Notre Dame Journal of Formal Logic
Volume 34, Number 1, Winter 1993

An Arithmetical Completeness Theorem for Pre-permutations

FRIEDEMANN TUTTAS

Abstract We prove an extension of an arithmetical completeness theorem for the system $\mathbf{R}^\omega$ with respect to pre-permutational arithmetic interpretations to all modal sentences. Hitherto, this type of completeness theorem has only been given for modal sentences with no nestings of witness comparisons.

In their joint paper [1], Guaspari and Solovay provide a modal analysis of Rosser sentences. Their results are presented, discussed, and somewhat complemented in great detail in Chapter 6 of Smoryński's recently published book [2]. Both for the sake of shortness and convenience, we assume full familiarity with this exposition and will refer to it throughout this paper.

Among other results, Guaspari and Solovay prove an Arithmetical Completeness Theorem (ACT) for the modal system $\mathbf{R}^\omega$ which is briefly described as follows ([2], p. 259–262).

The *language* of $\mathbf{R}^\omega$ is the usual one for propositional logic but equipped with witness comparisons $\preceq$ and $\prec$.

The *axioms* of $\mathbf{R}^\omega$ are all sentences (A1–A7) together with the necessitations of (A1–A6).

(A1) All tautologies
(A2) $\Box A \wedge \Box(A \to B) \to \Box B$
(A3) $\Box A \to \Box\Box A$
(A4) $\Box(\Box A \to A) \to \Box A$
(A5) $A \to \Box A$, for all Σ-formulas ([2], p. 260)
(A6) Order axioms for $\preceq$ and $\prec$ ([2], p. 261)
(A7) $\Box A \to A$.

(The necessitation of a sentence A is $\Box A$.)

The only *rule of inference* is modus ponens.

An *arithmetic interpretation* is an assignment $*$ of arithmetic sentences p^* to modal atoms p that extends as follows:

Received March 3, 1986; revised February 3, 1989

$$t^* = (\bar{0} = \bar{0}), \quad f^* = (\bar{0} = \bar{1})$$
$$(\neg A)^* = \neg A^*$$
$$(A \circ B)^* = A^* \circ B^*, \text{ for } \circ \in \{\wedge, \vee, \rightarrow, \preceq, \prec\}$$
$$(\Box A)^* = \mathrm{Th}(\ulcorner A^* \urcorner).$$

Here $\mathrm{Th}(v)$ is a standard provability predicate for PRA (primitive recursive arithmetic), i.e., a Σ_1-formula with only v as free variable such that

$$\mathrm{PRA} \vdash \forall v (\mathrm{Pr}_{\mathrm{PRA}}(v) \leftrightarrow \mathrm{Th}(v)).$$

The witness comparison formulas are defined by

$$\exists v_0 \varphi \preceq \exists v_1 \psi : \exists v_0 (\varphi v_0 \wedge \forall v_1 < v_0 \neg \psi v_1)$$
$$\exists v_0 \varphi \prec \exists v_1 \psi : \exists v_0 (\varphi v_0 \wedge \forall v_1 \leq v_0 \neg \psi v_1).$$

We can now state Guaspari and Solovay's ACT:

Theorem *For every modal sentence A,*

$$\mathbf{R}^\omega \vdash A \text{ iff } \forall^* \ (A^* \text{ is true}).$$

In some sense one might regard this as unsatisfactory. One is interested in the usual provability predicate $\mathrm{Pr}_{\mathrm{PRA}}(v)$ ($\mathrm{Pr}(v)$ in shortened form) rather than any predicate $\mathrm{Th}(v)$ provably equivalent to $\mathrm{Pr}(v)$. However, if we choose the interpretation of $\Box$, $\preceq$, and $\prec$ according to:

$$(\Box A)^* = \mathrm{Pr}(\ulcorner A^* \urcorner)$$
$$(1) \qquad (\Box A \circ \Box B)^* = \mathrm{Pr}(\ulcorner A^* \urcorner) \circ \mathrm{Pr}(\ulcorner B^* \urcorner), \text{ for } \circ \in \{\preceq, \prec\},$$

then it turns out that the above ACT for $\mathbf{R}^\omega$ does not hold. Put, for example, $\tau = (t \wedge t \rightarrow t)$ and $\sigma = (\Box\tau \prec \Box(\tau \wedge \tau))$. Then $\tau^* = (\bar{0} = \bar{0} \wedge \bar{0} = \bar{0} \rightarrow \bar{0} = \bar{0})$ is an axiom of PRA ([2], p. 19), and the Gödel number of $(\tau \wedge \tau)^* = \tau^* \wedge \tau^*$ is greater than that of τ^*. Therefore, σ^* is true. On the other hand, consider the Kripke model $\mathbf{K} = \{\alpha_0\}$ and declare $\neg\sigma$ to be forced at α_0 (this is possible). Hence by completeness of $\mathbf{R}^\omega$ ([2], p. 271), $\mathbf{R}^\omega \nvdash \sigma$. (We may also declare σ to be forced at α_0, whence $\mathbf{R} \nvdash \neg\sigma$. If we fix a special standard provability predicate $\mathrm{Th}(v)$ and denote again the corresponding arithmetic interpretation by $*$, then σ^* or $\neg\sigma^*$ must be true. So the above ACT does not even hold for a particular standard provability predicate.)

We can get around this difficulty by modifying the interpretation of $\preceq$ and $\prec$ in (1). Again we interpret $\Box$ by $\mathrm{Pr}(\cdot)$, but allow different orderings of the proofs to enter the interpretations of the witness comparisons.

Definition

(i) $D_x = \{y \mid [x/2^y] \text{ is odd}\}$, i.e., the finite set with canonical index x.

(ii) A recursive function H is a *pre-permutation* (of the natural numbers) if the following hold:

a. $\mathrm{PRA} \vdash \forall v_0 \exists v_1 (v_0 \in D_{H(v_1)})$

b. $\mathrm{PRA} \vdash \forall v_1 \exists v_0 (v_0 \in D_{H(v_1)})$

c. $\mathrm{PRA} \vdash \forall v_0 \forall v_1 (v_0 \neq v_1 \rightarrow D_{H(v_0)} \cap D_{H(v_1)} = \emptyset)$.

(iii) Let H be a pre-permutation and $^+$ an assignment of arithmetical sentences p^+ to atoms p. The *pre-permutational arithmetic interpretation* (ppi) $^+$ based on H and $^+$ is the extension of $^+$ by the following clauses:

$$t^+ = (\bar{0} = \bar{0}), f^+ = (\bar{0} = \bar{1})$$

$$(\neg A)^+ = \neg A^+, (\Box A)^+ = \mathrm{Pr}(\ulcorner A^+ \urcorner)$$

$$(A \circ B)^+ = A^+ \circ B^+, \text{ for } \circ \in \{\wedge, \vee, \rightarrow\}$$

$$(\Box A \circ \Box B)^+ = \mathrm{Pr}(\ulcorner A^+ \urcorner) \circ_H \mathrm{Pr}(\ulcorner B^+ \urcorner), \text{ for } \circ \in \{\leq, <\}.$$

Here, e.g., $\exists v \varphi v \leq_H \exists v \psi v$ is $\exists v(\varphi v \wedge \forall w <_H v \neg \psi w)$ and $w <_H v$ is defined to be $\exists u(w \in D_{H(u)} \wedge \forall z \leq u\ v \notin D_{H(z)})$ (see [2], p. 287).

Now, in [2] (p. 288) it is shown that an ACT for any ppi can be obtained essentially by rewriting the proof of Guaspari and Solovay's ACT above. But, alas, in doing so we can deal only with modal sentences having no nestings of witness comparisons. To be more precise, first inductively define, for modal A and B,

$$c(t) = c(f) = 0$$

$$c(\neg A) = c(\Box A) = c(A)$$

$$c(A \circ B) = \max\{c(A), c(B)\}, \text{ for } \circ \in \{\wedge, \vee, \rightarrow\}$$

$$c(\Box A \circ \Box B) = 1 + \max\{c(A), c(B)\}, \text{ for } \circ \in \{\leq, <\}.$$

We then have ([2], Theorem 6.2.11):

Theorem *For all modal sentences A with $c(A) \leq 1$:*

(2) $$\mathbf{R}^\omega \vdash A \text{ iff } \forall^+(A^+ \text{ is true}).$$

It is the aim of this note to prove (2) without any restriction on A.

Theorem *For all modal sentences A:*

$$\mathbf{R}^\omega \vdash A \text{ iff } \forall^+(A^+ \text{ is true}).$$

Proof: To some extent, the proof runs along the lines of the proof of Guaspari and Solovay's ACT. Therefore, if not otherwise stated, we adopt the notation of [2], p. 280 f. It suffices to show:

If A is a modal sentence and $\mathbf{K}$ an A-sound Kripke model for $\mathbf{R}^-$ in which A is true, then there is a ppi $^+$ such that A^+ is true.

Suppose $\mathbf{K} = (\{1, \ldots, n\}, R, 1, \Vdash)$ is A-sound and $1 \Vdash A$. Then define S as in [2], p. 280, introduce a new root 0 below $\mathbf{K}$ and define for atoms p:

$$\text{for } p \in S\text{: } 0 \Vdash p \text{ iff } 1 \Vdash p$$

$$\text{for } p \notin S\text{: } 0 \Vdash p.$$

This extends to an A-sound Kripke model of $\mathbf{R}^-$ in which, for all $B \in S$:

$$0 \Vdash B \text{ iff } 1 \Vdash B.$$

Then define Solovay's recursive function F by

$$F(0) = 0$$

$$F(x+1) = \begin{cases} \text{least } y[F(x)Ry \;\&\; \mathrm{Prov}(x+1, \ulcorner L \neq \bar{y} \urcorner)], & \text{if such exists} \\ F(x), & \text{otherwise.} \end{cases}$$

(L denotes the limit of F. For the basic properties of L, see [2], Chapter 3, Sections 1 and 2.)

Now define the interpretation $^+$ as outlined in [2], p.281, however, with the following modifications:

$$(\Box B)^+ = \mathrm{Pr}(\ulcorner B^+ \urcorner)$$

$$(\Box B \circ \Box C)^+ = \mathrm{Pr}(\ulcorner B^+ \urcorner) \circ_H \mathrm{Pr}(\ulcorner C^+ \urcorner), \text{ for } \circ \in \{\leq, <\}.$$

Here H is as yet undefined and supposed to be recursive. We do not provide a formal definition of H, but rather we will proceed informally. Nevertheless, a complete formalization within PRA is not problematic and can be obtained as a standard application of the Recursion Theorem.

We proceed in stages and define simultaneously H together with the auxiliary functions k and l. After each stage m, k will tell us that H is defined for all $x \leq k(m)$, whereas l keeps track of how many proofs of formulas belonging to a set specified below are already listed by H.

Stage $m = 0$: Put $k(0) = 0$, $H(0) = 2^0$, $l(0,j,x) = 0$, all j,x.

Stage $m > 0$: H is defined for all $x \leq k(m-1)$.

i. If m is no proof or if $\mathrm{Prov}(m, \ulcorner B^+ \urcorner)$ for some $\Box B \notin S$, then put $k(m) = k(m-1) + 1$, $H(k(m)) = 2^m$, and $l(m,j,x) = l(m-1,j,x)$.

ii. $\mathrm{Prov}(m, \ulcorner B^+ \urcorner)$ for some $\Box B \in S$.

Let $Y_m = \{C^+ \mid F(m) \Vdash \Box C \;\&\; \Box C \in S\}$, and let $E_m^0, \ldots, E_m^{s_{m-1}}$ be the equivalence classes "mod $\leq$" of Y_m ordered according to $<$ (see [2], p. 282). We define the partial recursive function

$$P(x,y) = \begin{cases} x\text{-th proof of } y, & \text{if } y \text{ codes a provable formula} \\ \text{undefined}, & \text{otherwise,} \end{cases}$$

and put $c_m = \max\{c(C) \mid C^+ \in Y_m\}$. Introduce, for $j \geq 0$,

$$E_m^{i,j} = \{C^+ \in E_m^i \mid c(C) \leq j\}.$$

We can assume that all these sets are nonempty. (For otherwise drop the empty sets and list only the remaining ones.)

Assume for a moment that $\Box C$ varies over S. Then define

$$l(m,0,\ulcorner C^+ \urcorner) = \begin{cases} 1 + l(m-1, c_{m-1}, \ulcorner C^+ \urcorner), & \text{if } C^+ \in Y_m \\ l(m-1, c_{m-1}, \ulcorner C^+ \urcorner), & \text{otherwise,} \end{cases}$$

and, for $j > 0$,

$$l(m,j,\ulcorner C^+ \urcorner) = \begin{cases} 1 + l(m, j-1, \ulcorner C^+ \urcorner), & \text{if } C^+ \in Y_m \;\&\; c(C) \leq c_m - j \\ l(m, j-1, \ulcorner C^+ \urcorner), & \text{otherwise.} \end{cases}$$

In all other cases, i.e., if x does not code a sentence C^+ for some $\Box C \in S$, just put $l(m,j,x) = 0$.

This completes the definition of l at stage m.

Next introduce

$$M(m,j,i) = \begin{cases} \text{code of } \{P(l(m,j,\ulcorner C^{+}\urcorner),\ulcorner C^{+}\urcorner) \mid C^{+} \in E_m^{i,c_m-j}\}, & \\ & \text{if all these sentences have proofs} \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

Armed with this notation we are now prepared to define H at stage m:

$$k(m) = k(m-1) + (c_m + 1)s_m$$

$$(3)\quad H(k(m-1) + 1 + js_m + i) = \begin{cases} M(m,j,i), \text{ if } M(m,j,i)\downarrow, & \text{all } i < s_m \\ \text{undefined,} & \text{otherwise} \end{cases}$$

($\downarrow$ denoting definedness). This ends the definition of H thereby ending that of $^+$, too.

Let us remark at this point that the main difficulty in the construction of H is to guarantee that H is total. For this purpose we have split up the sets E_m^i according to the complexity of their elements as described above. This is the key idea that will enable us to prove the recursiveness of H. Up to now, however, we can say only that H is a partial recursive function, the only arguments at which it might be undefined being those displayed in (3). For the following we fix a recursive predicate $S(m,j)$ stating that m is a proof of a sentence C^+ for some $\Box C \in S$ and $j \leq c_m$.

Lemma *Let* $0 \leq j \leq c_m$. *We have, for* $0 \leq x \leq n$ *and* $B \in S$ *such that* $c(B) \leq j$:

$$(4_j)\qquad x \Vdash B \Rightarrow \mathrm{PRA} \vdash L = \bar{x} \rightarrow B^+$$

$$(5_j)\qquad x \nVdash B \Rightarrow \mathrm{PRA} \vdash L = \bar{x} \rightarrow \neg B^+$$

$$(6_j)\qquad \mathrm{PRA} \vdash S(\bar{m},\bar{j}) \rightarrow \forall i < \bar{s}_m\, H(k(\bar{m} - \bar{1}) + \bar{1} + (\bar{c}_m - \bar{j})\bar{s}_m + \bar{i})\downarrow.$$

Proof of the lemma: By induction on j. Again we argue informally. For $j = 0$, (4_0) and (5_0) are well known assertions of a lemma of Solovay's ([3], §4; also [2], Lemma 3.1.10 and Lemma 3.2.3). To show (6_0), observe that the limit L of F exists and equals some $x \leq n$. So assume $L = x$ and let m be such that $S(m,0)$. If $F(m) = x$, then by (4_0), for all $i < s_m$, $H(k(m-1) + 1 + c_m s_m + i)\downarrow$. If $F(m)Rx$, then, for any modal sentence B,

$$F(m) \Vdash \Box B \Rightarrow x \Vdash \Box B.$$

Hence, according to (4_0), if $B^+ \in Y_m$ and $c(B) = 0$, then B^+ is provable. Thus H is again defined at all arguments of form $k(m-1) + 1 + c_m s_m + i$ $(i < s_m)$. This is (6_0).

Suppose now that the lemma has already been proved for $j < n \leq c_m$. Let B be $\Box C \preceq \Box D$ where $c(C) < n$ and $c(D) < n$, and suppose $x \Vdash \Box C \preceq \Box D$. Then $x \Vdash \Box C$, and by induction hypothesis, PRA $\vdash L = \bar{x} \rightarrow \mathrm{Pr}(\ulcorner C^{+}\urcorner)$. If $x \nVdash$ $\Box D$, then PRA $\vdash L = \bar{x} \rightarrow \neg\mathrm{Pr}(\ulcorner D^{+}\urcorner)$ and clearly PRA $\vdash L = \bar{x} \rightarrow (\Box C \preceq$

$\Box D)^+$. Now assume $x \Vdash \Box D$; then PRA $\vdash L = \bar{x} \to \mathrm{Pr}(\ulcorner D^+ \urcorner)$. Let m be minimum such that $\mathrm{Prov}(m, \ulcorner G^+ \urcorner)$ for some $G \in S$ and $F(m) \Vdash \Box C$. Two things can happen. Suppose first $F(m) \Vdash \Box D$. Then there are $a, b < s_m$, $a \leq b$, such that $C^+ \in E_m^a$ and $D^+ \in E_m^b$. Also, $C^+ \in E_m^{a,n-1}$ and $D^+ \in E_m^{b,n-1}$. By definition of H, for $n' \geq n$,

$$H(k(m-1)+1+(c_m - n')s_m + a)\downarrow \text{ iff } H(k(m-1)+1+(c_m-n')s_m+b)\downarrow.$$

Hence,

$$u \in D_{H(w)} \wedge \mathrm{Prov}(u, \ulcorner D^+ \urcorner) \Rightarrow \exists z \leq w \exists y\ (y \in D_{H(z)} \wedge \mathrm{Prov}(y, \ulcorner C^+ \urcorner)).$$

Since this assertion obviously holds if $F(m) \nVdash \Box D$, we can conclude

$$\text{PRA} \vdash L = \bar{x} \to (\Box C \preceq \Box D)^+.$$

This is (4_n) for $B = (\Box C \preceq \Box D)$. Along the same lines (4_n) and (5_n) can be obtained for this B as well as for $B = (\Box C \prec \Box D)$. Now, by induction hypothesis, (4_{n-1}) and (5_{n-1}). We just proved (4_n) and (5_n) for $B = (\Box C \circ \Box D)$, where $\circ \in \{\preceq, \prec\}$. The proof of Solovay's Lemma 3.2.3 in [2] then easily furnishes (4_n) and (5_n) for all $B \in S$ with nestings of witness comparisons at most n. In the same manner we have proved (6_0) above, we can now derive (6_n). This completes the proof of the lemma.

We are almost done. As outlined in [2] (p. 283), an easy application of the lemma yields that A^+ is true. Thus all what remains to show is that $^+$ is a ppi, and this amounts to a proof that H is a pre-permutation. H is recursive by the lemma and obviously satisfies clause (iib) of the definition above. A quick look at the definition of l will convince the reader that (iic) is fulfilled as well. To show (iia), suppose $\mathrm{Prov}(\bar{m}, \ulcorner B^+ \urcorner)$ is true for some $\Box B \in S$. We claim $0 \Vdash \Box B$. For otherwise the lemma yields PRA $\vdash L = \bar{0} \to \neg\mathrm{Pr}(\ulcorner B^+ \urcorner)$ and thus PRA $\vdash L \neq \bar{0}$, which is absurd. Therefore $B^+ \in Y_0 \subseteq Y_m$, and m gets listed by H at stage m or earlier. In all other cases we have $m \in D_{H(k(m))}$. This shows that H is a pre-permutation. The theorem is now proved.

Acknowledgment I would like to thank Craig Smoryński for much help and discussion on this and related topics.

REFERENCES

[1] Guaspari, D. and R. M. Solovay, "Rosser sentences," *Annals of Mathematical Logic*, vol. 16 (1979), pp. 81-99.

[2] Smoryński, C. A., *Self-Reference and Modal Logic*, Springer-Verlag, Berlin, 1985.

[3] Solovay, R. M., "Provability interpretations of modal logic," *Israel Journal of Mathematics*, vol. 25 (1976), pp. 287-304.

Neuberg 1a
D-6200 Wiesbaden
Federal Republic of Germany