

The Amalgamation Property in Normal Open Induction

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Abstract It is known that open induction (OI), the fragment of Peano arithmetic, fails to have the joint embedding property, a result due to Wilkie. On the other hand we have proved that if we require our models to be normal, that is, to be integrally closed in their fraction fields, the corresponding theory NOI extending OI, has the joint embedding property. Here we prove NOI does not have the amalgamation property.

1 Introduction Let \mathcal{L} denote the first-order language of ordered rings based on the symbols $0, 1, +, -, \cdot, <$. *Open Induction* (OI for short) is the fragment of Peano Arithmetic axiomatized by the axioms for discretely ordered rings together with the following axiom-scheme:

$$\forall \bar{x}((\theta(\bar{x}, 0) \wedge \forall y \geq 0(\theta(\bar{x}, y) \rightarrow \theta(\bar{x}, y + 1))) \rightarrow \forall y \geq 0 \theta(\bar{x}, y)),$$

where $\theta(\bar{x}, y)$ is a quantifier-free \mathcal{L} -formula.

In this article rings are always commutative with 1.

A first point in the study of models of OI is the following algebraic characterization.

Theorem 1.1 (Shepherdson) *Let M be a discretely ordered ring. M is a model of open induction if and only if for all r in the real closure of the fraction field of M there is an $a \in M$ such that $|r - a| < 1$.*

Shepherdson [6] proved several independence results for OI using this criterion. In particular, he showed that OI does not prove the normality axioms. We remind the reader that a domain is called normal if it is integrally closed in its fraction field.

Let NOI denote the extension of OI by adding the normality axioms, that is, for each $n \in \mathbb{N}^*$

$$\forall \bar{z}, x, y(x, y \neq 0 \wedge x^n + z_1 x^{n-1} y + \cdots + z_n y^n = 0 \rightarrow \exists w(x = wy)).$$

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Associated with OI we have the following two theories.

- DOR: axioms for discretely ordered rings.
(Ordered domains satisfying $\neg \forall x(0 < x < 1)$.)
- ZR: DOR together with axioms for \mathbf{Z} -rings.
(M is a \mathbf{Z} -ring if for every $n \in \mathbf{N}^*$, $M/nM \cong \mathbf{Z}/n\mathbf{Z}$.)

We also have their normal counterparts NDOR and NZR, respectively.

Requiring normality for a discretely ordered ring, we get an arithmetic closer to that of \mathbf{Z} . We remind the reader that \mathbf{Z} is normal and \mathbf{Q} is algebraically closed in the fraction field of any normal discretely ordered ring. More remarkably van den Dries proved in [1] that Hilbert's Irreducibility Theorem holds for normal discretely ordered rings.

Using Shepherdson's criterion, it is easy to see that every (normal) model of OI is a (normal) \mathbf{Z} -ring. Although the extension is strict, Wilkie proved in [7] that every model of ZR can be embedded in a model of OI. The key lemma in his construction is the following.

Lemma 1.2 (Wilkie) *Let M be a model of ZR, and F its fraction field. Let L be an ordered field dense in its real closure, extending M and $|M|^+$ -saturated. Let r be an element of the real closure of F such that $|r - a| > 1$ for all $a \in M$. Then there is a z in L transcendental over F such that $|r - z| < 1$ and $M[z]$ is discretely ordered.*

With the notation of the lemma we say that $M[z]$ has been obtained from M by the *Wilkie-construction* (against r).

On the other hand, provided we have ring homomorphisms $\varphi_p : M \rightarrow \mathbf{Z}_p$ for each prime p (where \mathbf{Z}_p denotes the ring of p -adic integers) we define

$$M_\varphi = \left\{ \frac{a}{n} : a \in M, n \in \mathbf{N}^* \text{ and } n \mid \varphi_p(a) \text{ for each prime number } p \right\}.$$

If M is a model of DOR then M_φ is a model of ZR (see [1]), and we say M_φ has been obtained from M by the *$\hat{\mathbf{Z}}$ -construction*.

Note that both the Wilkie- and the $\hat{\mathbf{Z}}$ -constructions preserve normality. That is, with the above notation, if M is normal then $M[z]$ and M_φ are both normal (van den Dries [1]).

Now to prove Wilkie's embedding theorem we start with a (normal) model of ZR and iterate both the Wilkie- and $\hat{\mathbf{Z}}$ -constructions to get a (normal) model of OI.

In [2], Macintyre and Marker do a systematic study of primes and their residue rings in (normal) models of OI. We shall use the following result later on.

Theorem 1.3 (Macintyre-Marker) *Suppose M is a (normal) discretely ordered ring equipped with homomorphisms $\varphi_p : M \rightarrow \mathbf{Z}_p$ for each prime number p , and q a positive prime element of M with $\varphi_p(q)$ a unit of \mathbf{Z}_p for each prime number p . Then there is an embedding $M \rightarrow R$, where R is a (normal) model of OI, such that the ideal qR is maximal. Moreover any other prime q' of M_1 satisfying $\varphi_p(q')$ a unit of \mathbf{Z}_p for each prime number p , keeps prime in R .*

In Otero [5] we prove that models of NOI have the joint embedding property (JEP for short). Among the usually studied fragments of arithmetic this

property is specific for NOI. For, as Wilkie showed, OI does not have JEP (see for example Otero [4]) and in [4] we proved that fragments of Peano Arithmetic where induction with (bounded) quantifiers is allowed do not have JEP either.

In [5] a natural question is left open: *Does NOI have the amalgamation property?* Here we give a negative answer to this question.

Theorem 1.4 *The theory of normal open induction does not have the amalgamation property.*

As is remarked in [5], to prove JEP for NOI (that is, amalgamation over \mathbf{Z}) we made use of the fact that any model of NOI is an end extension of \mathbf{Z} and a flat \mathbf{Z} -module. Here we get a counterexample to amalgamation producing two models M and M_1 of NOI, with M_1 an extension of M which is neither flat as M -module nor an end extension of M . Then for a suitable model M_2 of NOI extending also M we show that M_1 and M_2 cannot be amalgamated over M into a discretely ordered ring.

2 The construction We need the following two lemmas to prove the theorem.

Lemma 2.1 *There is a normal model of open induction M and q_1, q_2 distinct positive infinite primes in M such that (q_1, q_2) is a prime ideal of M and $(q_1, q_2) \cap \mathbf{Z} = \{0\}$.*

Lemma 2.2 *Let M be a discretely ordered ring and F its fraction field. Let L be an ordered field dense in its real closure, extending M and $|M|^+$ -saturated. Let q_1 and q_2 be two distinct positive infinite primes in M . Then*

- (i) *If $(q_1, q_2) \cap \mathbf{Z} = \{0\}$ then there is an x in L transcendental over F such that $M[x, q_1/x, q_2/x]$ is a discretely ordered ring.*
- (ii) *If (q_1, q_2) is a radical ideal of M , M is normal and x is transcendental over F , then $M[x, q_1/x, q_2/x]$ is also normal.*

Proof of Theorem 1.4: Let M be as in Lemma 2.1. By Lemma 2.2 there is a normal discretely ordered ring $R_1 = M[x, q_1/x, q_2/x]$ extending M with x transcendental over the fraction field of M (in particular $|x| > 1$). Being a model of OI, M is endowed with ring homomorphisms $\varphi_p: M \rightarrow \mathbf{Z}_p$ for each prime number p . We extend them to R_1 by $\varphi_p(x) = 1$ for each such p . Now we iterate the \mathbf{Z} - and Wilkie-constructions to get a model M_1 of NOI extending R_1 .

On the other hand q_1 and q_2 are distinct positive primes in M . So by Theorem 1.3 there is a normal model of open induction M_2 extending M in which q_2 is maximal and q_1 is prime. Hence since they are distinct and positive, there exist u and v in M_2 such that $uq_1 + vq_2 = 1$.

Let M' be any ordered ring amalgamating M_1 and M_2 over M . Then $1/x = u(q_1/x) + v(q_2/x)$ in M' ; therefore M' cannot have a discrete order.

Remark As we have mentioned above, M_1 is neither an end extension of M nor a flat M -module. The first assertion is clear. Let us prove the second one. Suppose first R_1 is a flat M -module. Then

$$(I_1 \cap I_2)R_1 = I_1R_1 \cap I_2R_1$$

for any pair of ideals I_1, I_2 of M (see, for example, Theorem 2.7.4 in Matsu-
mura [3]).

Let $I_i = (q_i)$ ($i = 1, 2$). Then $I_1 \cap I_2 = (q_1 q_2)$ for q_1 and q_2 are prime in M .
Now, $q_1 q_2 / x = q_1 \cdot q_2 / x = q_2 \cdot q_1 / x$ belongs to $q_1 R_1 \cap q_2 R_1$, hence also belongs
to $q_1 q_2 R_1$. Therefore

$$q_1 q_2 = q_1 q_2 x \sum_{(i,j,k) \in J} a_{ijk} x^{k-(i+j)} q_1^i q_2^j = q_1 q_2 \sum_{k-(i+j)=-1} a_{ijk} q_1^i q_2^j$$

for some $a_{ijk} \in M$ (where J is a finite subset of \mathbb{N}^3). Therefore $1 = a q_1 + b q_2$
with $a, b \in M$. This is a contradiction, hence R_1 is not flat over M .

On the other hand $(q_1, q_2) \cap \mathbf{Z} = \{0\}$. Let S denote \mathbf{Z}^* . Then

$$M_1 \subset S^{-1} R_1 [z_\nu; \nu \in \Delta]$$

with the z_ν 's algebraically independent over the fraction field of $S^{-1} R_1$, so a
similar argument shows that M_1 is not M -flat.

This last fact was the idea behind getting a counterexample for the amalgam-
ation property for NOI; however, it is not used later on.

To prove Lemma 2.1 we need the following.

Lemma 2.3 *Let B be a model of ZR and q_1, q_2 distinct positive infinite
primes in B . Then*

$$(q_1, q_2) S^{-1} B \cap B = (q_1, q_2) B.$$

Proof: For the nontrivial inclusion suppose $a \in B$ is such that $an = u_1 q_1 + u_2 q_2$
for some $n \in \mathbb{N}^*$ and u_1, u_2 in B . Since q_1 and q_2 are prime in B , their images
 $\varepsilon_{1,p}$ and $\varepsilon_{2,p}$ are units in \mathbf{Z}_p for each p . Let $n = p_1^{m_1} \cdot \dots \cdot p_l^{m_l}$ be the prime fac-
torization of n in \mathbb{N} . Get $s_p \in \mathbf{Z}$, $s^p \equiv (\varepsilon_{1,p})^{-1} \pmod{p^m}$ in \mathbf{Z}_p for each $p^m \in$
 $\{p_1^{m_1}, \dots, p_l^{m_l}\}$.

Let $\alpha_{i,p} = \varphi_p(u_i) \in \mathbf{Z}_p$ ($i = 1, 2$) and get $b \in \mathbf{Z}$, $b \equiv \alpha_{2,p} s_p \pmod{p^m}$ in \mathbf{Z}_p
for each p^m as above.

Since $a = (u_1/n)q_1 + (u_2/n)q_2 = [(u_1 + bq_2)/n]q_1 + [(u_2 - bq_1)/n]q_2$, it
suffices to prove that $\alpha_{2,p} - b\varepsilon_{1,p} \equiv 0 \pmod{p^m}$ and $\alpha_{1,p} + b\varepsilon_{2,p} \equiv 0 \pmod{p^m}$
for each p^m as above.

The first assertion is clear by the choice of b . For the second one note
that $a \in B$ so $\alpha_{1,p}\varepsilon_{1,p} + \alpha_{2,p}\varepsilon_{2,p} \equiv 0 \pmod{p^m}$, hence $\alpha_{1,p} + b\varepsilon_{2,p} \equiv \alpha_{1,p} +$
 $\alpha_{2,p} s^p \varepsilon_{2,p} \equiv \alpha_{1,p} - s_p \alpha_{1,p} \varepsilon_{1,p} \equiv 0 \pmod{p^m}$.

Proof of Lemma 2.1: Fix some ordered field L dense in its real closure and \aleph_1 -
saturated. We shall construct such M inside L . We begin with $A_0 = \mathbf{Z}[q_1, q_2]_\varphi$
with $q_1, q_2 \in L$ algebraically independent over \mathbf{Q} , $q_1 > n$ for all $n \in \mathbb{N}$, and
 $q_2 > |f(q_1)|$ for all $f(v) \in \mathbf{Z}[v]$. We define, for each prime number p , homo-
morphisms $\varphi_p: \mathbf{Z}[q_1, q_2] \rightarrow \mathbf{Z}_p$ by $\varphi_p(q_i) = \varepsilon_{i,p}$ for some fixed $\varepsilon_{i,p}$ unit of \mathbf{Z}_p
($i = 1, 2$). So A_0 is obtained from $\mathbf{Z}[q_1, q_2]$ and these φ_p 's using the $\hat{\mathbf{Z}}$ -con-
struction.

Now, starting with this A_0 we get M by iterating the Wilkie- and $\hat{\mathbf{Z}}$ -construc-
tion; thus $A_{i+1} = A_i[z_i]_\varphi$ with $\varphi(z_i) = 0$ and taking unions at limit stages. By
stage $\omega \cdot \omega$ we have M , a model of OI which is normal since $\mathbf{Z}[q_1, q_2]$ is so and
the Wilkie- and $\hat{\mathbf{Z}}$ -constructions preserve normality.

We prove by induction that M has the required conditions. First, note that $\varphi_p(q_1)$ and $\varphi_p(q_2)$ are units of \mathbf{Z}_p for each p , so q_1 and q_2 are also prime in M (see Lemmas 3.3 and 3.11b in [2]). Hence we have left to prove:

- (a) $(q_1, q_2)A_i \cap \mathbf{Z} = \{0\}$ and
- (b) $(q_1, q_2)A_i$ prime.

Let S denote, as above \mathbf{Z}^* .

$j = 0$:

- (a) $(q_1, q_2)A_0 \cap \mathbf{Z}^* \subset (q_1, q_2)\mathbf{Q}[q_1, q_2] \cap \mathbf{Z}^* = \emptyset$.

(b) $(q_1, q_2)S^{-1}A_0 = (q_1, q_2)\mathbf{Q}[q_1, q_2]$ is a prime ideal, so if u, v are in A_0 and $uv \in (q_1, q_2)A_0$, then $u \in (q_1, q_2)S^{-1}A_0$, say.

Therefore $u \in (q_1, q_2)S^{-1}A_0 \cap A_0 = (q_1, q_2)A_0$ by Lemma 2.3.

$j = i + 1$: $A_{i+1} = A_i[z_i]_\varphi$.

(a) Suppose $n \in (q_1, q_2)A_i[z_i]_\varphi$. Then $n = u(z_i)q_1 + v(z_i)q_2$, so $n = u(0)q_1 + v(0)q_2$. Now applying Lemma 2.3 and induction we get $n = 0$.

(b) By induction (q_1, q_2) is a prime ideal of A_i , so $(q_1, q_2)A_i[z_i]$ is also prime in $A_i[z_i]$. On the other hand, since $(q_1, q_2)A_i[z_i] \cap \mathbf{Z}^* = \emptyset$ we have $(q_1, q_2)S^{-1}A_i[z_i]$ prime. By Lemma 2.3

$$(q_1, q_2)A_i[z_i]_\varphi = (q_1, q_2)S^{-1}A_i[z_i] \cap A_i[z_i]_\varphi,$$

so $(q_1, q_2)A_{i+1}$ is a prime ideal.

Clearly (a) and (b) are preserved under taking unions.

Finally we prove Lemma 2.2. It is a generalization of Lemmas 3.22 and 3.23 in [2]. Their aim with those lemmas was to make reducible (in some extension) an irreducible element of M . In ours the aim is to have in some extension of M a common factor of two primes of M .

Proof of Lemma 2.2: Get M , q_1 and q_2 as in Lemma 2.1 and $L \supset M$ as in the hypothesis of Lemma 2.2.

The proof of (i) is similar to that of Lemma 3.22 in [2]. We first get x transcendental over F satisfying the standard cut, that is, $x > m$ for all $m \in \mathbf{Z}$, and $x < a$ for all those $a \in M$ which satisfy $a > m$ for all $m \in \mathbf{N}$.

Suppose there is $f(u, v, w) \in M[u, v, w]$ such that $0 < f(x, q_1/x, q_2/x) < 1$. We write $f(x, q_1/x, q_2/x) = 1/x^s g(x)$ where $g(v) \in M[v]$ and $g(0) \neq 0$.

Let $f(x, q_1/x, q_2/x) = \sum_{l=-s}^n d_l x^l$ where $d_l = \sum_{k-(i+j)=l} a_{ijk} q_1^i q_2^j$, then $g(v) = \sum_{i=0}^{n+s} d_{i-s} x^i$.

Next we prove that there is an l such that $|d_l| > m$ for all $m \in \mathbf{N}$. First, suppose there is $d_l \neq 0$ for some $l < 0$. Then all the (i, j) occurring in the summation of this d_l must be either $i < 0$ or $j < 0$. This implies $d_l \in (q_1, q_2)$. Hence, either $d_l = 0$ or $d_l \notin \mathbf{Z}$. By discreteness of the order this last one implies $|d_l| > m$ for all $m \in \mathbf{Z}$. Now suppose $d_l = 0$ for all $l < 0$, then $f(x, q_1/x, q_2/x) \in M[x]$ so one of the d_l 's must not be in \mathbf{Z} . Otherwise $f(x, q_1/x, q_2/x) \in \mathbf{Z}[x]$ which is a discretely ordered ring, contradicting $0 < f(x, q_1/x, q_2/x) < 1$.

The rest of the proof follows as in 3.22 of [2]. It is based on quantifier elimination in K the real closure of F , and it also makes use of the prime factorization of $g(v)$ in $K(i)[v]$.

To prove (ii) we first make use of Lemma 3.23 in [2] to get $R = M[x, q_1/x]$ normal. To prove that $R_1 = R[q_2/x]$ is also normal suffices to prove that xR_1 is a radical ideal of R_1 (see for example 3.17 in [2]).

We have

$$R_1/xR_1 \cong (M[x, u, v]/(ux - q_1, ux - q_2))/(x) \cong M[u, v]/(q_1, q_2).$$

Hence $R_1/xR_1 \cong (M/(q_1, q_2))[u, v]$.

Now, since (q_1, q_2) is a radical ideal of M , $M/(q_1, q_2)$ does not have nilpotent elements, therefore $(M/(q_1, q_2))[u, v]$ does not have nilpotents either. This means xR_1 is radical.

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