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On the Maximality of Some Pairs of p-t Degrees

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Abstract This paper discusses the properties of polynomial time Turing degrees. It is shown that there exist recursive p-t degrees $a > a_0$ and $b > b_0$, for any recursive p-t degrees a_0 and b_0, such that $\{a, b\}$ and $\{a_0, b_0\}$ have the same low bound set of the degrees. Hence, there is neither maximal minimal pair, maximal exact pair, nor maximal branching pair of p-t degrees.

Zheng [7] proved that there is no maximal p-m minimal pair. This paper will show a similar (in fact somewhat more general) result about p-t degrees.

The concept of the polynomial time Turing reducibility (abbrev. p-t reducibility) was introduced by Cook in [3]. A set A is polynomial time Turing reducible to B (denoted by $A \leq_t^p B$) if there is a polynomial time bounded Turing machine M^B with oracle B such that M^B accepts A. A is p-t equivalent to B if $A \leq_t^p B$ and $B \leq_t^p A$, which is denoted by $A \equiv_t^p B$. The p-t degree of set A (denoted by $\deg(A)$) is the class of all sets that are equivalent to A, i.e., $\deg(A) = \{B : A \equiv_t^p B\}$. Below, the p-t degrees are denoted by $a, b, c, \ldots$. For any p-t degrees a and b, a is called (p-t) reducible to b (denoted by $a \leq b$) if there are sets $A \in a$ and $B \in b$ such that $A \leq_t^p B$. In this paper, the degree always means the p-t degree, and the "p-t" is often omitted.

Like the Turing reducibility which gives a natural classification of the unsolvable problems according to their relative difficulty, p-t reducibility provides a natural tool for classifying the solvable but not efficient solvable (so-called intractable) problems according to their relative complexity. So the discussion of polynomial time degrees concerns mainly the recursive ones. In this paper, we deal only with recursive sets and degrees.

The study of the structural properties of p-t degrees was initiated by Ladner [5],[6]. Ambos-Spies [1] offers a broader discussion. Ladner proved in [6] that there are degrees a, b with infimum 0, the degree of polynomial time computable sets, i.e., $a \cap b = 0$. A pair a, b with this property is called a minimal pair. Furthermore, Ambos-Spies has shown in [1] that any nonzero degree is one-half of some minimal pair. Then it is natural to ask whether there is a minimal pair which is maximal, i.e., is there a minimal pair a, b such that any pair a_1, b_1 with

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$a_1 > a$ and $b_1 > b$ is not a minimal pair. This question for the recursively enumerable Turing degrees has recently been answered negatively by Harrington and Soare [4]. This paper answers this question negatively for the p-t degrees and shows that no minimal pair is maximal. Similar results will also be obtained for the exact pair and the branching pair. A pair of degrees a, b is called an exact pair if there is an increasing sequence of degrees $\{c_n\}_{n\in\omega}$ such that $\forall n(c_n \le a, b)$ & $\forall d(d \le a,\ b \to \exists n(d \le c_n))$, and a, b is a branching pair of degree c if $c = a \cap b$. In fact, we will prove a more general result which asserts that there are $a_1 > a$ and $b_1 > b$, for any p-t degrees a and b, such that $\{a_1, b_1\}$ and $\{a, b\}$ have the same set of low bound degrees.

Let Σ be an alphabet with two elements 1 and 0. Σ^* is the set of all finite strings over Σ. Our discussion will be restricted to Σ^*, and all sets constructed will be subsets of Σ^*. For any strings $x, y \in \Sigma^*$, xy is the concatenation of x followed by y. $|x|$ is the length of x. $\langle , \rangle$ is a one-to-one polynomial time computable pairing function from $\Sigma^* \times \Sigma^*$ onto Σ^* with polynomial time computable inverse functions (see Balcazar et al. [2]). Any natural number n can be written in binary and thought of as an element of Σ^*. Then we can define the nth section of A for any set A and natural number n as follows. $A^{[n]} = \{x : \langle n, x\rangle \in A\}$ and $A^{[\le n]} = \bigcup_{i \le n} A^{[i]}$. We do not distinguish between a set and its characteristic function, so $x \in A$ iff $A(x) = 1$ and $x \notin A$ iff $A(x) = 0$. A function f is polynomial honest if f is recursive and there is a polynomial p such that $f(n)$ can be computed in $p(f(n))$ steps for all $n \in \omega$. (Formally, function f from Σ^* to Σ^* is polynomial honest if there exists a polynomial p such that $p(|f(x)|) > |x|$ for all $x \in \Sigma^*$.) Note that for any recursive function g and natural number n, there is a strictly increasing polynomial honest function f with $g(n) \le f(n)$, which we denote by "f dominates g". For any sets A and B, the join of A and B is defined by $A \oplus B = \{0x : x \in A\} \cup \{1x : x \in B\}$.

Other unexpressed concepts and notations are all from [1] and [2].

Now we can prove the main theorem.

Theorem. *There are degrees $a > a_0$ and $b > b_0$ for any degrees a_0 and b_0 such that $\{a, b\}$ and $\{a_0, b_0\}$ have the same low bound set of degrees, i.e.,*

$$\forall d(d \le a, b \leftrightarrow d \le a_0, b_0).$$

(*Note: The result of the theorem is also true for all* p-t *degrees including nonrecursive ones.*)

Proof: Fix $A' \in a_0$, $B' \in b_0$ and define A_0, B_0 as follows.

$$A_0(\langle n, x\rangle) = \begin{cases} 0 & \text{if } |x| > n \\ A'(x) & \text{if } |x| \le n \end{cases}$$

$$B_0(\langle n, x\rangle) = \begin{cases} 0 & \text{if } |x| > n \\ B'(x) & \text{if } |x| \le n. \end{cases}$$

It is obvious that $A_0 \in a_0$, $B_0 \in b_0$ and $A_0^{[n]}, B_0^{[n]} \in 0$ for any n.

Note that it is sufficient for us to construct $a > a_0$ such that $\{a, b_0\}$ has the same low bound set of degrees with $\{a_0, b_0\}$. Thus we need only to construct a recursive set A satisfying the following requirements for all e:

$$R_{2e}: \quad A \neq P_e^{A_0}$$

$$R_{2e+1}: \quad D = M_e^{A_0 \oplus A} = N_e^{B_0} \rightarrow D \leq_t^p A_0, B_0$$

where $\{P_e^X\}_{e\in\omega}$ is an effective enumeration of all polynomial time bounded deterministic Turing machines with oracle X, and $\{M_e^X, N_e^X\}$ is the effective enumeration of all pairs of such machines. Let $\{p_e\}_{e\in\omega}$ be a recursive sequence of polynomials such that p_e bounds the running time of M_e^X for all X. If set A meets the requirements R_e for all e, then $A_0 <_t^p A_0 \oplus A$ by all R_{2e} and $\forall D(D \leq_t^p B_0 \;\&\; D \leq_t^p A_0 \oplus A \rightarrow D \leq_t^p A_0 \;\&\; D \leq_t^p B_0)$ by all R_{2e+1}. Hence the degree $a = \deg(A_0 \oplus A)$ is what we need.

The strategy for meeting the requirements R_{2e} for all e is a typical diagonal method. To meet R_{2e+1}, we try to ensure that $M_e^{A_0\oplus A} \neq N_e^{B_0}$ by choosing an appropriate extension of the given part of A. If it fails, then the set $D = M_e^{A_0\oplus A} = N_e^{B_0}$ must be p-t reducible to both A_0 and B_0.

The set A will be constructed by an initial segment argument. While we construct A we also construct a uniformly recursive sequence of sets $\{A_e\}$ and functions $\{f_e\}$. At the end of stage s, all the sets and functions constructed are determined on all strings of length $\leq s$. Each stage will be effective and finite, hence A (as well as A_e and f_e) are recursive. Furthermore, because we are interested only in the recursive sets, the step-counting function of the computation of B_0 must be a recursive function which can be dominated by a polynomial honest function, say, b. Then $b(|x|)$ is always greater than the number of the steps computing the $B_0(x)$ for all x.

The construction of A:

Stage 0: Proceed to stage 1.

Stage $s + 1$: Given $A_e(x)$, $A(x)$, and $f_e(x)$ for all e and x with $|x| < s$. For $e \geq s$ and x with $|x| = s$, let $A_e(x) = B_0(x)$ and $f_{e+1}(s) = b(s)$. For $e < s$ and x with length s, $A_e(x)$, $f_{e+1}(s)$, and $A(x)$ will be defined by the following $s + 1$ substages from s to 0.

Substage $e(0 \leq e \leq s)$: Let $f_e(s)$ be the least number $n > b(s)$ such that the construction up to now can be performed in n steps.

Now, we say the requirement R_{2e} is satisfied at stage $s + 1$ if the following hold:

$$\exists x(|x| < s \;\&\; A(x) \neq P_e^{A_0}(x)). \tag{1}$$

Requirement R_{2e+1} is satisfied at stage $s + 1$ if

$$\exists y(p_e(|y|) \leq s \;\&\; M_e^{A_0\oplus A\restriction s}(y) \neq N_e^{B_0}(y)) \tag{2}$$

where $A \restriction s = \{x : x \in A \;\&\; |x| < s\}$.

We say requirement R_{2e} requires attention at stage $s + 1$ if it is not yet satisfied and there is an x such that

$$x \notin \Sigma^{*[\leq e]} \;\&\; |x| = s. \tag{3}$$

Requirement R_{2e+1} requires attention if it is not yet satisfied and there are x, t, and finite set C satisfying the following:

$$|x| < f_e(s) \;\&\; t = p_e(|x|), \tag{4}$$

$$C \restriction s = (A_0 \oplus A) \restriction s, \tag{5}$$

(6) $\forall y(y \in \Sigma^{*[\leq e]} \oplus \Sigma^{*[\leq e]} \ \&\ s \leq |y| < t \rightarrow C(y) = (A_0 \oplus B_0)(y))$

(7) $M_e^{C \upharpoonright t}(x) \neq N_e^{B_0}(x).$

The definition of $A_{e-1}(x)$ for $|x| = s$ is distinguished into the following three cases.

Case 1. If R_{2e} requires attention at stage $s+1$, let x_0 be the least x satisfying (3) under the lexicographical order of Σ^*. For all x with length s, define

$$A_{e-1}(x) = \begin{cases} B_0(x) & \text{if } x \neq x_0 \\ 1 \dot{-} P_e^{A_0}(x_0) & \text{if } x = x_0 \end{cases}.$$

Then we say that R_{2e} receives attention at stage $s+1$.

Case 2. If R_{2e+1} requires attention at stage $s+1$ but R_{2e} does not, then let x_0, t_0, C_0 be the least x, t, C satisfying (4)–(7), under the assumption that the class of all finite sets has been effectively coded, and hence there is a well ordering on it. Define $A_{e-1}(x) = C(1x)$ for all x with length s and then say that R_{2e+1} receives attention at stage $s+1$.

Case 3. If neither R_{2e} nor R_{2e+1} requires attention at stage $s+1$, then define $A_{e-1}(x) = A_e(x)$ for all x with length s.

In all three cases, if $e > 0$, proceed to substage $e-1$. At the last substage (i.e., $e = 0$), we will define the $A_{-1}(x)$ for all x with length s. Then let $A(x) = A_{-1}(x)$ for $|x| = s$ and proceed to stage $s+2$.

This ends the construction.

It is worth noting that our construction is a priority argument within a stage. That is, during each stage the requirements all try to become satisfied. Some of the requirements may receive attention. But only the highest priority one that receives attention receives really effective treatment for our set A by the reverse order of the substages from s to 0.

We say that requirement R_m is active at stage $s+1$ if R_m receives attention but no requirements R_n with $n < m$ require, hence receive, attention. It is obvious that the construction is effective, and so A, A_e, and f_e are all recursive. And f_e is a strictly increasing polynomial honest function such that $A_e(x)$ and $A(x)$ can be computed in $f_e(|x|)$ and $f_e(|x|+1)$ steps, respectively, for all x and e.

To see that A succeeds we need only prove the following five lemmas.

Lemma 1. *For any e, if R_{2e} is active at stage $s+1$, then R_{2e} is satisfied at stage $s+2$ and is met.*

Proof: By the construction, if R_{2e} is active at stage $s+1$, then $A_{e'}(x) = A_e(x)$ for all $e' < e$ and all x with length s. Particularly, we have $A(x_0) = A_{e-1}(x_0) = 1 \dot{-} P_e^{A_0}(x_0) \neq P_e^{A_0}(x_0)$ for some x_0 with length s. Hence R_{2e} is satisfied at stage $s+2$ and then R_{2e} is met.

Lemma 2 *For any e, if R_{2e+1} is active at stage $s+1$ and there is no requirement $R_{m'}$ with $m' < 2e+1$, that requires attention after stage $s+1$, then R_{2e+1} is satisfied at some stage after stage $s+1$.*

Proof: Suppose that R_{2e+1} is active at stage $s+1$, then there are x_0, t_0, C_0 satisfying (4)–(7). Note that $e \le s+1 \le t_0$. So, if R_{2e+1} is not satisfied before stage t_0, then R_{2e+1} will be active whenever it requires attention after stage $s+1$ because there is no $R_{m'}$ with $m' < 2e+1$ that will require attention. Hence $A_{e-1}(x) = A(x)$ for any x with $|x| \ge s+1$. This means that R_{2e+1} will require attention, hence be active, at stage t for all t with $s+1 \le t \le t_0$, and even the x_0, t_0, C_0 being chosen at stage t are the same as those of stage $s+1$. Then $C_0 \upharpoonright t_0 = A_0 \oplus A \upharpoonright t_0$, and R_{2e+1} is satisfied at stage t_0 by (7).

Lemma 3. *Each requirement R_m requires attention at most finitely often.*

Proof: By induction on m. For $m = 0$, it is obvious by Lemma 1 and the construction.

Suppose that $m > 0$, then there is s_0, by the induction hypothesis, such that there is no $R_{m'}$ with $m' < m$ which will require attention after stage s_0. So, if R_m requires attention after stage s_0, then R_m will be active and will be satisfied at some later stage t by Lemma 2. Then it is not difficult to see that R_m will not require attention after stage t. So R_m requires attention at most finitely often.

Based on the above technical lemmas, our main results can be shown as the following two lemmas.

Lemma 4. *Requirement R_{2e} is met for all e.*

Proof: Firstly, it is obvious by (1) that if R_{2e} is satisfied at some stage, then it is met. Now suppose by way of contradiction that R_{2e} is never satisfied at any stage. Note that there are infinitely many s such that $\exists x(x \notin \Sigma^{*[\le e]} \;\&\; |x| = s)$, and for every such s, R_{2e} will require attention at stage $s+1$. This means that R_{2e} requires attention infinitely often, contrary to Lemma 3.

Lemma 5 *R_{2e+1} is met for every e.*

Proof: Suppose by way of contradiction that R_{2e+1} is not met and has the least index among the requirements which are not met. Then we have at first that

(8) $$D = M_e^{A_0 \oplus A} = N_e^{B_0}.$$

Choose s_0 by Lemma 3 such that, for any $s \ge s_0$, there are no x, t, C satisfying (4)–(7) at stage s and no requirement R_m with $m < 2e+1$ that requires attention at stage s. Then, by the construction, we have

(9) $$\forall x(|x| \ge s_0 \to A(x) = A_e(x)), \text{ and}$$

(10) $$\forall x(|x| \ge s_0 \;\&\; x \in \Sigma^{*[\le e]} \to A(x) = B_0(x)).$$

It follows from (9) and the definition of f_e that

(11) $$\forall x(|x| \ge s_0 \to A(x) \text{ can be computed in } f_e(x) \text{ steps})$$
$$\& \; \forall x(|x| < s_0 \to A(x) \text{ can be computed in } f_e(s_0) \text{ steps}).$$

Now we can show that $D \le_t^p A_0, B_0$ as follows. $D \le_t^p B_0$ is from (8) immediately. To show that $D \le_t^p A_0$, we need only show that D can be computed in polynomial time with oracle A_0.

Note that there are only finite many x with $|x| \leq f_e(s_0)$. We can consider only such x that $|x| > f_e(s_0)$. Given x with $|x| > f_e(s_0)$, find the maximal s such that $f_e(s-1) \leq |x|$. This can be done in polynomial time by the polynomial honesty of f_e. Since $s > s_0$, by the choice of s_0 we have that $M_e^{C_1 \upharpoonright t}(x) = N_e^{B_0}(x) = M_e^{C_2 \upharpoonright t}(x)$ for $t = p_e(|x|)$ and any two finite sets C_1, C_2 satisfying (5) and (6) at stage s. Note that, by (10), $A_0 \oplus A$ does satisfy (5) and (6) at stage s. Hence we have $M_e^{C \upharpoonright t}(x) = M_e^{A_0 \oplus A \upharpoonright t}(x) = M_e^{A_0 \oplus A}(x) = D(x)$ whenever C satisfies (5) and (6). Particularly, let C satisfy (5) and the following:

$$\forall y (s \leq |y| < t \rightarrow C(y) = A_0^{[\leq e]} \oplus B_0^{[\leq e]}(y)). \tag{12}$$

Then C must satisfy (6). So it is sufficient to compute $M_e^{C \upharpoonright t}(x)$ for the computation of $D(x)$. Now, in the computation of $M_e^{C \upharpoonright t}(x)$, any query $z \in C$? can be replaced by the query $z \in A_0^{[\leq e]} \oplus B_0^{[\leq e]}$? if $|z| \geq s$, or by $z \in A_0 \oplus A$? if $|z| < s$. Note that $z \in A_0 \oplus A \leftrightarrow \exists z_1(z = 0z_1 \;\&\; z_1 \in A_0)$ or $\exists z_2(z = 1z_2 \;\&\; z_2 \in A)$, $A(z_2)$ can be computed in $\max(f_e(|z_2|), f_e(s_0)) \leq f_e(s-1) \leq |x|$ steps, and $A_0^{[\leq e]} \oplus B_0^{[\leq e]} \in \mathit{0}$ by the choice of A_0 and B_0. So, $D(x)$ can be computed in polynomial time with oracle A_0, hence $D \leq_t^p A_0$.

This completes the proof of the theorem.

Corollary 1. *For any minimal pair a_0, b_0, there are $a > a_0$ and $b > b_0$ such that a, b is a minimal pair. Hence there is no maximal minimal pair of* p-t *degrees.*

Corollary 2. *For any exact pair a_0, b_0 of an increasing sequence of degrees $\{c_n\}_{n \in \omega}$, there are $a > a_0$ and $b > b_0$ such that a, b is also an exact pair of $\{c_n\}_{n \in \omega}$. Hence, there is no maximal exact pair.*

Corollary 3. *If a_0, b_0 is a branching pair of c, then there are degrees $a > a_0$ and $b > b_0$ such that a, b is also a branching pair of c. Hence there is no maximal branching pair for any degree.*

Corollary 4. *For any degrees a, b with $a < b$, there exists degree $d > a$ such that $a = b \cap d$, i.e., b is a-cappable.*

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