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Is It Possible that Belief Isn't Necessary?

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Abstract There has been a tradition in the history of doxastic logic of treating belief as analogous to necessity. The resulting logics presuppose that believers are "ideal", which is unacceptable in light of various counterexamples discussed in the literature. It is argued that Rantala's proposals to salvage the alleged analogy between necessity and belief fail. In addition, a logic that treats belief as analogous to possibility and a corresponding semantics motivated by Stalnaker's claim that agents can be in more than one belief state are developed. Although this logic and semantics are inconsistency-tolerant, new problems arise. Finally a modest though nontrivial belief logic is proposed which does not treat belief as possibility or necessity and which does not presuppose that agents' beliefs are consistent or deductively closed.

1 The alleged analogy between necessity and belief Beginning with Hintikka's discussion of epistemic and doxastic logics in Hintikka [7], the tradition in the literature has been to treat the belief operator '$\boldsymbol{B}$' (x believes that) as a kind of necessity operator (see also Hintikka [8], Rantala [14],[15], and Rescher [16]). That is, sentential and quantified doxastic logics have traditionally been regarded as normal[1] alethic modal systems where the necessity operator is informally construed as 'x believes that'. Pushing this analogy between necessity and belief has invited disaster, at least if we regard doxastic logics as embodying principles of belief attribution.

In particular, all instances of the following schemata are derivable in any 'normal' doxastic system, although T2 is derivable only for systems containing the doxastic version of D, $\boldsymbol{B}\alpha \supset \neg\boldsymbol{B}\neg\alpha$:

T1: $(\boldsymbol{B}\alpha \;\&\; \boldsymbol{B}\beta) \supset \boldsymbol{B}(\alpha \;\&\; \beta)$ adjunction schema

T2: $\neg(\boldsymbol{B}\alpha \;\&\; \boldsymbol{B}\neg\alpha)$ consistency schema.

Informally, T1 says that agents always conjoin beliefs and T2 asserts that agents' beliefs are always consistent. The principles of belief attribution embodied in these schemata have been rejected for the most part in the literature (see, for ex-

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ample, Barcan Marcus [12], Rescher and Brandom [17], and Stalnaker [20]). Presumably, agents may hold inconsistent beliefs in different contexts without conjoining them, thus casting doubt on both T1 and T2 qua principles of belief attribution. If Barcan Marcus is right, Kripke's now famous puzzling Pierre case and his Paderewski example (see Kripke [10]) could be regarded as hypothetical cases where an agent fails to conjoin inconsistent beliefs held in different contexts (see [12], pp. 506–507).

Further, the following inference rule derivable in any normal doxastic system has been a source of embarrassment to those who wish to exploit the apparent analogy between belief and necessity:

DR1: $\vdash \alpha \supset \beta \rightarrow \vdash \boldsymbol{B}\alpha \supset \boldsymbol{B}\beta$ omnidoxasticity rule.

Informally, DR1 says that agents believe whatever is logically classically implied by what they believe—they are logically 'omnidoxastic' (or in the case of epistemic logic, omniscient) with respect to what they believe. This principle expressed by DR1 has been discussed in Cresswell [5], Hintikka [9], Rantala [14],[15], and Stalnaker [19],[20]. All instances of the equivalential variant of DR1, $\vdash \alpha \equiv \beta \rightarrow \vdash \boldsymbol{B}\alpha \equiv \boldsymbol{B}\beta$ are also derivable in normal doxastic systems. What is objectionable about DR1 and its equivalential variant qua principles of belief attribution is that agents will end up believing all truths of classical logic or mathematics if they believe at least one such truth since any mathematical truth logically implies and is implied by any other.

Despite these various difficulties, rather than abandoning the attempt to exploit the supposed parallel between necessity and belief, some authors, most notably Rantala, have tried to salvage this parallel. Rantala suggests that on the syntactic front we could simply restrict the application of $\vdash \alpha \rightarrow \vdash \boldsymbol{B}\alpha$ to some recursive set Ω, so that depending on how this rule is restricted, various instances of DR1, T1, and T2 can be rendered underivable. So it would seem that the analogy between necessity and belief can be salvaged after all. Rantala's proposal will be discussed and criticized in Section 2.

As will be argued in Section 2, there are two difficulties with Rantala's solution to the problem of deduction on the *semantic* side of things. First, there is the practical difficulty that in order to invalidate any instance of DR1, T1, or T2, it will be necessary to exclude an unspecifiably infinite number of theorems from Ω if it is not a logic. Matters are worse if Ω is a logic. Second, the semantics that makes use of 'impossible worlds'[2] involves an equivocation with respect to the classical extensional connectives.

If we wish to treat doxastic logics as variants of normal alethic modal logics, perhaps the more fruitful approach is to treat belief as *possibility* rather than necessity. (Barcan Marcus alludes to the similarity between belief and possibility in [12].) We thus avoid the embarrassing result that agents always conjoin their beliefs, and we also avoid the result that agents always have consistent beliefs. This approach is discussed in Section 3.

The proposal to treat belief as analogous to possibility can be made intelligible on the semantic front by formalizing Stalnaker's suggestion that agents can be in more than one 'belief state' (where a belief state is the set of possible situations where some of an agent's belief contents obtain). An agent who is in more than one belief state may fail to conjoin his/her beliefs that α and that $\neg\alpha$ if these contents obtain at all members of distinct states. It will be shown how this informal suggestion can be understood within the context of a relational semantics for belief.

However, treating belief as analogous to possibility is at best a partial solution to the 'problem of deduction' for doxastic logic since the omnidoxasticity feature is retained. Further, there are some additional problems that this approach creates, as will be noted in Section 3. It would then seem that it is a mistake to treat belief as analogous to either necessity or possibility, at least if we are dealing with normal systems or with variants of normal systems. Then belief is not an alethic modal operator.

Moreover, it will be argued in Section 4 that on the semantic side of things, if we wish to preserve the intuition that belief is a relation obtaining between agents and alternatives, a more appropriate semantics of belief can be developed along the following lines: A content is *believed* just in case it is a member of the agent's alternative 'partial description'. A partial description is a finite set of propositions that may fail to be consistent and deductively closed. The intuitive idea here is that when an agent believes that α, he or she considers a finite descriptive alternative to the 'actual world' where α is a member. As will be shown, these descriptive alternatives are like Kripkean stipulative 'words' although they are not the 'situations' of Barwise and Perry. By imposing the right sorts of restrictions on partial descriptions, we attain a nontrivial *logic* of belief that does not contain versions of T1, T2, and DR1 with respect to the belief operator (see Section 5 of this article). The belief operator in this setup is not reducible to either necessity or possibility for normal systems of belief logic.

These remarks having been made, we shall now turn our attention to Rantala's attempts to salvage the analogy between belief and necessity.

2 Impossible worlds and semantic equivocation Rantala has argued that emendations can be made to the syntax and the corresponding semantics of normal epistemic and doxastic systems where necessity is construed as knowledge or belief in order to avoid the embarrassing result that agents are omniscient or omnidoxastic (see [14] and [15]).

On the syntactic front, Rantala's proposal to avoid omnidoxasticity in the case of doxastic logics is to restrict the rule $\vdash\alpha \rightarrow \vdash\boldsymbol{B}\alpha$ (which we shall call 'doxastic necessitation') to some arbitrary recursive subset 'Ω' of the set of wffs. As Rantala notes, Ω may be a logic (such as the intuitionistic calculus) although it need not be given that "it is hardly adequate to suppose that a person's attitudes are necessarily guided by a logic" ([14], p. 108). The derivation of any instance of DR1, $\vdash\alpha \supset \beta \rightarrow \vdash\boldsymbol{B}\alpha \supset \boldsymbol{B}\beta$, depends on the application of the doxastic necessitation rule to $\alpha \supset \beta$. In light of Rantala's proposal, we could block the derivation of some instance of DR1 by stipulating that the corresponding instance of the appropriate thesis-schema $\alpha \supset \beta$—the one to which the doxastic necessitation rule applies—is *not* in Ω. However, as we shall see, matters are not so clear-cut in the corresponding impossible worlds semantics.

Rantala's strategy of restricting the doxastic necessitation rule to avoid omnidoxasticity can be extended to block the derivation of any instance of the adjunction schema T1, $(\boldsymbol{B}\alpha \,\&\, \boldsymbol{B}\beta) \supset \boldsymbol{B}(\alpha \,\&\, \beta)$. The derivation of any instance of T1 depends on applying the doxastic necessitation rule to the appropriate instance of the classical PC thesis-schema $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$. So by excluding some instance of $\alpha \supset (\beta \supset (\alpha \,\&\, \beta))$ from the set Ω, we effectively block the derivation of the corresponding instance of T1 as well as $\vdash \alpha \supset (\beta \supset (\alpha \,\&\, \beta)) \rightarrow \vdash \boldsymbol{B}\alpha \supset \boldsymbol{B}(\beta \supset (\alpha \,\&\, \beta))$.

Further, the derivation of any instance of the consistency schema T2, $\neg(\boldsymbol{B}\alpha \,\&\, \boldsymbol{B}\neg\alpha)$, for normal systems with D depends on applying the doxastic necessitation rule to the appropriate instance of $\alpha \supset (\neg\alpha \supset (\alpha \,\&\, \neg\alpha))$. So the derivation of some instance of T2 can be blocked by excluding from Ω the relevant instance of $\alpha \supset (\neg\alpha \supset (\alpha \,\&\, \neg\alpha))$.[3]

And so modal logics where the necessity operator is construed as 'x believes that' need not contain every instance of T1, T2, or DR1, although these logics will not be 'normal' since the doxastic necessitation rule is restricted. The problem is to find a characteristic semantics for these systems. Rantala suggests emendations to the semantics for *normal* systems with this end in mind.

Rantala's suggestion on the semantic front for restricting validity-preservingness of DR1 to only those implicational theses contained in Ω (and by extension, for invalidating any or all instances of T1 and T2) is to allow 'impossible' worlds to serve as doxastic alternatives. In particular, he proposes that a model for the appropriate restricted (sentential) system is a 4-tuple, $\langle \boldsymbol{W}, \boldsymbol{W}^*, R, V\rangle$. Thus, in addition to the set $\boldsymbol{W}$ of 'possible' or standard worlds, a model will contain a set $\boldsymbol{W}^*$ of so-called impossible or nonstandard worlds. And what is nonstandard about members of $\boldsymbol{W}^*$ is that $V_{\mathcal{M}}$ is not defined inductively for such indices, though $V_{\mathcal{M}}$ assigns only '1' or '0' to wffs, not both. (This conception of 'nonclassical' worlds is also discussed in Cresswell [4].) The doxastic accessibility relation R is then redefined as ranging over members of $\boldsymbol{W} \cup \boldsymbol{W}^*$ although any restrictions imposed on R are imposed on it only for members of $\boldsymbol{W}$ (which is important in terms of the completeness results). In this semantics, validity in a model is truth at all *normal* worlds.

Rantala imposes the following restrictions on $V_{\mathcal{M}}$ for members of $\boldsymbol{W}^*$ to ensure soundness:

(1) For any wff $\alpha \in \Omega$ true at all normal worlds in $\boldsymbol{W}$, $V_{\mathcal{M}}(\alpha, w_i^*) = 1$ for all w_i^* in $\boldsymbol{W}^*$.
(2) For any $w_i^* \in \boldsymbol{W}^*$, if $V_{\mathcal{M}}(\alpha, w_i^*) = V_{\mathcal{M}}(\alpha \supset \beta, w_i^*) = 1$ then $V_{\mathcal{M}}(\beta, w_i^*) = 1$.

This second stricture ensures that impossible worlds are closed under detachment, thus validating K, $(\boldsymbol{B}\alpha \,\&\, \boldsymbol{B}(\alpha \supset \beta)) \supset \boldsymbol{B}\beta$. The first stricture ensuring that valid wffs in Ω are true at all impossible worlds is important for invalidating instances of T1, T2, and DR1. Belief wffs are evaluated as follows at members of $\boldsymbol{W}$:

$$V_{\mathcal{M}}(\boldsymbol{B}\alpha, w_i) = 1 \text{ iff for all } w_j \in \boldsymbol{W} \cup \boldsymbol{W}^* \text{ where } w_i R w_j,\ V_{\mathcal{M}}(\alpha, w_j) = 1.$$

These truth conditions allow that some of the doxastic alternatives to a world that an agent inhabits may be nonstandard in the above sense.

This semantics does in fact provide countermodels for various instances of DR1, T1, and T2, at least where Ω is not a logic, but at a very high cost. For example, suppose we wish to find a countermodel for the following instance of DR1, $\vdash p \supset (p \vee q) \rightarrow \vdash \boldsymbol{B}p \supset \boldsymbol{B}(p \vee q)$ for some sentential logic. Let $\mathcal{M}$ be a model $\langle \boldsymbol{W}, \boldsymbol{W}^*, R, V\rangle$ for some restricted K-extension such that $\boldsymbol{W} = \{w_1\}$,

$W^* = \{w_2^*\}$, and such that $\langle w_1, w_2^* \rangle \in R$. We then exclude from Ω the classical thesis $p \supset (p \vee q)$. Let $V(p, w_2^*) = V_{\mathcal{M}}(p, w_2^*) = 1$. Since $V_{\mathcal{M}}$ is not defined inductively for w_2^* and supposing that Rantala's closure restriction (2) is not violated, then $V_{\mathcal{M}}(p \supset (p \vee q), w_2^*) = 0$ would *appear* to be an admissible valuation. Also, $V_{\mathcal{M}}(p \vee q, w_2^*) = 0$ would *appear* to be an admissible valuation since once again $V_{\mathcal{M}}$ is not defined inductively for w_2^* and since the closure restriction for implication is presumably not violated.

Unfortunately, all is not well in the semantics. Suppose that Ω is a logic that contains as a thesis $p \supset (p \vee q)$. Then the above would not be a countermodel to the instance of DR1 in question. In fact, no countermodel to this instance of DR1 would be possible. Or, suppose that Ω is not a logic. Consider any theorem α of the restricted normal K-extension in question. Since the system is classical and since any theorem classically implies any other, then it will also be a theorem that $\alpha \supset (p \supset (p \vee q))$. Thus, even if $p \supset (p \vee q)$ is excluded from Ω, if either α or $\alpha \supset (p \supset (p \vee q))$ are not excluded from Ω then $V_{\mathcal{M}}(\alpha, w_2^*) = V_{\mathcal{M}}(\alpha \supset (p \supset (p \vee q)), w_2^*) = 1$ by restriction (1). Then, by restriction (2), $V_{\mathcal{M}}((p \supset (p \vee q)), w_2^*) = V_{\mathcal{M}}(p \vee q, w_2^*) = 1$. So all theorems of the form $\alpha \supset (p \supset (p \vee q))$ will need to be excluded from Ω if we wish to show that the rule $\vdash p \supset (p \vee q) \rightarrow \vdash \boldsymbol{B}p \supset \boldsymbol{B}(p \vee q)$ does not preserve validity.[4] But for any theorem α, it will also be a theorem that $p \supset (\alpha \supset (p \vee q))$. In the end, it would seem that the only way to get around all possible difficulties of the above sort is to stipulate that *all* theorems of the appropriate system must be excluded from Ω. But then the restricted rule of doxastic necessitation would be rendered inapplicable if we wished to invalidate any instance of DR1 or for that matter T1 and T2.

Even if it were possible to surmount the above technical difficulties with Rantala's proposed semantics, there is an additional problem. It could be objected that since the extensional connectives $\neg$, &, $\vee$, $\supset$, and $\equiv$ are redefined for impossible indices, they 'mean something different' for nonstandard worlds—they do not represent classical negation, conjunction, disjunction, implication, and equivalence, respectively. Then such a semantics does not show how classical negation, etc., can misbehave at nonstandard worlds. Classical negation cannot misbehave and remain classical. This objection has been discussed by a number of authors and most notably by Cresswell in Chapter 3 of [4].

This sort of objection would have more force if it were emphasized that classical negation (definable in terms of two-valued truth matrices) is not in any sense privileged; i.e., the objection is not that $\neg$, &, $\vee$, $\supset$, and $\equiv$ do not represent 'real' negation, etc., for impossible worlds, but that we are equivocating with respect to the connectives $\neg$, &, $\vee$, $\supset$, and $\equiv$. They mean one thing for impossible worlds and something else for normal worlds.

And this equivocation is not benign for the reason that an impossible worlds semantics is supposed to show, for example, how agents can fail to classically conjoin believed contents which obtain at nonstandard alternatives. But if $\alpha \,\&\, \beta$ is false at some impossible alternative to a world even though the 'conjuncts' α and β are true, then '&' in $\alpha \,\&\, \beta$ is not classical conjunction. So it has not been shown how some instance of T1, $(\boldsymbol{B}\alpha \,\&\, \boldsymbol{B}\beta) \supset \boldsymbol{B}(\alpha \,\&\, \beta)$, is invalid if '&' in the content $\alpha \,\&\, \beta$ is classical conjunction.

It could be countered that even though there is admittedly an equivocation in an impossible worlds *semantics* with respect to the connectives $\neg$, &, $\vee$, $\supset$, and $\equiv$, there is no corresponding equivocation in the *syntax*, since even for a system with a restricted rule of doxastic necessitation, all the 'laws' of the classical propositional calculus remain theses. Then assuming that the connectives $\neg$, &, $\vee$, $\supset$, and $\equiv$ are definable purely syntactically (in terms of certain classical theses), the equivocation with respect to them in the impossible worlds semantics is innocuous. (This response is discussed by Rescher and Brandom in [17].)

The problem with this rejoinder is that it is wrongly assumed that the only theses that matter in terms of characterizing the so-called extensional connectives are nondoxastic theses. For example, it could be argued that the adjunction schema, $(\boldsymbol{B}\alpha \mathbin{\&} \boldsymbol{B}\beta) \supset \boldsymbol{B}(\alpha \mathbin{\&} \beta)$, is the principle that doxastic necessity factors out of *classical* conjunction. Then systems for which some or all instances of the adjunction schema fail will involve an equivocation with respect to '&' since '&' behaves standardly for classical theses of PC but nonstandardly with respect to certain modal theses.

3 Believing in possibility Rantala's semantics for his restricted doxastic systems which makes use of impossible worlds does not invalidate any instance of T1, T2, and DR1 if the extensional connectives are interpreted *classically.* In the absence of a characteristic semantics for these systems which is not beside the point, the long-standing tradition of construing the necessity operator for alethic modal systems as 'x believes that' is best seen as a degenerating research program—at least if we are interested in developing a logic for believers who do not always conjoin their beliefs, whose beliefs are not always consistent, and who sometimes fail to believe the consequences of what they believe. It is time to consider another approach.

One way of construing Kripke's puzzling Pierre case is that Pierre in one context believes that London is pretty and in another context he believes that London is not pretty; i.e., Pierre holds contradictory beliefs in different contexts (though Kripke himself is not willing to accept this construal). Also, supposing that agents do not believe self-contradictory propositions, then this can also be construed as a case where the agent Pierre fails to conjoin his beliefs. Barcan Marcus argues that one moral to be drawn from the puzzling Pierre case so construed is ". . . that belief, like possibility, does not always factor out of a conjunction" (see [12], p. 507). And in fact, for normal alethic modal systems where '$\boldsymbol{M}$' is the possibility operator, $(\boldsymbol{M}\alpha \mathbin{\&} \boldsymbol{M}\beta) \supset \boldsymbol{M}(\alpha \mathbin{\&} \beta)$ is not a thesis-schema.

Another moral to be drawn from the Kripke puzzle is that belief is like possibility in the sense that for normal alethic systems, $\neg(\boldsymbol{M}\alpha \mathbin{\&} \boldsymbol{M}\neg\alpha)$ is not a thesis-schema. What all this seems to point to is that if we are going to adopt normal alethic systems as doxastic logics then we should construe the *possibility* operator and not the necessity operator as 'x believes that'.

However, if we were to construe the *possibility* operator as 'x believes that', we are still left with the consequence that agents are logically omnidoxastic since the following is an inference rule for normal alethic systems:

DR2: $\vdash \alpha \supset \beta \rightarrow \vdash \boldsymbol{M}\alpha \supset \boldsymbol{M}\beta$.

But given that agents need not conjoin their beliefs in a logic where the possibility operator is construed as 'x believes that', then even though x believes that α and x believes that β, it does not follow that x believes that Q such that $\vdash(\alpha \,\&\, \beta) \supset Q$. But if we were to construe *necessity* as belief then x would believe that Q—so in a sense, although agents are omnidoxastic if we construe the possibility operator as belief, they believe 'less'. In short, the failure to conjoin beliefs mitigates the omnidoxasticity feature.

The *necessity* operator for normal systems adopted as doxastic logics could be construed as '*x ideally* believes that' ($\boldsymbol{B_I}$). The 'ideal' believer always conjoins beliefs and has only consistent beliefs.

We already have a characteristic semantics for normal doxastic systems where possibility is construed as belief and necessity is construed as ideal belief, viz., the semantics described above where a model is a triple $\langle \boldsymbol{W}, R, V\rangle$. However, we shall require that our semantics is not only technically adequate, but that it makes some sort of intuitive sense out of the claim that belief is analogous to possibility. Our motivation for this semantics is Stalnaker's notion of 'belief state' which he discusses in [20].

A belief state s_i is the set of possible situations such that all the contents of some of an agent's beliefs obtain at each situation in the set. As a way of handling examples (such as the puzzling Pierre case) where agents hold inconsistent beliefs in different contexts and where agents fail to conjoin beliefs, Stalnaker suggests that an agent can be in more than one belief state at the same time (see [20], pp. 82–84). Thus, x believes that α at w_i iff for at least one belief state, α obtains at every member of the state. So, puzzling Pierre can believe that London is pretty and he can also believe that London is not pretty if he is in at least two distinct belief states such that the former content obtains at all members of one state and the latter content obtains at all members of a distinct state. Also, Pierre does not conjoin these beliefs since in neither state is $\alpha \,\&\, \neg\alpha$ true at any member.

A relational semantics for normal systems motivated by the proposal that agents can be in more than one belief state would involve defining a model as a 4-tuple $\langle \boldsymbol{W}, f, \boldsymbol{S}, V\rangle$ such that $\boldsymbol{W}$ is a nonempty set of worlds and f is a function that takes members of $\boldsymbol{W}$ into the power set of $\boldsymbol{W}$, $\wp \boldsymbol{W}$. For each $w_i \in \boldsymbol{W}$, $f(w_i) \subseteq \boldsymbol{W}$. The function f determines for each world a set of doxastic alternatives and so f is equivalent to the relation R. $f(w_i) = \{w_j \in \boldsymbol{W} \mid w_i R w_j\}$ (see Chellas [2]). The special twist to this semantics is the set $\boldsymbol{S}$ whose members are sets of belief states—which are themselves sets of worlds. Each member of $\boldsymbol{S}$ is associated with exactly one world via the 'alternativeness' function f as follows: $S_j \in \boldsymbol{S}$, where $\boldsymbol{S} = \wp f(w_j)$, for exactly one member of $\boldsymbol{W}$, w_j. Every S_j in $\boldsymbol{S}$ is a set of belief states whose members are drawn from the doxastic alternatives of exactly one member of $\boldsymbol{W}$, w_j.

V is an assignment function and $V_{\mathcal{M}}$ is defined inductively with the following truth conditions for ideal and nonideal belief:

$$V_{\mathcal{M}}(\boldsymbol{B_I}\alpha, w_i) = 1 \text{ iff for all } w_j \in \boldsymbol{W} \text{ such that } w_j \in f(w_i),\ V_{\mathcal{M}}(\alpha, w_j) = 1.$$

$$V_{\mathcal{M}}(\boldsymbol{B}\alpha, w_i) = 1 \text{ iff for at least one nonempty } s_{ik} \in S_i \text{ where } s_{ik} \subseteq f(w_i),$$
$$V_{\mathcal{M}}(\alpha, w_j) = 1 \text{ for all } w_j \in s_{ik}.$$

$\mathcal{M}$-validity is truth at all worlds. The truth conditions for 'ideal' belief are simply those for belief in the old semantics. The idea here is that the ideal believer has integrated his or her belief states into one system which is consistent and closed under deduction. For nonideal belief, the idea is that the content α will be true at all members of at least one nonempty belief state—and the nonideal believer will be in more than one belief state.

Finally, soundness is easily verified by showing that the axioms are valid and the rules of inference preserve validity. Completeness is guaranteed by the fact that the equivalences $\boldsymbol{B}\alpha \equiv \neg\boldsymbol{B_I}\neg\alpha$ and $\boldsymbol{B_I}\alpha \equiv \neg\boldsymbol{B}\neg\alpha$ are both valid. Then, from a technical point of view, the semantics herein proposed is adequate.

However, if we construe the possibility operator as belief for normal systems, the resulting logics will contain the following as a thesis-schema:

DS: $\boldsymbol{B}(\alpha \vee \beta) \supset (\boldsymbol{B}\alpha \vee \boldsymbol{B}\beta)$ Disjunction Schema.[5]

DS is implausible because an agent can believe that a disjunction obtains without thereby believing that each disjunct obtains. Further, for normal systems containing D, $\boldsymbol{B_I}\alpha \supset \boldsymbol{B}\alpha$, the following is a thesis-schema for such systems:

BC: $\boldsymbol{B}\alpha \vee \boldsymbol{B}\neg\alpha$ Belief Completeness Schema.

The schema BC is implausible since it says that for every proposition α, any agent will either believe that it or its negation obtains.

Thus we have avoided problems associated with conjunctive belief at the cost of incurring new problems associated with disjunctive belief in construing possibility as 'x believes that'. Then given that omnidoxasticity is retained, even if mitigated, for systems where possibility is construed as belief, it would seem that the only reason for favoring systems where belief is regarded as possibility over systems where belief is regarded as necessity is that the former are inconsistency-tolerant with respect to agents' beliefs.

4 Total belief in partial descriptions And so, it would seem that the possibility operator does not fare much better than the necessity operator vis à vis characterizing the 'nonideal' believer. This is not to say that normal modal logics where either necessity or possibility are construed as 'x believes that' cannot be lucrative in terms of solving certain puzzles about belief. However, these solutions to various puzzles about belief presuppose that agents believe all the consequences of what they believe. It is therefore arguable whether these logics shed much light on puzzles about belief for agents who are not 'ideal'.[6]

The intuitive idea behind the possible worlds *semantics* for normal doxastic logics is that belief is a relation between an agent at a world and a set of alternative possible worlds. Any given belief content obtains at each of these alternatives if necessity is construed as belief, or at some of them if possibility is construed as belief. Although the Kripkean semantics is mute concerning the metaphysics of these 'worlds', it is clear that minimally they can be described as *complete* in the sense that for any given proposition α and for any given world w in a normal model $\mathcal{M}$, either α is true at w or it is false. This is because the valuation function, which evaluates formulas relative to worlds, is not partial. In addition, since the valuation function is defined classically, each world is consistent and closed under implication.

Then on the possible worlds account of belief, it follows that whenever an agent believes that α and α (classically) logically implies β—in which case, both $\alpha \supset \beta$ and α are true at each doxastically accessible world—the agent in question cannot fail to believe that β. This is because β will also be true at any accessible world since each world is both complete in the sense specified above and is closed under implication. This is what we have called the logical omnidoxasticity feature of normal modal logics and their semantics. Construing possibility rather than necessity as belief does not rid the semantics of this feature and, as was noted, allowing alternatives to be logically impossible does not work either.

However, if we wish to preserve the intuition that belief is a relation between agents and alternatives, there are alternative conceptions of alternatives which do not give rise to semantics with the omnidoxasticity feature. The current trend in the literature seems to be in the direction of talk about *partial* worlds or 'situations'. In short, the completeness feature of possible worlds discussed above is abandoned. Ironically, Kripke, who is one of the originators of possible worlds semantics for modal logics, seems to have been one of the first to consider this approach in Kripke [11].

Kripke's maxim is that worlds are not discovered, they are stipulated. Further, these stipulated worlds—or more precisely, these stipulated alternative 'descriptions'—are not complete in the sense of giving a Yes/No answer as to whether any given proposition obtains (see [11], Lecture I).[7] For example, if I consider what Nixon would have done if he had been a gardener, it is not necessary to imagine a 'complete' possible world in the sense that it provides a Yes/No answer concerning each fact. Rather, I merely stipulate that the very same individual (in the sense of being born to such-and-such parents) who was *actually* President of the U.S. *might have been* something else, such as a gardener. It may then be asked what, in this situation so stipulated, Nixon would have done. Presumably, an answer to this question does not require a complete specification of all 'irrelevant' details—although admittedly, what counts as relevant here is open to debate. This also suggests a new semantics for counterfactuals, one that does not require considering the 'closest' possible *world* where the antecedent obtains in order to evaluate the truth of the conditional in question, but rather 'close' stipulated partial descriptions.

Kripke's informal work on counterfactuals in [11] could easily be extended to propositional attitudes. If possible worlds are stipulated alternative partial *descriptions* rather than concrete particulars, then we shall say that a Kripkean possible world or partial description is simply a set of *propositions*, some of which may obtain at the 'actual world' and some of which may not. Then suppose that an agent stipulates a descriptive alternative to the world the agent inhabits. If α is a member of this descriptive alternative, then the agent believes that α. It can now be shown that this informal semantic account of belief avoids the problem of deductive closure of beliefs, given that alternative descriptions are partial.

First of all, consider the problem of logical omnidoxasticity. Suppose that α is a member of an agent's stipulated descriptive alternative. Further, suppose that α logically implies some proposition β. Since this alternative is partial, it does not follow that $\alpha \supset \beta$ is a member of it and so it does not follow that β is a member of this alternative—even if alternatives are closed with respect to implication. Then on this model of belief, agents do not always believe the consequences of what they believe.

Perry suggests a solution to the logical omnidoxasticity problem along similar lines in Perry [13]. According to Perry, propositions are not sets of worlds but rather sets of both worlds (in the standard sense) and 'partial ways'. A partial way does not give a Yes/No answer to every question concerning whether or not some proposition obtains. Thus the proposition that is expressed by '$2 + 2 = 4$' is not necessarily identical to the proposition expressed by '$367 + 345 = 712$' since there may be partial ways that give a Yes/No answer to whether the former obtains without giving any answer as to whether the latter obtains. So an agent may believe that $2 + 2 = 4$ without thereby believing that $367 + 345 = 712$. However, our Kripkean partial descriptions are not Perry's partial ways nor are they the situations of Barwise and Perry's situation semantics (see Barwise and Perry [1]). Propositions have truth values at situations, whereas at our Kripkean 'worlds', propositions are not assigned truth values. They simply are or are not members of a given partial description. This feature of partial descriptions becomes important later when we attempt to develop a formal semantics based on this account of belief. This formal semantics avoids the logical omniscience problem without thereby equivocating with respect to the logical connectives.

Next, with respect to the problem of conjunctive closure, suppose that both α and β obtain at the alternative partial description which an agent stipulates. Then the agent believes that α and that β. However, the agent may not believe that $\alpha \mathbin{\&} \beta$ if $\alpha \mathbin{\&} \beta$ fails to be a member of this alternative. This situation is possible since a stipulated alternative may not give a Yes/No answer concerning the status of every proposition and its negation. So even though both α and β are members of the agent's partial description, it may be 'silent' with respect to the status of $\alpha \mathbin{\&} \beta$. So on this model of belief, agents do not always believe the conjunction of what they believe.

Finally, there is no reason to suppose that agents' alternatives in the sense of partial descriptions will be consistent. Then there is nothing to exclude the possibility that an agent's alternative is 'weakly' inconsistent in the sense that for at least one proposition α, both it and its negation are members of this partial description—in which case, the agent both believes that α obtains and that its negation obtains. (This notion of a world's being 'weakly inconsistent' is discussed in [17].) However, it does not follow that the agent thereby believes that $\alpha \mathbin{\&} \neg\alpha$ obtains, given that these alternatives are partial. In fact, in the formal semantics to be developed below, we shall require that partial descriptions are strongly consistent in the sense that they cannot contain any proposition of the form $\alpha \mathbin{\&} \neg\alpha$.

A simple logic of belief will now be proposed which does not treat belief as either the possibility or the necessity operator of alethic normal systems. This logic does not presuppose that agents' beliefs are consistent or deductively closed. Further, quantificational extensions of this logic are possible, although complexities can come later. The corresponding semantics will be based on our account of belief as a relation between an agent and a partial description.

5 The axiom system BEL The language $\mathcal{L}$ for the axiom system BEL corresponding to the semantics to be described consists of a denumerable set $\boldsymbol{C}$ of constants $\{t_1, t_2, \ldots\}$, a denumerable set $\boldsymbol{P}$ of predicate variables $\{F, G, H, \ldots, F_1, G_1, H_1, \ldots\}$, a set of truth-functional connectives $\neg$, &, $\vee$, $\supset$, and $\equiv$, and finally a belief operator, $\boldsymbol{B}$. The set $\boldsymbol{F}$ of well-formed formulas is defined recursively in the usual manner along with the clause that if α is a wff, so is $\boldsymbol{Bt_i}\alpha$. $\boldsymbol{Bt_i}\alpha$ can be read as 't_i believes that α'. The axiom-schemata for our system BEL are as follows:

AS1: α where α is any PC thesis.
AS2: $(\boldsymbol{Bt_i}\alpha \,\&\, \boldsymbol{Bt_i}(\alpha \supset \beta)) \supset \boldsymbol{Bt_i}\beta$
AS3: $\boldsymbol{Bt_i}(\alpha \,\&\, \beta) \supset (\boldsymbol{Bt_i}\alpha \,\&\, \boldsymbol{Bt_i}\beta)$
AS4: $\neg\boldsymbol{Bt_i}(\alpha \,\&\, \neg\alpha)$.

The rule of inference for BEL is:

R1: Modus Ponens.

What is noteworthy about the axiom-schemata is that there is no analogue of the necessitation rule for the belief operator. And, given that there is no necessitation rule for the belief operator, the logical omnidoxasticity rule for the belief operator is not derivable. For the same reason, the adjunction schema is not derivable and even if we were to add D, $\boldsymbol{Bt_i}\alpha \supset \neg\boldsymbol{Bt_i}\neg\alpha$, the consistency-schema would also not be derivable. However, our logic is nontrivial with respect to the belief operator since it is an axiom-schema that conjoined beliefs are believed separately (AS3), that beliefs are closed under implication, albeit not under logical entailment (AS2)[8] and that agents cannot believe self-contradictory propositions (AS4).

It could be objected at this point that if the purpose of our logical system BEL is to embody principles of belief attribution for nonideal believers, then certain 'key' schemata that express some of these principles have been omitted from the axiom set. Thus, for example, we may want to include as axiom-schemata one or more of the following:

S5: $\boldsymbol{Bt_i}(\alpha \,\&\, \beta) \supset \boldsymbol{Bt_i}\alpha$
S6: $\boldsymbol{Bt_i}(\alpha \,\&\, \beta) \supset \boldsymbol{Bt_i}(\beta \,\&\, \alpha)$
S7: $\boldsymbol{Bt_i}(\alpha \vee \beta) \supset \boldsymbol{Bt_i}(\beta \vee \alpha)$.

The schema S5 expresses the principle that conjunctive beliefs simplify and the schemata S6 and S7 express the principles that conjunctive and disjunctive beliefs commute.

In response to this objection, I am not claiming that AS2–AS4 are the only possible or even the best axiom-schemata for a system of doxastic logic that does not treat agents as 'hyper-rational' creatures. The point of the exercise is to show that a doxastic logic for nonideal agents is possible. We can therefore regard the system BEL as tentative. Any or all of the axiom-schemata AS2–AS4 could be replaced or supplemented with other schemata such as S5–S7 (along with appropriate alterations to strictures on partial descriptions in the semantics) that turn out to express plausible principles of belief attribution for nonideal agents.

As Cherniak argues in [3], a theory of belief attribution which treats agents as 'minimally rational' and hence as nonideal will rely on a theory of feasible inferences.[9] Cherniak characterizes minimally rational agents as those who only make *some* inferences from their beliefs that are deductively appropriate (such as deducing the logical consequences of beliefs) and resolve only *some* inconsistencies in their beliefs. Such a theory would tell us which sorts of inferences from existing beliefs a 'typically' minimally rational agent would be most likely to accomplish. Further, this theory would rank inferences in terms of their ease of accomplishment so that, for example, modus ponens would be 'easier' to accomplish than a complex inference of first order logic. Thus the principles of belief attribution we regard as plausible will be a function of what sorts of inferences are feasible for minimally rational and hence nonideal agents. However, if Cherniak is right, what these feasible inferences turn out to be is an empirical rather than an a priori matter (see Chapter 2 in [3]). Thus, until it is determined which sorts of inferences are feasible, any attempt at a plausible doxastic logic for nonideal agents will have to remain tentative.

It could also be objected that a doxastic logic embodying principles of belief attribution should include schemata which state that agents believe the consequences of what they believe and that their beliefs are consistent. If we did not assume these things about agents, belief attribution and hence the prediction of behavior would be impossible. However, this objection misses the mark entirely since agents are in fact not ideal. Therefore, if we were to assume that agents are maximally rational when in fact they are not, then belief attribution and prediction of behavior for such agents would generally be mistaken, a point which Cherniak emphasizes in Chapter 1 of [3].

By way of illustration of this last point, in the case of logical or mathematical truths, most agents will not know that one such truth entails or is equivalent to another. For example, suppose we attribute to Jones the belief that $\alpha \vee \neg\alpha$. Unless Jones is a brilliant classical logician, we would likely be mistaken in attributing to Jones the further belief that $(\alpha \supset \beta) \supset ((\gamma \supset \delta) \supset ((\alpha \,\&\, \gamma) \supset (\beta \,\&\, \delta)))$, even though the two contents are classically logically equivalent. Thus it would be a mistake to include in a doxastic logic an inference rule such as DR1 to the effect that agents believe the consequences of what they believe.

Also, in the case of conjunctive belief, consider the lottery paradox discussed by Stalnaker in [20]. Suppose that we attribute to Jones the belief that each ticket will probably lose. If we assumed as a principle of belief attribution that agents believe the conjunction of what they believe, then we would be forced to attribute to Jones the further belief that all tickets will lose. But this attribution is most likely mistaken. Thus we should not include in a doxastic logic a schema such as T1 which states that agents believe the conjunction of what they believe.

Finally, vis à vis doxastic consistency, it would seem that nonideal agents can believe that p and also believe that not-p at the same time. The Paderewski example devised by Kripke in [10] illustrates this type of situation. Peter believes that no politicians are musicians. Suppose Peter believes that Paderewski was a politician and he also believes that someone else who was also named Paderewski had musical talent. Then it would seem fair in this case to attribute to Peter the further belief that Paderewski (the politician) had no musical talent. But it happens to be the case that Paderewski the politician and Paderewski the musician were one and the same person. Then we have attributed to Peter inconsistent beliefs. Barcan Marcus in [12] seems to advocate a similar construal of the Kripke puzzle about belief. Admittedly, Kripke in this case would say that it is unfair to attribute to Peter inconsistent beliefs since even if Peter is a brilliant logician, he could not possibly detect the inconsistency in the contents of his alleged beliefs without appeal to additional relevant information.

However, the assumption which Kripke employs in his argument against construing the Paderewski case in the way we did, viz., that *an agent can be charged with inconsistencies in his or her beliefs only if the agent is in a position to detect these inconsistencies without appeal to additional information*, is faulty. It could be countered that it is in just those sorts of cases where a minimally rational agent cannot detect inconsistencies without appeal to additional information that we would be *most* inclined to attribute to the agent inconsistent beliefs; i.e., minimally rational agents would not hold inconsistent beliefs *unless* the inconsistency were undetectable by logic alone. And so, if I am right here, the Kripke puzzles illustrate that it is sometimes plausible to attribute to agents inconsistent beliefs. Thus it is a mistake to include a consistency schema such as T2 in a doxastic logic, if it is intended that such a logic embodies principles of belief attribution.

With this defense of our logical system BEL, I shall now present a semantics for BEL which appeals to the idea that belief is a relation between agents and partial descriptions.

A model for the semantics for BEL is a 4-tuple, $\langle \boldsymbol{D}, \boldsymbol{S}, g, V\rangle$. $\boldsymbol{D}$ is a nonempty set of 'individuals', and $\boldsymbol{S}$ is a set of sets of propositions or 'partial descriptions'. Further, to guarantee the soundness of BEL relative to this semantics, the following restrictions apply to all members of $\boldsymbol{S}$. Where s_i is any member of $\boldsymbol{S}$ and α, β are any wffs of the language $\mathcal{L}$ for BEL:

(1) If $\alpha \in s_i$ and $\alpha \supset \beta \in s_i$ then $\beta \in s_i$.
(2) If $\alpha \,\&\, \beta \in s_i$ then $\alpha \in s_i$ and $\beta \in s_i$.
(3) For no s_i is it the case that $\alpha \,\&\, \neg\alpha \in s_i$.

As will be shown, (1) validates AS3, (2) validates AS4, and (3) validates AS5. Also, g is a function that assigns to each member of $\boldsymbol{D}$ exactly one member of $\boldsymbol{S}$; i.e., for any $d_i \in \boldsymbol{D}$, $g(d_i) \in \boldsymbol{S}$. Thus, intuitively, the function g assigns to an agent a partial description which is a set of propositions. The idea here is that what the agent believes (propositionally) is determined by what is contained in the agent's associated partial description. To avoid the type of situation where a partial description assigned to an agent contains only one proposition, so that the agent has only one belief,[10] we shall stipulate that each member of $\boldsymbol{S}$ must be at least of some requisite cardinality c.

Finally, the assignment function V can be defined as follows:

(i) For any constant t_i in $\boldsymbol{C}$, $V(t_i) \in \boldsymbol{D}$.
(ii) For any n-ary predicate variable F in $\boldsymbol{P}$, $V(F) \subseteq \boldsymbol{D}^n$.

That is, V assigns members of $\boldsymbol{D}$ to constants and V assigns to any n-ary predicate a set of n-tuples of members of $\boldsymbol{D}$.

Further, $V_{\mathcal{M}}$ is a valuation over a model $\mathcal{M}$ such that:

(1) a. $V_{\mathcal{M}}(Ft_1 \ldots t_n) = 1$ iff $\langle V(t_1), \ldots, V(t_n)\rangle \in V(F)$.
b. $V_{\mathcal{M}}(\boldsymbol{B}t_i\alpha) = 1$ iff $\alpha \in g(V(t_i))$.

Suppose that $V_{\mathcal{M}}(\alpha)$ and $V_{\mathcal{M}}(\beta)$ are already defined. Then,

(2) $V_{\mathcal{M}}(\neg\alpha) = 1$ iff $V_{\mathcal{M}}(\alpha) = 0$.
(3) $V_{\mathcal{M}}(\alpha \mathbin{\&} \beta) = 1$ iff $V_{\mathcal{M}}(\alpha) = V_{\mathcal{M}}(\beta) = 1$.
(4) $V_{\mathcal{M}}(\alpha \supset \beta) = 1$ iff either $V_{\mathcal{M}}(\alpha) = 0$ or $V_{\mathcal{M}}(\beta) = 1$.

BEL validity is then defined as truth in all BEL models.

The base clause in the inductive definition of the valuation function $V_{\mathcal{M}}$ includes the case where α is of the form $\boldsymbol{B}t_i\beta$. This is so because the truth of a belief wff does not depend on what $V_{\mathcal{M}}$ assigns to its content at a given partial description, but merely on whether or not its content is a member of the partial description that is assigned to $V(t_i)$ by the function g. The valuation function $V_{\mathcal{M}}$ does not assign truth values to wffs at partial descriptions in $\boldsymbol{S}$. The reason for this feature of the semantics is that we can invalidate the omnidoxasticity rule as well as both the adjunction and consistency schemata while avoiding the major downfall with the impossible worlds semantics for belief logic, viz., equivocation with respect to the extensional connectives of the logic.

For example, consider the following instance of the omnidoxasticity rule, $\vdash Ft_2 \supset (Ft_2 \vee Gt_2) \rightarrow \vdash \boldsymbol{B}t_1Ft_2 \supset \boldsymbol{B}t_1(Ft_2 \vee Gt_2)$. The wff $Ft_2 \supset (Ft_2 \vee Gt_2)$ is a thesis of BEL, and it is also valid relative to the semantics now being proposed. But $\boldsymbol{B}t_1Ft_2 \supset \boldsymbol{B}t_1(Ft_2 \vee Gt_2)$ is not valid relative to this semantics, as will now be shown. Consider the model where $\boldsymbol{D} = \{d_1, d_2\}$ and $\boldsymbol{S} = s_1$. Further, suppose that $s_1 = \{Ft_2\}$. Also, $g(d_1) = s_1$, $V(t_1) = d_1$, $V(t_2) = d_2$, and $V(F) = d_2$. Regardless of what V is, $V_{\mathcal{M}}(Ft_2 \supset (Ft_2 \vee Gt_2)) = 1$. However, although $Ft_2 \in s_1$, neither $Ft_2 \supset (Ft_2 \vee Gt_2)$ nor $Ft_2 \vee Gt_2$ are in s_1 – given that s_1 is partial. Thus, $V_{\mathcal{M}}(\boldsymbol{B}t_1Ft_2) = 1$, whereas $V_{\mathcal{M}}(\boldsymbol{B}(Ft_2 \vee Gt_2)) = 0$.

Unlike the impossible worlds semantics discussed above, the formula $\boldsymbol{B}t_1Ft_2 \supset \boldsymbol{B}t_1(Ft_2 \vee Gt_2)$ is not invalidated by allowing accessible alternatives where the content Ft_2 is true and yet where $Ft_2 \vee Gt_2$ is false, thereby involving an equivocation with respect to '$\vee$'. Rather, it is invalidated by allowing there to be partial alternatives which fail to include certain propositions as members. But wffs are neither true nor false at such alternatives, and so the truth conditions for the classical connectives have not been altered if a classical thesis fails to be a member of a partial description.

The *soundness* of BEL relative to this semantics is guaranteed by the strictures imposed on the members of $\boldsymbol{S}$. These strictures ensure the validity of all instances of the axiom-schemata AS3–AS5 pertaining to the belief operator, '$\boldsymbol{B}$'. To illustrate this, consider first AS3, $(\boldsymbol{B}t_i\alpha \mathbin{\&} \boldsymbol{B}t_i(\alpha \supset \beta)) \supset \boldsymbol{B}t_i\beta$. Suppose that some instance of this schema is invalid; i.e., suppose that for some instance of this schema and for some BEL model, $\mathcal{M}$, that instance is false. Thus, $V_{\mathcal{M}}(\boldsymbol{B}t_i\alpha) = V_{\mathcal{M}}\boldsymbol{B}t_i(\alpha \supset \beta) = 1$ but $V_{\mathcal{M}}(\boldsymbol{B}t_i\beta) = 0$. Then for $g(V(t_i))$, $\alpha \in g(V(t_i))$ and $\alpha \supset \beta \in g(V(t_i))$ but $\beta \notin g(V(t_i))$. But this is impossible in light of stricture (1) on members of $\boldsymbol{S}$, viz., if $\alpha \in s_i$ and $\alpha \supset \beta \in s_i$ then $\beta \in s_i$. Similar considerations show that AS4, $\boldsymbol{B}t_i(\alpha \mathbin{\&} \beta) \supset (\boldsymbol{B}t_i\alpha \mathbin{\&} \boldsymbol{B}t_i\beta)$ is validated given stricture (2) on members of $\boldsymbol{S}$, viz., if $\alpha \mathbin{\&} \beta \in s_i$ then $\alpha \in s_i$ and $\beta \in s_i$. Finally, AS5, $\neg\boldsymbol{B}t_i(\alpha \mathbin{\&} \neg\alpha)$ is validated in light of stricture (3) for members of $\boldsymbol{S}$, viz., for no s_i is it the case that $\alpha \mathbin{\&} \neg\alpha \in s_i$.

The *completeness* of BEL with respect to the semantics herein proposed can be proven by the Henkin method. First of all, we shall say that a wff α is BEL-consistent iff $\neg\alpha$ is not a thesis of BEL. A set of wffs $\{\alpha_1,\dots,\alpha_n\}$ is BEL-consistent iff $\neg(\alpha_1 \& \dots \& \alpha_n)$ is not a thesis of BEL. Supposing that the wffs of BEL can be ordered, then we can prove Lindenbaum's lemma, viz., that every consistent set of wffs has a maximal consistent extension. The proof of Lindenbaum's lemma is standard. Suppose further that for some arbitrary consistent wff, α, we construct its maximal consistent extension, Γ. We then construct the following BEL model, $\mathcal{M} = \langle \boldsymbol{D}, \boldsymbol{S}, g, V\rangle$ such that:

(a) $\boldsymbol{D}$ is a set of constants.
(b) $\boldsymbol{S}$ is a set of sets of wffs where strictures (1)–(3) mentioned above apply.
(c) For any $t_i \in \boldsymbol{D}$, $g(t_i) = \{\alpha \mid \boldsymbol{B}t_i\alpha \in \Gamma\}$, where $g(t_i) \in \boldsymbol{S}$.
(d) (i) For any constant t_i and supposing that the constants of $\mathcal{L}$ can be ordered:

$$V(t_i) = t_i.$$

(ii) For any n-ary predicate variable F, for any constants $t_1,\dots,t_n$:

$$\langle t_1,\dots,t_n\rangle \in V(F) \text{ iff } Ft_1\dots t_n \in \Gamma.$$

What must now be proven is the fundamental theorem of Henkin models, viz., that for the valuation over the Henkin model, $V_{\mathcal{M}}$:

$$V_{\mathcal{M}}(\alpha) = 1 \text{ iff } \alpha \in \Gamma.$$

The fundamental theorem is proven by mathematical induction.

Basis: (i) Suppose that α is of the form $Ft_1\dots t_n$:

$$\begin{aligned} V_{\mathcal{M}}(Ft_1\dots t_n) = 1 &\text{ iff } \langle V(t_1),\dots,V(t_n)\rangle \in V(F)\\ &\text{ iff } \langle t_1,\dots,t_n\rangle \in V(F)\\ &\text{ iff } Ft_1\dots t_n \in \Gamma.\end{aligned}$$

(ii) Suppose that α is of the form $\boldsymbol{B}t_i\beta$:

$$V_{\mathcal{M}}(\boldsymbol{B}t_i\beta) = 1 \text{ iff } \beta \in g(t_i) \text{ iff } \boldsymbol{B}t_i\beta \in \Gamma \quad (\text{def. } `g(t_i)\text{'}).$$

The *inductive hypothesis* is that the fundamental theorem holds for all wffs which have 'k' or fewer occurrences of connectives/operators and which are not of the form $\boldsymbol{B}t_1\beta$. It is then shown that the theorem in question holds for all wffs having '$k + 1$' occurrences of connectives/operators that are not of the form $\boldsymbol{B}t_i\beta$. The proof of the inductive step is standard. It therefore follows that for any non-theorem α of BEL, $\mathcal{M}$ serves as a countermodel since $\neg\alpha$ is BEL-consistent and $\neg\alpha \in \Gamma$ iff $V_{\mathcal{M}}(\neg\alpha) = 1$ iff $V_{\mathcal{M}}(\alpha) = 0$. Thus any valid wff is a theorem of BEL.

NOTES

1. As a kind of rough definition of 'normal', any normal modal system contains all PC tautologies, any wff of the form $(L\alpha \;\&\; L(\alpha \supset \beta)) \supset L\beta$, viz., K and has as rules of inference material detachment and $\vdash\alpha \rightarrow \vdash L\alpha$.

2. Impossible worlds are distinguished from possible worlds in that for the former, the connectives are not defined inductively.

3. Excluding some instance of $\alpha \supset (\neg\alpha \supset \neg(\alpha \,\&\, \neg\alpha))$ from Ω will also result in rendering the appropriate instances of T1 and DR1 underivable.

4. This difficulty with Rantala's semantics was made apparent to me in a number of discussions I had with Professor Storrs McCall.

5. This was pointed out to me by Professor John Kearns who commented on an earlier draft of this paper which I read at the Eastern APA meetings in Boston, 1990.

6. For example, Seager [18] has recently proposed a logic and characteristic semantics that aims to make sense of certain puzzles concerning 'self-locating' belief. His system LB presupposes that agents are logically omnidoxastic since it contains the doxastic necessitation rule.

7. In particular, see pp. 44–45. Kripke says that "in theory, everything needs to be decided to make a total description of the world. We can't really imagine that except in part; that, then, is a 'possible world'."

8. Granted, Dretske [6] has cast some doubt on AS3, at least in the case of epistemic logics. However, even if we dispense with AS3, we would still have a nontrivial logic of belief.

9. I decided to read Cherniak after a discussion with my colleague Professor Murray Clarke on the problem of rationality.

10. My thanks to the referee for this article who brought this difficulty to my attention.

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