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On the Question 'Do We Need Identity?'

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Abstract Sommers posed the question 'Do We Need Identity?' and answered in the negative. According to Sommers, the need for a special identity relation resulted from an arbitrary distinction between concept and object introduced by Frege and retained in modern predicate logic (MPL). This is reflected in the syntactic distinction between predicate and individual constant. Traditional formal logic (TFL) does not respect this distinction and, as a consequence, has no need for a special identity relation. But Sommers's position has not gained wide acceptance. While it is conceded that TFL can express the identity of *individual constants*, it is quickly pointed out that this falls far short of providing the expressiveness of the logical identity relation. But the precise extent of the deficit in expressiveness has not been determined. It appears that Sommers's position on identity has not been adequately formalized to permit such a determination. This paper formalizes and extends Sommers's position on identity. This formalization is compared with MPL to define precisely the difference in expressive power. The formal language defined for this investigation is similar to the language of MPL. The similarity will not only facilitate comparison, but perhaps will also make this formal language more palatable to readers whose experience and/or predisposition favors MPL.

1 Introduction The question 'Do We Need Identity?' was raised by Sommers [4],[5]. He answered that a special identity relation is not needed in traditional formal logic (TFL), since predication and the laws governing it already allow identity to be expressed. But Frege injected a new, and arbitrary, distinction into modern predicate logic (MPL), which gave rise to the need for an identity relation.

The new distinction is between concept and object, reflected in the syntactic distinction between predicate and individual constant (or name). Its import is that a predicate can predicate, but an individual constant cannot. Consequently, two individual constants can be related only under a binary predicate. In particular, two individual constants can be declared identical only by a binary identity relation.

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TFL does not respect this distinction. In TFL an individual constant, denoting an object, can occupy the predicate position. For example, 'Hans is John' predicates the property (concept) of being John to Hans. But if 'John' is a predicate in 'Hans is John', consistency dictates that it is a predicate also in 'John is kind', and hence can be quantified. Thus 'some John is kind' must be well-formed, and must assert that the denotations of the predicates 'John' and 'kind' have a nonempty intersection. Since 'John' is singular (i.e., denotes a singleton set), this is tantamount to asserting that the unique element in the set denoted by 'John' is a member of the set denoted by 'kind'. Therefore, 'John is kind' can be viewed as an abbreviation for 'some John is kind'. Because of the singularity of the predicate 'John', 'some John is kind' is equivalent to 'all John is kind'. To indicate that 'John' is thus simultaneously universally and existentially quantified, Sommers writes '*John is kind'. This he calls 'wild quantity'. When the arbitrary distinction between object and concept is eliminated, Sommers argues, the need for a special identity relation disappears. Thus '*Hans is John' asserts that the denotations of the predicates 'Hans' and 'John' have a nonempty intersection (equivalently, the denotation of 'Hans' is a subset of the denotation of 'John'), that is, are *identical*. Sommers gives a demonstration that for individual constants a and b, the unary predication '$*a$ is b' in TFL has all the properties ascribed to the binary predication '$a = b$' in MPL.

Sommers's position has not gained wide acceptance. Neither has it received a full evaluation. This is at least in part due to inadequate formalization and/or failure to completely exploit the potential of Sommers's views. The objective of this paper, and the claim made for it, is a precise and complete explication of Sommers's position on identity.

Sommers's theory of identity is formalized and extended, and then compared with MPL to determine precisely the difference in expressive power. The formal language defined for this investigation (hereinafter 'PCS') differs from the language of MPL (hereinafter 'PCI') principally in that the distinction between predicate and individual constant is not present; in other respects they are similar. The similarity will not only facilitate comparison, but perhaps will also make PCS more palatable to readers whose experience and/or predisposition favors MPL.

In the following sections, the syntax and semantics of PCS are defined. Then the essential properties of singular expressions are established. To facilitate comparison, a conventional definition of PCI is provided. Translation from PCS to PCI demonstrates that PCS is equivalent to a subset of PCI. However, translation from PCI to PCS is only partial, suggesting a deficit in expressiveness of PCS relative to PCI. It is shown that there are wffs in PCI for which there are no semantically equivalent wffs in PCS.

It is further shown that these wffs are exactly those containing a subexpression of the form $x = y$ (it is required that *both* arguments of the identity relation are *variables*) and for which no equivalent wff exists without such a subexpression. Any wff in PCI not containing a subexpression of this form has a semantically equivalent wff in PCS. This deficit is a fundamental limitation of Sommers's approach, but a weaker one than generally attributed to it.

The use of *schemas* is considered as a way to overcome the deficit. This reveals another limitation of languages based on Sommers's approach. Those properties not expressible by wffs in PCS cannot in general be expressed by schemas either. The reason is that for a schema to be effective here, it is necessary that all the elements of the universe of interpretation be named. Certain well-known methods such as expansion of the language by adjunction of constants to name the elements of the universe can be employed. Then, given a structure and an axiomatization in PCI of the theory of that structure, the theory can be axiomatized with axiom schemas in an appropriate expansion of PCS. In this limited sense, it is possible to compensate for the deficit in expressiveness. But these methods are linked to the interpretation, and so do not constitute a general solution.

The treatment throughout is semantic; however, an axiomatic treatment can also be given (Purdy [3]).

2 Definition of PCS This section defines PCS, a first-order language that formalizes and extends Sommers's ideas regarding singular terms. PCS resembles PCI, the language of MPL, with the following difference. Singular predicates supplant individual constants and functions. It is not unusual to treat individual constants as nullary functions, nor to treat n-ary functions as $(n + 1)$-ary predicates. But it appears that these devices have not been used together. When they are, the result is a uniformity in the treatment of individual constants, functions, and predicates. While PCS does not have an identity relation, identity of singular expressions, which correspond to terms in PCI, can be expressed. Moreover, deduction with identicals can be performed conveniently in PCS.

2.1 Syntax The vocabulary of PCS is listed first. Let $\omega_+ := \omega - \{0\}$.

1. Predicate symbols $\mathcal{P}$ of two kinds
   (a) ordinary predicate symbols $\mathcal{R} = \bigcup_{n\in\omega_+}\mathcal{R}_n$, where $\mathcal{R}_n = \{R_i^n : i \in \omega\}$, and
   (b) singular predicate symbols $\mathcal{S} = \bigcup_{n\in\omega_+}\mathcal{S}_n$, where $\mathcal{S}_n = \{S_i^n : i \in \omega\}$
2. Variable symbols $\mathcal{V} = \{v_i : i \in \omega\}$
3. Boolean operators $\wedge$ and $\neg$
4. Quantifier $\exists$
5. Parentheses (and)
6. Comma , .

There are no terms in PCS. In their stead, singular expressions are used. These are defined as follows:

1. If $S^1 \in \mathcal{S}_1$ and $x \in \mathcal{V}$ then $S^1(x)$ is a singular expression.
2. If $S^{n+1} \in \mathcal{S}_{n+1}, x, x_1, \ldots, x_n \in \mathcal{V}$ are distinct and $S_1, \ldots, S_n$ are singular expressions, then $\exists x_1 (S_1(x_1) \wedge \cdots \wedge \exists x_n (S_n(x_n) \wedge S^{n+1}(x_1, \ldots, x_n, x)) \cdots)$ is a singular expression.
3. Nothing else is a singular expression.

Expressions in PCS are defined as follows:

1. If $P^n \in (\mathcal{R}_n \cup \mathcal{S}_n)$ and $x_1, \ldots, x_n \in \mathcal{V}$, then $P^n(x_1, \ldots, x_n)$ is an expression.
2. If φ is an expression then $\neg\varphi$ is an expression.

3. If φ, ψ are expressions then $(\varphi \wedge \psi)$ is an expression.
4. If φ is an expression and $x \in \mathcal{V}$ occurs free in φ, then $\exists x\varphi$ is an expression.
5. Nothing else is an expression.

Free and bound variables are defined in the usual way. When a list of variable symbols follows an expression symbol, e.g., $\varphi(x_1, \ldots, x_n)$, these variables are all the free variables and only free variables in the expression. When the expression symbol is used without a list of variable symbols, it is left open which variables are free in that expression. As a general rule, it is assumed that all expressions are rectified. Since the intended interpretation of $\exists x\varphi(x_1, \ldots, x_n, x)$ is identical to that of $\exists y\varphi(x_1, \ldots, x_n, y)$, PCS expressions are defined to be equivalence classes with each equivalence class consisting of all alphabetic variants. This equivalence can be defined formally (e.g., see Barnes and Mack [1]), but this will not be done here. Any member of a given equivalence class will be used to represent the class. Hence the two forms given above represent the same PCS expression.

In the sequel, parentheses are dropped whenever no confusion can result. Metavariables are used as follows: R^n ranges over $\mathcal{R}_n$; S^n ranges over $\mathcal{S}_n$; P^n ranges over $\mathcal{R}_n \cup \mathcal{S}_n$; x, y, z range over $\mathcal{V}$; S ranges over singular expressions; and φ, ψ, θ range over expressions. Applying subscripts to these symbols does not change their ranges.

2.2 Semantics An *interpretation* of PCS is a pair $\mathcal{I} = \langle \mathfrak{D}, \mathcal{G} \rangle$ where $\mathfrak{D}$ is a nonempty set and $\mathcal{G}$ is a mapping defined on $\mathcal{P}$ satisfying:

1. If $R^n \in \mathcal{R}_n$, then $\mathcal{G}(R^n) \subseteq \mathfrak{D}^n$.
2. If $S^{n+1} \in \mathcal{S}_{n+1}$, then $\mathcal{G}(S^{n+1}) \subseteq \mathfrak{D}^{n+1}$ such that for all $d_1, \ldots, d_n \in \mathfrak{D}$ there exists $d \in \mathfrak{D}$ with $\langle d_1, \ldots, d_n, d \rangle \in \mathcal{G}(S^{n+1})$ and for all $d' \in \mathfrak{D}$, $\langle d_1, \ldots, d_n, d' \rangle \in \mathcal{G}(S^{n+1})$ implies $d' = d$.

Let $g \in \mathfrak{D}^{\mathcal{V}}$ be an *assignment of values to variables*, and φ be an expression of PCS. Then φ *is satisfied by g in $\mathcal{I}$* (written $\mathcal{I} \models \varphi[g]$) iff one of the following holds:

1. $\varphi = P^n(x_1, \ldots, x_n)$ and $\langle g(x_1), \ldots, g(x_n) \rangle \in \mathcal{G}(P^n)$
2. $\varphi = \neg\psi$ and $\mathcal{I} \not\models \psi[g]$
3. $\varphi = \psi \wedge \theta$ and $(\mathcal{I} \models \psi[g]$ and $\mathcal{I} \models \theta[g])$
4. $\varphi = \exists x\psi$, where x occurs free in ψ, and there exists $g' \in \mathfrak{D}^{\mathcal{V}}$ that agrees with g off x such that $\mathcal{I} \models \psi[g']$.

An expression φ is *true in* $\mathcal{I}$, written $\mathcal{I} \models \varphi$, iff for all $g \in \mathfrak{D}^{\mathcal{V}}$, $\mathcal{I} \models \varphi[g]$. φ is *valid*, written $\models\varphi$, iff φ is true in every interpretation.

2.3 Abbreviations It is convenient to extend PCS by introducing the following abbreviations.

1. $\psi \vee \theta := \neg(\neg\psi \wedge \neg\theta)$
2. $\psi \rightarrow \theta := \neg(\psi \wedge \neg\theta)$
3. $\psi \leftrightarrow \theta := (\psi \rightarrow \theta) \wedge (\theta \rightarrow \psi)$
4. $\forall x\psi := \neg\exists x\neg\psi$.

The semantics for these abbreviations can be given directly as follows:

1. If $\varphi = \psi \vee \theta$ then $\mathfrak{I} \vDash \varphi[g]$ iff ($\mathfrak{I} \vDash \psi[g]$ or $\mathfrak{I} \vDash \theta[g]$).
2. If $\varphi = \psi \rightarrow \theta$ then $\mathfrak{I} \vDash \varphi[g]$ iff ($\mathfrak{I} \vDash \psi[g]$ implies $\mathfrak{I} \vDash \theta[g]$).
3. If $\varphi = \psi \leftrightarrow \theta$ then $\mathfrak{I} \vDash \varphi[g]$ iff ($\mathfrak{I} \vDash \psi[g]$ iff $\mathfrak{I} \vDash \theta[g]$).
4. If $\varphi = \forall x \psi$, where x occurs free in ψ, then $\mathfrak{I} \vDash \varphi[g]$ iff for all $g' \in \mathfrak{D}^{\mathfrak{V}}$ that agree with g off x, $\mathfrak{I} \vDash \psi[g']$.

3. Properties of singular expressions Singular expressions play a central role in PCS. The denotation of a singular expression is a single (though not necessarily unique) individual. Singular expressions commute in a certain way with the Boolean operators. The principal result is that not only unary singular predicates, corresponding to individual constants in PCI, but more generally singular expressions, exhibit 'wild quantity'. These results are established in this section.

In the following, if $\varphi(x_1, \ldots, x_n)$ is a wff, $\mathfrak{I} \vDash \varphi[d_1, \ldots, d_n]$ will abbreviate $\mathfrak{I} \vDash \varphi[g]$ where $g \in \mathfrak{D}^{\mathfrak{V}}$ such that $g(x_1) = d_1, \ldots, g(x_n) = d_n$.

Lemma 1 *There exists $d \in \mathfrak{D}$ such that $\mathfrak{I} \vDash S[d]$ and for all $d' \in \mathfrak{D}$, $\mathfrak{I} \vDash S[d']$ implies $d' = d$.*

Proof: Define the depth of a singular expression as follows: $\text{depth}(S^1(x)) := 0$. $\text{depth}(\exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge S^{n+1}(x_1, \ldots, x_n, x)) \cdots)) := 1 + \max\{\text{depth}(S_i(x_i)) : 1 \leq i \leq n\}$. The proof is a straightforward induction on the depth of $S(x)$.

In the following, Lemma 1 will be abbreviated $\exists! d \in \mathfrak{D} : \mathfrak{I} \vDash S[d]$.

Theorem 2 $\mathfrak{I} \vDash \exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \neg\varphi(x_1, \ldots, x_n)) \cdots)$ *iff* $\mathfrak{I} \vDash \neg\exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \varphi(x_1, \ldots, x_n)) \cdots)$.

Proof: $\mathfrak{I} \vDash \exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \neg\varphi(x_1, \ldots, x_n)) \cdots)$ iff $\exists! d_1 \cdots \exists! d_n : (\mathfrak{I} \vDash S_1[d_1]) \wedge \cdots \wedge (\mathfrak{I} \vDash S_n[d_n]) \wedge (\mathfrak{I} \vDash \neg\varphi[d_1, \ldots, d_n])$ iff $\exists! d_1 \cdots \exists! d_n : (\mathfrak{I} \vDash S_1[d_1]) \wedge \cdots \wedge (\mathfrak{I} \vDash S_n[d_n]) \wedge (\mathfrak{I} \nvDash \varphi[d_1, \ldots, d_n])$ iff $\mathfrak{I} \nvDash \exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \varphi(x_1, \ldots, x_n)) \cdots)$ iff $\mathfrak{I} \vDash \neg\exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \varphi(x_1, \ldots, x_n)) \cdots)$ (follows from the definition of satisfaction and Lemma 1).

Corollary 3 $\mathfrak{I} \vDash \exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \varphi(x_1, \ldots, x_n)) \cdots)$ *iff* $\mathfrak{I} \vDash \forall x_1(S_1(x_1) \rightarrow \cdots \rightarrow \forall x_n(S_n(x_n) \rightarrow \varphi(x_1, \ldots, x_n)) \cdots)$.

Using the notation of restricted quantification, this result can be recognized as asserting the 'wild quantity' of singular expressions, e.g., $(\exists x : S(x))(\varphi(x)) \leftrightarrow (\forall x : S(x))(\varphi(x))$.

Theorem 4 $\mathfrak{I} \vDash \exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \varphi(x_{i_1}, \ldots, x_{i_l}) \wedge \psi(x_{j_1}, \ldots, x_{j_m})) \cdots)$ *iff* ($\mathfrak{I} \vDash \exists x_{i_1}(S_{i_1}(x_{i_1}) \wedge \cdots \wedge \exists x_{i_l}(S_{i_l}(x_{i_l}) \wedge \varphi(x_{i_1}, \ldots, x_{i_l})) \cdots)$ *and* $\mathfrak{I} \vDash \exists x_{j_1}(S_{j_1}(x_{j_1}) \wedge \cdots \wedge \exists x_{j_m}(S_{j_m}(x_{j_m}) \wedge \psi(x_{j_1}, \ldots, x_{j_m})) \cdots)$), *where* $\{i_1, \ldots, i_l\} \cup \{j_1, \ldots, j_m\} = \{1, \ldots, n\}$.

Proof: $\mathfrak{I} \vDash \exists x_1(S_1(x_1) \wedge \cdots \wedge \exists x_n(S_n(x_n) \wedge \varphi(x_{i_1}, \ldots, x_{i_l}) \wedge \psi(x_{j_1}, \ldots, x_{j_m})) \cdots)$ iff $\exists! d_1 \cdots \exists! d_n : (\mathfrak{I} \vDash S_1[d_1]) \wedge \cdots \wedge (\mathfrak{I} \vDash S_n[d_n]) \wedge (\mathfrak{I} \vDash (\varphi(x_{i_1}, \ldots, x_{i_l}) \wedge \psi(x_{j_1}, \ldots, x_{j_m}))[d_1, \ldots, d_n])$ iff $\exists! d_1 \cdots \exists! d_n : (\mathfrak{I} \vDash S_1[d_1]) \wedge \cdots \wedge (\mathfrak{I} \vDash S_n[d_n]) \wedge (\mathfrak{I} \vDash \varphi[d_{i_1}, \ldots, d_{i_l}]) \wedge (\mathfrak{I} \vDash \psi[d_{j_1}, \ldots, d_{j_m}])$ iff ($\mathfrak{I} \vDash \exists x_{i_1}(S_{i_1}(x_{i_1}) \wedge \cdots \wedge$

$\exists x_{i_l}(S_{i_l}(x_{i_l}) \wedge \varphi(x_{i_1},\ldots,x_{i_l}))\cdots)) \wedge (\mathfrak{I} \models \exists x_{j_1}(S_{j_1}(x_{j_1}) \wedge \cdots \wedge \exists x_{j_m}(S_{j_m}(x_{j_m}) \wedge \psi(x_{j_1},\ldots,x_{j_m}))\cdots))$ (follows from the definition of satisfaction and Lemma 1).

Thus singular expressions distribute over conjunction. Examples, using the notation of restricted quantification, are: $(\exists x : S(x))(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\exists x : S(x))(\varphi(x)) \wedge (\exists x : S(x))(\psi(x)))$ and $(\forall x : S(x))(\varphi(x) \wedge \psi(x)) \leftrightarrow ((\forall x : S(x))(\varphi(x)) \wedge \psi(x))$.

4 PCS and PCI compared The expressiveness of PCS relative to PCI will be investigated through the use of meaning-preserving translations between the two languages. Translation from PCS to PCI is not surjective. The difference of PCI and the image of PCS in PCI will give the deficit in expressiveness.

To facilitate definition of a translation function, a brief definition of PCI will first be given. This definition is standard, but is chosen to parallel the definition of PCS given in Section 2.

4.1 Definition of PCI The vocabulary of PCI consists of the following:

1. Predicate symbols $\mathcal{R} = \bigcup_{n \in \omega_+} \mathcal{R}_n$, where $\mathcal{R}_n = \{R_i^n : i \in \omega\}$
2. Individual constant symbols $\mathcal{C} = \{c_i : i \in \omega\}$
3. Function symbols $\mathcal{F} = \bigcup_{n \in \omega_+} \mathcal{F}_n$, where $\mathcal{F}_n = \{f_i^n : i \in \omega\}$
4. Variable symbols $\mathcal{V} = \{v_i : i \in \omega\}$
5. Boolean operators $\wedge$ and $\neg$
6. Identity relation $=$
7. Quantifier $\exists$
8. Parentheses (and)
9. Comma , .

Terms in PCI are defined as follows:

1. Individual constant symbols and variable symbols are terms.
2. If $f^n \in \mathcal{F}_n$ and $t_1,\ldots,t_n$ are terms, then $f^n(t_1,\ldots,t_n)$ is a term.
3. Nothing else is a term.

In the following, t will be used as a metavariable ranging over terms of PCI. Expressions in PCI are defined as follows:

1. If $R^n \in \mathcal{R}_n$ and $t_1,\ldots,t_n$ are terms, then $R^n(t_1,\ldots,t_n)$ is an expression.
2. If t_1, t_2 are terms, then $t_1 = t_2$ is an expression.
3. If φ is an expression then $\neg\varphi$ is an expression.
4. If φ, ψ are expressions then $(\varphi \wedge \psi)$ is an expression.
5. If φ is an expression and $x \in \mathcal{V}$ occurs free in φ, then $\exists x\varphi$ is an expression.
6. Nothing else is an expression.

An *interpretation* of PCI is a pair $\mathfrak{I} = \langle \mathfrak{D}, \mathcal{G} \rangle$ where $\mathfrak{D}$ is a nonempty set and $\mathcal{G}$ is a mapping defined on $\mathcal{P}$ satisfying:

1. If $R^n \in \mathcal{R}_n$, then $\mathcal{G}(R^n) \subseteq \mathfrak{D}^n$.
2. If $c \in \mathcal{C}$, then $\mathcal{G}(c) \in \mathfrak{D}$.
3. If $f^n \in \mathcal{F}_n$, then $\mathcal{G}(f^n) \in \mathfrak{D}^{\mathfrak{D}^n}$.
4. $\mathcal{G}(=)$ is the diagonal relation on $\mathfrak{D}$.

Let $g \in \mathfrak{D}^{\mathcal{V}}$ be an assignment of values to variables. Define an extension g^* of g to the set of terms of PCI as follows:

1. If $x \in \mathcal{V}$, then $g^*(x) := g(x)$.
2. If $c \in \mathcal{C}$, then $g^*(c) := \mathcal{G}(c)$.
3. If $f^n \in \mathcal{F}_n$ and $t_1, \ldots, t_n$ are terms, then $g^*(f^n(t_1, \ldots, t_n)) := \mathcal{G}(f^n)(g^*(t_1), \ldots, g^*(t_n))$.

Let φ be an expression of PCI. Then φ *is satisfied by g in* $\mathcal{I}$ (written $\mathcal{I} \models \varphi[g]$) iff one of the following holds:

1. $\varphi = R^n(t_1, \ldots, t_n)$ and $\langle g^*(t_1), \ldots, g^*(t_n) \rangle \in \mathcal{G}(R^n)$
2. $\varphi = (t_1 = t_2)$ and $g^*(t_1) = g^*(t_2)$
3. $\varphi = \neg\psi$ and $\mathcal{I} \not\models \psi[g]$
4. $\varphi = \psi \wedge \theta$ and $(\mathcal{I} \models \psi[g]$ and $\mathcal{I} \models \theta[g])$
5. $\varphi = \exists x\psi$, where x occurs free in ψ, and there exists $g' \in \mathfrak{D}^{\mathcal{V}}$ that agrees with g off x such that $\mathcal{I} \models \psi[g']$.

The usual definitions and notational conventions defined for PCS carry over to PCI.

4.2 Translation to PCI A translation function τ from PCS into PCI is defined as follows. For atomic expressions:

1. $R_i^n(x_1, \ldots, x_n) \mapsto R_i^n(x_1, \ldots, x_n)$
2. $S_i^1(x) \mapsto c_i = x$
3. $S_i^{n+1}(x_1, \ldots, x_n, x) \mapsto f_i^n(x_1, \ldots, x_n) = x$.

This definition for atomic expressions is extended to a $\langle \wedge, \neg, (\exists x)_{x \in \mathcal{V}} \rangle$-homomorphism. Let $\mathcal{I} = \langle \mathfrak{D}, \mathcal{G} \rangle$ and $\mathcal{I}' = \langle \mathfrak{D}, \mathcal{G}' \rangle$ be interpretations of PCS and PCI, respectively, over the same universe. Then $\mathcal{I}$ and $\mathcal{I}'$ are *similar* iff

1. $\mathcal{G}(R_i^n) = \mathcal{G}'(R_i^n)$
2. $\mathcal{G}(S_i^1) = \{\langle d \rangle\}$ iff $\mathcal{G}'(c_i) = d$
3. $\langle d_1, \ldots, d_n, d \rangle \in \mathcal{G}(S_i^{n+1})$ iff $\mathcal{G}'(f_i^n)(d_1, \ldots, d_n) = d$.

Lemma 5 *Let $\mathcal{I}$ and $\mathcal{I}'$ be similar interpretations of PCS and PCI, respectively, over universe $\mathfrak{D}$. Let $g \in \mathfrak{D}^{\mathcal{V}}$ and $\varphi \in$ PCS. Then $\mathcal{I} \models \varphi[g]$ iff $\mathcal{I}' \models \tau(\varphi)[g]$.*

Proof: The proof is a straightforward induction on the structure of φ.

Thus τ is a mapping of PCS into PCI.

4.3 Translation from PCI Next consider a translation τ' of PCI into PCS, defined for atomic expressions:

1. $c_i = x \mapsto S_i^1(x)$
2. $c_i = t \mapsto \exists x(S_i^1(x) \wedge \tau'(t = x))$, where $t \notin \mathcal{V}$
3. $f_i^n(x_1, \ldots, x_n) = x \mapsto S_i^{n+1}(x_1, \ldots, x_n, x)$
4. $f_i^n(x_1, \ldots, x_n) = t \mapsto \exists x(S_i^{n+1}(x_1, \ldots, x_n, x) \wedge \tau'(t = x))$, where $t \notin \mathcal{V}$
5. $f_i^n(t_1, \ldots, t_n) = t \mapsto \exists x_{k_1}(\tau'(t_{k_1} = x_{k_1}) \wedge \cdots \wedge \exists x_{k_m}(\tau'(t_{k_m} = x_{k_m}) \wedge \exists x(\tau'(t = x) \wedge S_i^{n+1}(x_1, \ldots, x_n, x)) \cdots)$, where $t, t_{k_1}, \ldots, t_{k_m} \notin \mathcal{V}$ and $(\{t_1, \ldots, t_n\} - \{t_{k_1}, \ldots, t_{k_m}\}) \subseteq \mathcal{V}$

6. $R_i^n(t_1,\ldots,t_n) \mapsto \exists x_{k_1}(\tau'(t_{k_1}=x_{k_1}) \wedge \cdots \wedge \exists x_{k_m}(\tau'(t_{k_m}=x_{k_m}) \wedge R_i^n(x_1,\ldots, x_n))\cdots)$, where $t_{k_1},\ldots,t_{k_m} \notin \mathcal{V}$ and $(\{t_1,\ldots,t_n\} - \{t_{k_1},\ldots,t_{k_m}\}) \subseteq \mathcal{V}$.

As with τ, this definition of τ' for atomic expressions is extended to a $\langle \wedge, \neg, (\exists x)_{x\in\mathcal{V}}\rangle$-homomorphism. Note that τ' is *partial* since $\tau'(x_1 = x_2)$ is not defined. Let PCI_1 be the domain of τ'.

Lemma 6 *Let $\mathcal{I}$ and $\mathcal{I}'$ be similar interpretations of PCS and PCI, respectively, over universe $\mathfrak{D}$. Let $g \in \mathfrak{D}^{\mathcal{V}}$ and $\psi \in PCI_1$. Then $\mathcal{I}' \models \psi[g]$ iff $\mathcal{I} \models \tau'(\psi)[g]$.*

Proof: The proof is a straightforward induction on the structure of ψ.

Therefore, PCS and PCI_1 are equivalent in expressiveness, and any deficit in expressiveness of PCS is restricted to the difference PCI $-$ PCI_1. More precisely, any deficit in expressiveness of PCS is restricted to those wffs of PCI $-$ PCI_1 containing noneliminable occurrences of atomic expressions of the form $x_1 = x_2$. Occurrences of atomic expressions of the form $x_1 = x_2$ in a wff ψ are *eliminable* iff there exists a wff ψ' such that for any interpretation $\mathcal{I}$ of PCI, $\mathcal{I} \models \psi'$ iff $\mathcal{I} \models \psi$. Let PCI_2 be the set of wffs containing noneliminable occurrences of expressions of the form $x_1 = x_2$. That PCI_2 is not empty is shown next.

Consider the unary predicate $R_0^1 \in$ PCI and let $\psi = \exists x_1 \forall x_2 (R_0^1(x_2) \leftrightarrow (x_2 = x_1))$. Then in any interpretation $\mathcal{I}' = \langle \mathfrak{D}, \mathcal{G}\rangle$ of PCI, $\mathcal{I}' \models \psi$ only if $card(\mathcal{G}(R_0^1)) = 1$. The next lemma shows that PCS is indifferent to this property.

Lemma 7 *There is no closed wff $\varphi \in PCS$ such that for every interpretation $\mathcal{I} = \langle \mathfrak{D}, \mathcal{G}\rangle$ of PCS, $\mathcal{I} \models \varphi$ only if $card(\mathcal{G}(R_0^1)) = 1$.*

Proof: Let $\varphi \in$ PCS and let $n \in \omega$ such that if S_j^l occurs in φ then $j < n$. Let $\mathcal{I}_1 = \langle \omega, \mathcal{G}_1\rangle$ and $\mathcal{I}_2 = \langle \omega, \mathcal{G}_2\rangle$ be interpretations of PCS, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined as follows. $\mathcal{G}_1(R_0^1) = \{\langle n\rangle\}$ and $\mathcal{G}_2(R_0^1) = \{\langle n\rangle, \langle m\rangle\}$ for $n < m$, and for all other predicates R_j^l of PCS, $\mathcal{G}_1(R_j^l) = \mathcal{G}_2(R_j^l) = \varnothing$. For all singular predicates S_j^l of PCS, $\mathcal{G}_1(S_j^l) = \mathcal{G}_2(S_j^l) = \{\langle i_1,\ldots,i_{l-1},j\rangle : i_1,\ldots,i_{l-1} \in \omega\}$.

It suffices to show the following. If φ is any rectified wff of PCS with free variables $x_1,\ldots,x_l$, then $\exists i_1,\ldots,i_l \in \omega : \mathcal{I}_1 \models \varphi[i_1,\ldots,i_l]$ iff $\exists j_1,\ldots,j_l \in \omega : \mathcal{I}_2 \models \varphi[j_1,\ldots,j_l]$. The proof is by induction on the structure of φ.

For the basis, let $\varphi = P^l(x_1,\ldots,x_l)$ where P^l is an ordinary or singular predicate of PCS. First suppose that $\mathcal{I}_1 \models P^l[i_1,\ldots,i_l]$. Define $j_1,\ldots,j_l$ as follows. For $1 \le k \le l$, if $i_k \neq m$ then $j_k = i_k$ and if $i_k = m$ then $j_k = m + 1$. It follows from the definitions of $\mathcal{G}_1$ and $\mathcal{G}_2$ that $\mathcal{I}_2 \models P^l[j_1,\ldots,j_l]$. For the converse, suppose that $\mathcal{I}_2 \models P^l[j_1,\ldots,j_l]$. Define $i_1,\ldots,i_l$ as follows. For $l \le k \le l$, if $j_k \neq m$ then $i_k = j_k$ and if $j_k = m$ then $i_k = n$. Again it follows from the definitions of $\mathcal{G}_1$ and $\mathcal{G}_2$ that $\mathcal{I}_1 \models P^l[i_1,\ldots,i_l]$. Hence $\mathcal{I}_1 \models P^l[i_1,\ldots,i_l]$ iff $\mathcal{I}_2 \models P^l[j_1,\ldots,j_l]$.

The induction step is straightforward.

It remains to show that the deficit in expressiveness of PCS relative to PCI is exactly PCI_2.

Theorem 8 *Let $\mathfrak{I}'$ and $\mathfrak{I}$ be similar interpretations of PCI and PCS, respectively, and let ψ be a wff of PCI. There exists a wff φ of PCS such that* ($\mathfrak{I}' \models \psi[g]$ *iff* $\mathfrak{I} \models \varphi[g]$) *iff* $\psi \notin PCI_2$.

Proof: The 'if' direction is an immediate corollary of Lemma 6. For the 'only if' direction, suppose φ is a wff of PCS such that $\mathfrak{I}' \models \psi[g]$ iff $\mathfrak{I} \models \varphi[g]$. By Lemma 5, $\mathfrak{I}' \models \tau(\varphi)[g]$ iff $\mathfrak{I} \models \varphi[g]$. By definition, $\tau(\varphi)$ has no occurrences of atomic expressions of the form $x_1 = x_2$. Therefore, $\psi \notin \mathrm{PCI}_2$.

Theorem 8 shows that Sommers's theory of identity as implemented in PCS cannot duplicate the expressiveness of the logical identity relation of PCI. A final question is whether it is possible to compensate for this deficit through use of axiom *schemas*. Consider the binary relation R_0^2 defined by schema I:

$$[\mathrm{I.}] \qquad \exists x_1(S_1(x_1) \wedge \exists x_2(S_2(x_2) \wedge R_0^2(x_1,x_2))) \leftrightarrow \exists x(S_1(x) \wedge S_2(x)).$$

Theorem 9 *Let $\mathfrak{I} = \langle \mathfrak{D}, \mathfrak{G} \rangle$ be a PCS model of schema I. Let $\mathfrak{D}' = \{d \in \mathfrak{D} : \mathfrak{I} \models S[d]$, where S is a singular expression$\}$. Then $\mathfrak{G}(R_0^2)$ restricted to $\mathfrak{D}'$ is the diagonal relation.*

Proof: Let $d_1, d_2 \in \mathfrak{D}$ such that $\mathfrak{I} \models S_1[d_1]$ and $\mathfrak{I} \models S_2[d_2]$. Then $\mathfrak{I} \models R_0^2[d_1,d_2]$ iff $\mathfrak{I} \models \exists x_1(S_1(x_1) \wedge \exists x_2(S_2(x_2) \wedge R_0^2(x_1,x_2)))$ (definition of satisfaction) iff $\mathfrak{I} \models \exists x(S_1(x) \wedge S_2(x))$ (schema I) iff $\exists! d \in \mathfrak{D} : (\mathfrak{I} \models S_1[d]) \wedge (\mathfrak{I} \models S_2[d])$ (definition of satisfaction and Lemma 1) iff $d_1 = d_2$.

$\mathfrak{D}'$ is the subset of *named* elements of the universe $\mathfrak{D}$. Theorem 9 shows that, in the subuniverse of named elements, axiom schemas suffice to express any property expressible in PCI. However, not all elements of the universe of a PCS interpretation are named in general. The inability of PCS to predicate certain properties of the unnamed elements of the universe, even with schemas, is a fundamental limitation inherent to languages based on Sommers's approach.

The need to name elements of the universe arises in many contexts. Two familiar ones are the method of diagrams and the standard model of arithmetic. The following paragraphs will consider their connection with the present discussion. For a more complete account of these topics, see Chang and Keisler [2], Chapter 2.

The method of diagrams gives logical expression to certain model-theoretic notions. Let $\mathfrak{I} = \langle \mathfrak{D}, \mathfrak{F} \rangle$ be an interpretation of a language L and $A \subseteq \mathfrak{D}$. Then $L_A := L \cup \{c_a : a \in A\}$ is an expansion of L, and $\mathfrak{I}_A := (\mathfrak{I}, a)_{a \in A}$ is the obvious expansion of $\mathfrak{I}$ to L_A. The diagram of $\mathfrak{I}$ is the set of all atomic sentences and negations of atomic sentences that are true in $\mathfrak{I}_{\mathfrak{D}}$. Expansion of L is necessary to permit those properties of the structure to be expressed. In the present context, all elements of the universe $\mathfrak{D}$ are named in $\mathrm{PCS}_{\mathfrak{D}} := \mathrm{PCS} \cup \{S_a^1 : a \in \mathfrak{D}\}$ under the interpretation $\mathfrak{I}_{\mathfrak{D}}$. Therefore, given a structure $\mathfrak{I}$ and an axiomatization Γ in PCI of the theory of $\mathfrak{I}$, PCS can be expanded by adjoining unary singular predicates to name the elements of the universe of $\mathfrak{I}$. Then the theory of $\mathfrak{I}$ can be axiomatized by a set Γ' of axiom schemas in the expansion of PCS. Of course, this is not a general solution to the deficit in expressiveness of PCS, since the expansion is linked to the intended structure.

ω-logic originated in the study of the standard model of arithmetic. Let L be a first-order language with individual constants $\{c_i : i \in \omega\}$. ω-logic is formed by adding to a first-order axiomatization of L the distinctness of the individual constants and the ω-rule:

$$\text{from } \varphi(c_0), \varphi(c_1), \varphi(c_2), \ldots, \text{ infer } \forall x \varphi(x)$$

and allowing infinite proofs. In ω-logic, an expression that is true of each of the individual constants in L is true of everything. If the semantics of PCS were expanded to require a distinct denotation for each $S_i^1 \in \mathbb{S}_1$ and the condition that

$$\text{if } \mathcal{I} \models \exists x(S_0^1(x) \wedge \varphi(x)), \mathcal{I} \models \exists x(S_1^1(x) \wedge \varphi(x)),\ \mathcal{I} \models \exists x(S_2^1(x) \wedge \varphi(x)), \ldots,$$
$$\text{then } \mathcal{I} \models \forall x \varphi(x)$$

then all interpretations of PCS would be countably infinite and all elements of the domain would be named in PCS. In these interpretations, PCS can attain the expressiveness of PCI through the use of schemas. But again, this is not a general approach and so does not remove the fundamental deficit in expressiveness of PCS.

5 Conclusion Sommers's position on identity has not received a full evaluation. Part of the reason is perhaps that his argument was presented in the context of the Calculus of Terms (Sommers [6]), running counter to the prevailing bias that only MPL can be taken seriously. Further, the formalization of his position was incomplete, dealing only with individual constants. This paper gives a full answer to Sommers's question, 'Do We Need Identity?'. The argument is couched in MPL, modified only as much as necessary to eliminate the distinction between concept and object.

The results show that when Sommers's approach is extended as much as it can be, there still remain properties expressible in predicate calculus with identity that are not expressible in Sommers's approach. These are exactly those properties whose expression in PCI involves an occurrence of a subexpression of the form $x = y$ (where *both* arguments of the identity relation are *variables*), and for which there exists no equivalent expression without a subexpression of that form.

In interpretations in which every element of the universe is named in the language, this deficit can be overcome by the use of schemas. However, this technique cannot be said to eliminate the deficit because it is not general, being linked to the particular interpretations.

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