T_\gamma$.

Lemma 4.9 *Let η be a linear order of power ω_1.*
(i) *$U(\eta)$ is of power ω_1.*
(ii) *If η has κ Dedekind cuts of depth ω_1, then $U(\eta)$ has at least κ branches of length ω_1.*

Proof:

(i) Each level of $U(\eta)$ consists of disjoint subsets of η, and is therefore of power $\leq\omega_1$.

(ii) Suppose that (J, J') is a Dedekind cut of type (ω_1, ω), for example. From Construction 4.6 it is easy to see that on each level $\alpha < \omega_1$ of the tree $U(\eta)$ there is some node I_α such that I_α contains a nonempty end segment of J. Now $(I_\alpha)_{\alpha<\omega_1}$ is an ω_1-branch in $U(\eta)$ and it is rather easy to see that these branches are distinct for distinct Dedekind cuts.

Corollary 4.10 *If $A'(\omega_1, \omega_2)$ fails, then each linear order of power ω_1 has at most ω_1 Dedekind cuts of depth ω_1.*

Corollary 4.11 *In the model $\mathfrak{N}$ of Theorem 4.2, where $\theta = \omega$ and $\kappa = \omega_1$, the following holds: every linear order η of power ω_1 has a universal equivalence ω_2, ω_1-tree t which is of the form $t' + 1$, where t' is of power ω_1.*

4.12 Remark Instead of trees we can use linear orders to measure the length of Ehrenfeucht-Fraisse games; i.e., instead of choosing an increasing sequence of elements of a tree, $\forall$ chooses decreasing sequence of elements of a linear order. We say that a linear order ξ is a κ-well ordering if there is no decreasing sequence of length κ in ξ. Then we can formulate Corollary 4.11 in the following way. In the model $\mathfrak{N}$ of Theorem 4.2, where $\theta = \omega$ and $\kappa = \omega_1$, the following holds: every linear order η of power ω_1 has a universal equivalence linear order ξ such that ξ is an ω_1-well ordering and is of cardinality ω_1.

Acknowledgment Professor Shelah's work was partially supported by the United States Israel binational science foundation, publ. 428.

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