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# The Difference Model of Voting

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**Abstract** A participant in a voting procedure may be concerned not only that the outcome should be as good as possible but also that her own vote should be cast for some alternative that is as good as possible. A high priority for the latter means unwillingness to compromise. Two formal models, one ordinal and one cardinal, are developed for voting in which the individual preferences are combinations of these two types of values. A close relationship is shown to hold between the two models. Formal results are obtained on conditions for voting sequences to be "stable" in the sense that no collection of voters will have incentive to unilaterally defect from the way they voted (Nash equilibria). Furthermore, computer simulations of the three person case are reported. In these it is shown how the incidence of instability, multi-stability (more than one stable outcome), and tie-stability (tie as the only stable outcome) varies with different priorities between the two types of values.

***1 Introduction*** Social choice theory traditionally "conflates 'choice' and 'preference'", and treats them "as essentially synonymous concepts . . . A preference is a potential choice, whereas a choice is an actualized preference". (Reynolds and Paris [5], cf. Friedland and Cimbala [1]) In other words, the theory represents collective decision-making as a mechanical aggregation of individual preferences over the set of possible outcomes of the decision process.

This feature of the formalism impedes the representation of procedural preferences. A participant may, for instance, prefer a unanimous decision, prefer to vote in the same way as some other person, or prefer to be part of the majority. Her voting behavior will then depend both on her preferences over the possible outcomes and on her preferences over procedural properties of the decision process. In order to achieve a convenient representation of such combined preferences, an extension of the classical, Arrovian framework is needed.

One procedural factor that seems to be important to many participants in voting procedures is *her or his own vote*. The participant may be concerned not only that the outcome should be as good as possible (according to some stan-

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dard), but also that her own vote be cast for some alternative that is as good as possible (according to some, perhaps most often the same, standard).

This can be seen most clearly in cases where someone refuses to compromise to achieve a second-best alternative. Let us consider majority voting to decide between the three alternatives $X$, $Y$, and $Z$. A certain participant may at the same time (1) strictly prefer $X$ to $Y$ and $Y$ to $Z$, and (2) consider it to be against her principles to vote for anything but the best alternative, i.e., $X$. Then she will not join a coalition and vote for $Y$ even if this would change the outcome from $Z$ to $Y$. Such preference patterns can, to mention just one example, account for votes received by political candidates whose chances to get elected are negligible.

The present study will be devoted to decision processes where each participant has two concerns:

(1) She prefers the outcome to be as good as possible (according to her own standards).
(2) She prefers the alternative that she votes for to be as good as possible (according to her own standards).

A participant who gives greater priority to (1) rather than (2) is a *result-oriented* (or *compromise-willing*) participant. Someone with the opposite priorities is *principled* with respect to her own voting.

*2 Voting procedures* The classical Arrovian model will be extended as follows in order to accommodate procedural preferences.

**Definition 2.1** A *voting procedure* is a quadruple $\langle \mathfrak{I}, A, S, r \rangle$ such that:
(i) $\mathfrak{I} = \langle 1, \ldots n \rangle$ is an $n$-tuple of *individuals*, namely those individuals that take part in the voting procedure. $I$ is the (unordered) set of elements of $\mathfrak{I}$.
(ii) $A = \{x, y \ldots\}$ is the set of *alternatives* that the voting procedure aims at choosing between. $A \cup \{\lambda\}$ is the set of *outcomes*, where $\lambda$ is the *tie outcome*.
(iii) $S = \{s_1, \ldots s_m\}$ is the set of *strategies* that are available to the participants in the voting procedure.
(iv) An $n$-tuple $\pi = \langle \alpha_1, \ldots \alpha_n \rangle$ of elements of $S$ is a *voting pattern*. It denotes that for each $k$, the $k$th individual performs according to strategy $\alpha_k$.
(v) $r$, the *social choice function*, is a total function from the set of voting patterns to the set of outcomes.

This definition is uncommitted as to the structure of the set $S$ of strategies. That set may be identical to the set of alternatives, in which case each participant has to vote for one of the alternatives. The set of strategies may also consist of subsets of the set of alternatives, or of (ordinal or cardinal) rankings of the alternatives.

The preference relations of the participants will be constructed to refer to voting patterns instead of alternatives:

**Definition 2.2** A *preference profile* is an $n$-tuple $\langle R_1, \ldots R_n \rangle$, where each $R_k$ is a reflexive, transitive, and connected relation over the set of voting patterns. $R_k$ is the *preference relation* of the $k$th individual in $\mathfrak{I}$. The relations $P_k$ and $I_k$ are defined as follows:

$$\pi P_k \pi' \leftrightarrow \pi R_k \pi' \ \& \ \neg(\pi' R_k \pi)$$

$$\pi I_k \pi' \leftrightarrow \pi R_k \pi' \ \& \ \pi' R_k \pi.$$

The following shorthand notation will be used:

**Definition 2.3** Let $k \in I$ and let $\pi$ and $\pi'$ be voting patterns. Then:
(i) $v(k,\pi)$ is the $k$th element of $\pi$ (i.e., it denotes how $k$ votes in $\pi$).
(ii) $diff(\pi,\pi')$ is the set of elements $k$ of $I$ such that $v(k,\pi) \neq v(k,\pi')$.

Definitions 2.1 and 2.2 constitute a game format for voting. They make it possible to apply the traditional concept of Nash equilibria, i.e., strategy combinations from which no collection of voters will have incentive to defect (Luce and Raiffa [3], Myerson [4]). At least two versions of this notion of stability are plausible:

**Definition 2.4** A voting-pattern $\pi'$ *defeats* another voting-pattern $\pi$ iff $\pi' \neq \pi$ and for all $k \in diff(\pi,\pi')$: $\pi' P_k \pi$. A voting-pattern $\pi$ is *A-stable* (absolutely stable) iff there is no $\pi'$ such that $\pi'$ defeats $\pi$. It is *R-stable* (result-stable) iff there is no $\pi'$ such that $\pi'$ defeats $\pi$ and $r(\pi') \neq r(\pi)$. An outcome $x \in A \cup \{\lambda\}$ is *A-stable* (*R-stable*) iff there is a voting pattern $\pi$ such that $r(\pi) = x$ and that $\pi$ is *A-stable* (*R-stable*).

As should be obvious, the relation "defeats" is antireflexive and asymmetric. It is not in general transitive.

**Theorem 2.5** *A-stability and R-stability for outcomes do not coincide for voting procedures in general.*

(For proofs of this and following theorems, see the Appendix.)

The present study will be concerned with voting procedures whose strategies are the same as their alternatives (i.e., $S = A$). (For some basic results in the more general framework, see Hansson [2].) Furthermore, attention will be restricted to procedures that satisfy the following condition:

**Definition 2.6** A voting procedure satisfies *nonobliquity* iff (i) $S = A$, and (ii) for all voting patterns $\pi$ and $\pi'$, if $v(k,\pi) \neq r(\pi)$ for all $k \in diff(\pi,\pi')$, then $r(\pi') = r(\pi)$.

Nonobliquity implies that if only those who do not vote for the outcome change their votes, then this will not change the outcome. The outcome of $\pi$ cannot be changed in "oblique" ways, i.e., ways other than that some of those who voted for $r(\pi)$ in $\pi$ vote for something other than $r(\pi)$.

Nonobliquity implies, and is stronger than, the condition of non-negativity (also called monotonicity), namely that if someone who did not vote for the outcome changes her vote to vote for the outcome, then the outcome will not be changed.

Simple and qualified majority procedures (without abstentions) are among the voting procedures that satisfy this criterion.

***3 Aggregative and single-based preferences*** Classical social choice theory has been devoted to preferences that are consequentialist in the following sense:

**Definition 3.1** A preference relation $R_k$ over voting patterns is *consequentialist* iff for all $\pi$ and $\pi'$, if $r(\pi) = r(\pi')$ then $\pi I_k \pi'$.

The following wider class of preference relations includes the consequentialist preference relations:

**Definition 3.2** A preference relation $R_k$ over voting patterns is *aggregative* for the $k$th individual iff for all $\pi$ and $\pi'$, if $r(\pi) = r(\pi')$ and $v(k,\pi) = v(k,\pi')$, then $\pi I_k \pi'$.

A preference relation that is aggregative for the $k$th individual may only refer to the outcome and to the way in which the $k$th individual votes. The votes cast by others are taken into account only to the extent that they influence the outcome. (Not all realistic preference patterns are aggregative. As an example, preferences for unanimous decisions are not aggregative.)

The following definition provides a simplified notation for aggregative preferences:

**Definition 3.3** Let $R_k$ be an aggregative preference relation. Furthermore, let $x$ and $x'$ be outcomes, and let $s$ and $s'$ be strategies. Then $[x,s]R_k[x',s']$ is an abbreviation for: for all voting patterns $\pi,\pi'$, if $r(\pi) = x$, $v(k,\pi) = s$, $r(\pi') = x'$, and $v(k,\pi') = s'$, then $\pi R_k \pi'$. $[x,s]P_k[x',s']$ and $[x,s]I_k[x',s']$ are defined correspondingly.

One interesting class of aggregative preference relations are those that conform with an (underlying) preference relation over outcomes in the following way:

**Definition 3.4** An *outcome ordering* $\geq_k$ is a reflexive, transitive, and connected preference relation on $A \cup \{\lambda\}$, i.e., on the set of outcomes. Furthermore, to each outcome ordering $\geq_k$ are associated $>_k$ and $\equiv_k$, defined as follows:

$$x >_k y \leftrightarrow x \geq_k y \ \& \ \neg(y \geq_k x)$$

$$x \equiv_k y \leftrightarrow x \geq_k y \ \& \ y \geq_k x.$$

$B(k)$ is the set of best alternatives according to the $\geq_k$ ordering, i.e., $x \in B(k)$ iff for all $y \in A \cup \{\lambda\}$: $x \geq_k y$. $b(k)$ is an (arbitrary) element of $B(k)$.

**Definition 3.5** Let $R_k$ be an aggregative preference relation on the voting patterns, and let $\geq_k$ be an outcome ordering. Then $R_k$ *conforms with* $\geq_k$ iff for all $x,x' \in A \cup \{\lambda\}$ and $y,y' \in A$:
(i) $[x,y]R_k[x',y]$ iff $x \geq_k x'$, and
(ii) $[x,y]R_k[x,y']$ iff $y \geq_k y'$.

An aggregative preference relation is *single-based* iff there is an outcome ordering to which it conforms.

Single-based preference relations can be used as a model to represent the combination of, and conflict between, result-oriented and principled voting. (It should be noted that with a single-based preference relation, the voter applies the same standards of goodness to the outcome as to her own vote.)

In the Arrovian framework, the tie outcome is not covered by the individual preference relations. The present framework does not share this limitation. In particular, a single-based preference relation may very well conform with an outcome ordering such as $\lambda >_k x >_k y >_k z$ or $x >_k \lambda >_k y >_k z$. ($\lambda$ represents tie and $x, y, z$ the other possible outcomes.) In most of what follows, however, such preferences will be excluded by the following condition:

**Definition 3.6** An outcome ordering $\geq_k$ on $A \cup \{\lambda\}$ satisfies *tie avoidance* iff $x >_k \lambda$ for all $x \in A$.

The following result is obtained for single-based preferences that satisfy tie avoidance:

**Theorem 3.7** *Let r be a nonoblique social choice function. Furthermore, let all individuals have single-based preferences that satisfy tie avoidance. Then an outcome x is A-stable iff it is R-stable.*

When $A$-stability and $R$-stability coincide, the word "stability" will be used to cover both concepts.

In a given voting procedure, for each preference profile $\pi$ there is a (possibly empty) subset of $A \cup \{\lambda\}$ that is the set of stable outcomes for $\pi$. Two preference relations $R_k$ and $R'_k$ are *interchangeable* with respect to outcomes if the replacement of one of them by the other in a preference profile can never change the set of stable outcomes. The following theorem characterizes one class of pairs of preference relations that are interchangeable in this sense:

**Theorem 3.8** *Let $\langle R_1, \ldots R_k, \ldots R_n\rangle$ be a preference profile all of whose elements are single-based and satisfy tie avoidance. Furthermore, let $R'_k$ be a preference relation that satisfies the same two conditions and is such that:*
(1) *there is an outcome ordering $\geq_k$ with which both $R_k$ and $R'_k$ conform, and*
(2) *for all $x \in A$ and $y \in A \cup \{\lambda\}$*

$$[x,x] R_k [y, b(k)] \text{ iff } [x,x] R'_k [y, b(k)]$$

*and*

$$[y, b(k)] R_k [x,x] \text{ iff } [y, b(k)] R'_k [x,x].$$

*Furthermore let r be a nonoblique social choice function. Then $\langle R_1, \ldots R_k, \ldots R_n\rangle$ has exactly the same stable outcomes as $\langle R_1, \ldots R'_k, \ldots R_n\rangle$.*

**Example 3.9** Let $S = A = \{x, y\}$. Then the following two preference relations satisfy the conditions of Theorem 3.8.

| | |
|---|---|
| $[x,x]$ | $[x,x]$ |
| $[y,x]$ | $[x,y]$ |
| $[x,y]$ | $[y,x]$ |
| $[y,y]$ | $[y,y]$ |
| $[\lambda,x]$ | $[\lambda,x]$ |
| $[\lambda,y]$ | $[\lambda,y]$ |

***4 The difference model*** Under the conditions given in Theorem 3.8, the set of stable outcomes is uniquely determined if the following are known for each $R_k$: (1) the outcome ordering ($\geq_k$) to which it conforms, and (2) for all $x$ and $y$ whether or not $[x,x] R_k [y, b(k)]$ and whether or not $[y, b(k)] R_k [x,x]$.

If $x$ is not among the best alternatives (i.e., $x \notin B(k)$), then the formula

$[x,x]R_k[y,b(k)]$ can hold only if $x >_k y$. The formula $[x,x]R_k[y,b(k)]$ can be interpreted as "$x$ is so much better than $y$ that it is better to join a coalition and vote for $x$, if $x$ can then be achieved, than to vote for a best alternative if the outcome will then be $y$". In other words, for $[x,x]R_k[y,b(k)]$ to hold (when $x \notin B(k)$), $x$ must be *sufficiently much better* than $y$.

The phrase "sufficiently much better than" suggests the possibility of a cardinal model. Such a model can be constructed as follows:

**Definition 4.1** A *difference model* for (single-based preferences on) an alternative set $A$ is an ordered pair $\langle u,c \rangle$, where $u$ is a function that to each element $x$ of $A \cup \{\lambda\}$ assigns a real number $u(x)$, and $c$ is a non-negative real number. The number $c$ is called the *difference limit*. Let $\langle u,c \rangle$ be a difference model for the alternative set $A$. Furthermore, let $R_k$ be a single-based preference relation that conforms with the outcome ordering $\geq_k$. Then $R_k$ *conforms with* $\langle u,c \rangle$ iff for all $x \in A \backslash B(k)$ and $y,z \in A \cup \{\lambda\}$:
(1) $y \geq_k z$ iff $u(y) \geq u(z)$,
(2) $[x,x]R_k[y,b(k)]$ iff $u(x) - u(y) \geq c$, and
(3) $[x,x]I_k[y,b(k)]$ iff $u(x) - u(y) = c$.

A close relationship holds between difference models and single-based preference relations, as can be seen from the following two theorems:

**Theorem 4.2** *Let $\langle R_1,\ldots R_k,\ldots R_n \rangle$ be a preference profile, all of whose elements are single-based and satisfy tie avoidance. Furthermore, let $R'_k$ be a preference relation that satisfies the same two conditions and is such that $R_k$ and $R'_k$ both conform with the same difference model $\langle u,c \rangle$. Then, if $r$ is a non-oblique social choice function, $\langle R_1,\ldots R'_k,\ldots R_n \rangle$ has exactly the same stable outcomes as $\langle R_1,\ldots R_k,\ldots R_n \rangle$.*

**Theorem 4.3**
(1) *For each difference model $\langle u,c \rangle$ with $c > 0$, there is at least one single-based preference relation $R_k$ that conforms with $\langle u,c \rangle$.*
(2) *For each single-based preference relation $R_k$, there is at least one difference model $\langle u,c \rangle$ with $c > 0$ such that $R_k$ conforms with $\langle u,c \rangle$.*

*5 Computer simulations* The difference model is well suited for computer simulation studies.

*5.1 Standardization of the model* Since the stability properties are invariant under linear transformations, no generality is lost by the following standardization that is practical for simulation studies:

**Definition 5.1.1** Let $\langle u,c \rangle$ be a difference model for the alternative set $A$. Furthermore, let $g_{max}$ be the highest value of $u(x)$ assigned to any $x \in A \cup \{\lambda\}$, and let $g_{min}$ be the lowest value. Then $\langle u,c \rangle$ is a *standardized difference model* iff:
(1) $g_{min} = 0$, and
(2) $g_{max} = 0$ or $g_{max} = 1$.

The $g_{max} = 0$ case obtains only for difference models that correspond to completely indifferent outcome orderings (i.e., $x \equiv_k y$ for all $x, y \in A \cup \{\lambda\}$). When tie avoidance obtains, $g_{max} = 1$, and $u(x) = 0$ obtains only for $x = \lambda$.

*5.2 The simulation program* Having been given the input specifications, the simulation program produces, for each individual, a random (standardized) difference model that satisfies tie avoidance. It is then determined which (if any) of the outcomes are stable. The result is stored, and the process is repeated, with new random difference models for each case, until statistics from the required number of cases have been obtained.

The input specifications are the number of participants, the difference limit of each participant, the number of alternatives, and the type of voting procedure (simple majority or unanimity).

To assign a random (standardized) difference model to an individual, the program proceeds as follows: (1) Tie is assigned the value 0. (2) Each nontie outcome is assigned a random value between 0 and 1. (3) The model is standardized by multiplying all values with $1/g_{max}$, where $g_{max}$ is the highest of the values assigned in the second part of the procedure.

All runs were made on a personal computer. The program is available upon request.

*5.3 Results* Three series of runs are summarized in Diagrams 1, 2, and 3. Each diagram is based on a series of 11 runs, namely with the difference limits 0.0, 0.1, 0.2, ... 1.0. The curves are drawn by interpolation from the points determined in these runs. Each of the points from which Diagrams 1 and 3 are drawn has been determined with a standard deviation not bigger than 0.5%. Each of the points on which Diagram 2 was based has been determined with a standard deviation not bigger than 0.7%.

Diagram 1 shows the results when there are three participants and three alternatives, and the decision is made by simple majority. All three participants had the same difference limit, which is represented along the horizontal axis.

Furthest to the left in the diagram, where the difference limit is 0, we have a situation commonly referred to in Arrovian social decision theory: namely, three participants, three alternatives, simple majority, and no procedural preferences. There is either no stable outcome or exactly one stable outcome. The theoretically predicted probability of the former case (5.56%) was confirmed in the simulations (5.8%, standard deviation 0.2%).

As the difference limit is increased, the probability that there will be no stable outcome diminishes. When the difference limit is 0.5, this probability is around 0.1%. At that point, another occurrence has grown in frequency: namely, the existence of two or three stable outcomes. (The probabilities for this are around 11.6% and 0.7% respectively, when the difference limit is 0.5.)

With a further increase in the difference limit, the existence of more than one stable outcome is gradually replaced by another phenomenon: that the tie outcome is the only stable outcome. The probability for this is around 0.4% when the difference limit is 0.5 (simulation result) and 22.2% when it is 1.0 (theoretical prediction; the simulation result was 22.1%).

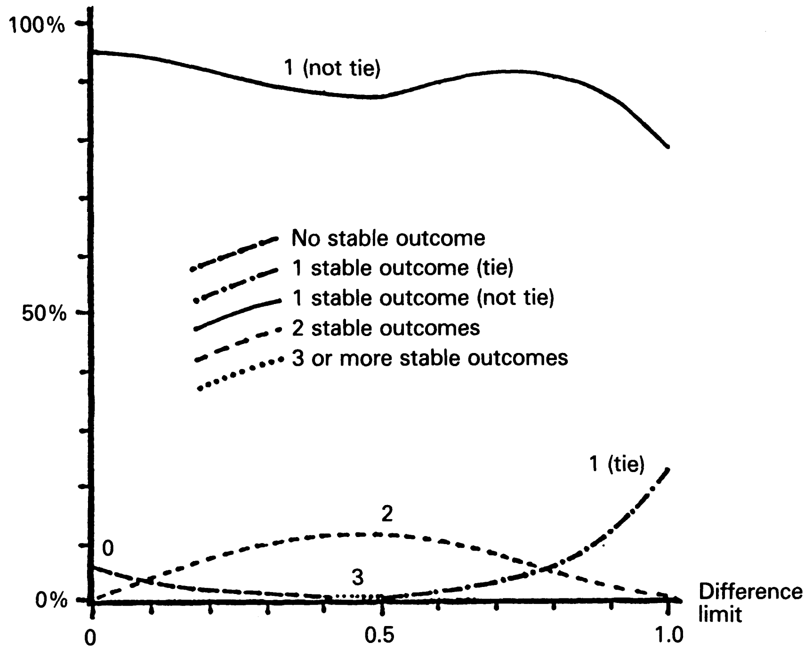


Diagram 1. 3 participants, $\frac{2}{3}$ majority, 3 alternatives.

In the series of runs summarized in Diagram 2, the input differed from that of Diagram 1 only in one respect: the number of alternatives was increased from 3 to 5. (Thus the input conditions were 3 participants, $\frac{2}{3}$ majority, and 5 alternatives.) The overall picture is much the same as in Diagram 1. However, the probability of exactly one stable outcome decreased, and increased probabilities were found for all three other possibilities: instability, more than one stable outcome, and tie as the only stable outcome.

In Diagram 3, the voting procedure is a unanimity procedure, and the rest of the input conditions are as in Diagram 1. (Thus the input conditions were 3 participants, rule of unanimity, and 3 alternatives.) Here, the general picture is much different from that of Diagrams 1 and 2. In particular, cases with no stable outcome do not occur.

***5.4 Discussion*** In social choice theory, it is commonly assumed that voting procedures should be as close as possible to a particular ideal or norm. According to this ideal, whatever the preference profile, there should be exactly one stable outcome, and it should not be the tie outcome.

Besides this ideal, there are three other possibilities:

(1) *Instability*: there is no stable outcome.
(2) *Multi-stability*: there are two or more stable outcomes.
(3) *Tie-stability*: there is exactly one stable outcome, which is the tie outcome.

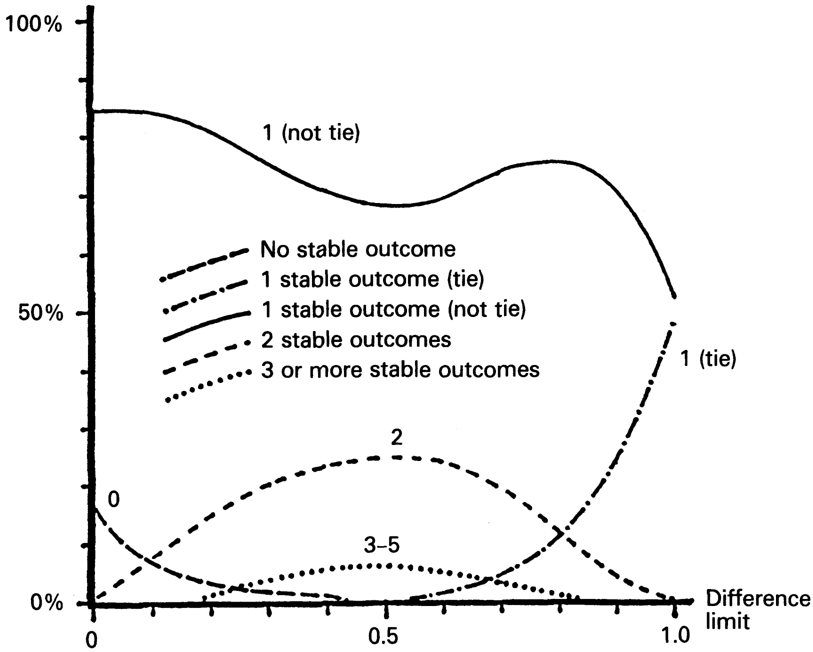


Diagram 2. 3 participants, $\frac{2}{3}$ majority, 5 alternatives.

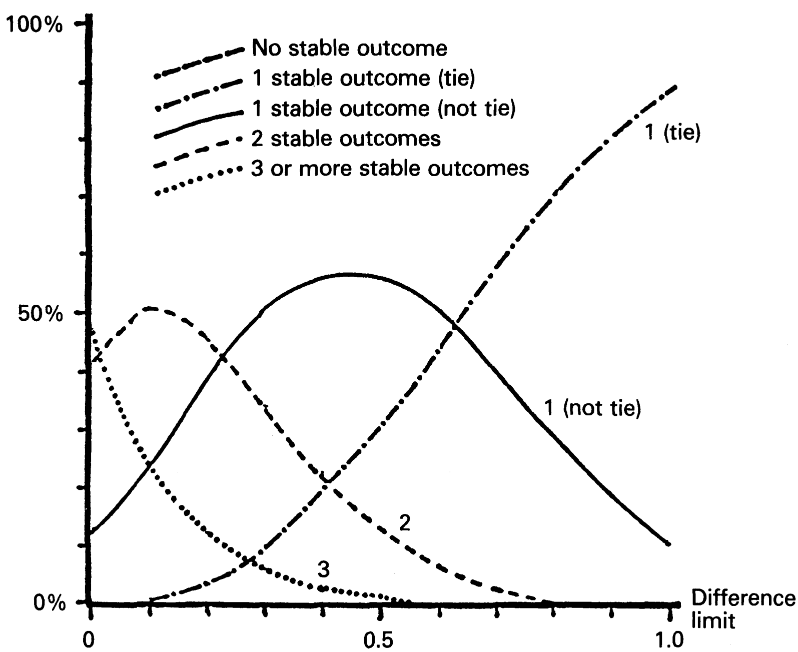


Diagram 3. 3 participants, rule of unanimity, 3 alternatives.

Instability is obviously a problematic situation. Whatever the outcome, there is a coalition that has something to gain by reopening the issue and repeating the voting procedure. Therefore, the decision process would seem to have a tendency to go on forever. In contrast, multi-stability and tie-stability involve no such tendency.

In actual political life, majority moves to reopen an issue seem to be fairly uncommon when preferences are unchanged. One possible explanation for this is that instability may have a smaller role in practical decision-making than what has been theoretically predicted.

However, this does not necessarily mean that the "ideal" case (one single, nontie, stable outcome) is more common than in theory. Instead, a decreased occurrence of instability may have been bought at the price of an increased occurrence of multi-stability and tie-stability. Unwillingness to take part in compromises is, as these simulation studies have shown, one possible mechanism for such an exchange.

## *6 Appendix*

Following are proofs of the theorems presented above.

*Proof of Theorem 2.5:* The theorem will be proved by the following example:

The set of alternatives is $\{x,y,z\}$, and the set of individuals is $\{1,2,3\}$. The voting procedure is by simple majority. Each individual has aggregative preferences (cf. Definitions 3.2 and 3.3).

Individual 1 ranks $[z,x]$ as best and $[y,y]$ as second-best.
Individual 2 ranks $[z,z]$ as best and $[y,y]$ as second-best.
Individual 3 ranks $[y,z]$ as best and $[y,x]$ as second-best.

All three individuals rank the rest of the possibilities lower than their second-best, in an order that need not be specified.

Outcome $y$ is $R$-stable since there is a voting pattern—namely, $\langle y,y,x\rangle$—that is not defeated by any $\pi$ with $r(\pi) \neq y$. However, outcome $y$ is not $A$-stable since all voting patterns with this outcome are defeated by $\langle y,y,z\rangle$ except $\langle y,y,z\rangle$ itself, which is defeated by $\langle x,z,z\rangle$.

*Proof of Theorem 3.7:* The proof consists of two main parts. In Part 1, it will be proved that if there is an $R$-stable voting pattern $\pi$ with $r(\pi) = \lambda$, then there is an $A$-stable voting pattern $\pi'$ with $r(\pi') = r(\pi) = \lambda$. In Part 2, it will be proved that if there is an $R$-stable voting pattern $\pi$ with $r(\pi) \neq \lambda$, then there is an $A$-stable voting pattern $\pi'$ with $r(\pi') = r(\pi)$. Since it is obvious that $A$-stability implies $R$-stability, this is sufficient to prove the theorem.

*Part 1:* Let $\pi$ be an $R$-stable voting pattern with $r(\pi) = \lambda$. $\pi'$ is constructed as follows: If $v(i,\pi) \in B(i)$, then $v(i,\pi') = v(i,\pi)$. If $v(i,\pi) \notin B(i)$, then $v(i,\pi') = b(i)$.

Let $i \in \mathit{diff}(\pi,\pi')$. Then $v(i,\pi') >_i v(i,\pi)$. Also, by $r(\pi) = \lambda$ and the tie avoidance condition, it follows that $r(\pi') \geq_i r(\pi)$. Thus $[r(\pi'),v(i,\pi')]$ $P_i[r(\pi),v(i,\pi)]$ for all $i \in \mathit{diff}(\pi,\pi')$. Thus if $\pi' \neq \pi$ then $\pi'$ defeats $\pi$. Since $\pi$ is $R$-stable, $r(\pi') = \lambda$.

Let $\pi''$ be any voting pattern that defeats $\pi'$.

Suppose that $r(\pi'') = \lambda$. Since $\pi''$ defeats $\pi'$, it follows from Definition 2.4 that $\mathit{diff}(\pi',\pi'') \neq \varnothing$, and that $[r(\pi''),v(i,\pi'')]P_i[r(\pi'),v(i,\pi')]$ holds for all

$i \in \mathit{diff}(\pi', \pi'')$. This, however, is impossible according to Definition 3.5, since $r(\pi'') = r(\pi')$ and $v(i, \pi') \in B(i)$. Thus $r(\pi'') \neq \lambda$.

Next, let $i$ be an element of $\mathit{diff}(\pi, \pi'')$. There are two cases according to whether or not $i \in \mathit{diff}(\pi', \pi'')$.

Case (i): $i \in \mathit{diff}(\pi', \pi'')$. Since $\pi''$ defeats $\pi'$, we have (by Definition 2.4) $[r(\pi''), v(i, \pi'')] P_i [r(\pi'), v(i, \pi')]$.

Case (ii): $i \notin \mathit{diff}(\pi', \pi'')$. It follows from $r(\pi) = \lambda$, $r(\pi'') \neq \lambda$ and tie avoidance that $r(\pi'') >_i r(\pi')$. From this and $v(i, \pi'') = v(i, \pi')$, it follows (by Definition 3.5) that $[r(\pi''), v(i, \pi'')] P_i [r(\pi'), v(i, \pi')]$.

Thus in both cases, i.e., for all $i \in \mathit{diff}(\pi, \pi'')$, it has been shown that $[r(\pi''), v(i, \pi'')] P_i [r(\pi'), v(i, \pi')]$.

Since $r(\pi') = r(\pi)$ and $v(i, \pi') \geq_i v(i, \pi)$, we have $[r(\pi'), v(i, \pi')] R_i [r(\pi), v(i, \pi)]$. From this and $[r(\pi''), v(i, \pi'')] P_i [r(\pi'), v(i, \pi')]$, it follows that $[r(\pi''), v(i, \pi'')] P_i [r(\pi), v(i, \pi)]$.

Thus if $\pi'$ can be defeated by any voting pattern $\pi''$, then $r(\pi'') \neq r(\pi)$ and $\pi''$ defeats $\pi$. Since $\pi$ is $R$-stable, this is impossible; so there can be no voting pattern that defeats $\pi'$. It follows that $\pi'$ is $A$-stable if $\pi$ is $R$-stable.

*Part 2:* Suppose that there is an $R$-stable $\pi$ with $r(\pi) \neq \lambda$. We need to find an $A$-stable $\pi'$ with $r(\pi') = r(\pi)$. This will be done in the following three steps:

(A) Construct a particular voting pattern $\pi'$ with $r(\pi') = r(\pi)$.
(B) Prove that for all $\pi''$, if $\pi''$ defeats $\pi'$, then $r(\pi'') \neq r(\pi')$.
(C) Prove that for all $\pi''$, if $\pi''$ defeats $\pi'$, then there is a voting pattern $\pi^*$ such that $r(\pi^*) = r(\pi'')$ and that $\pi^*$ defeats $\pi$.

Since these properties of $\pi^*$ contradict the $R$-stability of $\pi$, it can be concluded that there is no $\pi''$ that defeats $\pi'$, i.e., that $\pi'$ is $A$-stable.

*Part 2A:* Let $K(\pi)$ be the set of individuals such that $v(i, \pi) = r(\pi)$.

For each subset $w$ of $K(\pi)$, a voting pattern $\pi_w$ is defined as follows:

(1) If $v(i, \pi) \in B(i)$ or $i \in w$, then $v(i, \pi_w) = v(i, \pi)$.
(2) Otherwise, $v(i, \pi_w) = b(i)$.

According to nonobliquity (Definition 2.6), there is at least one $w \in K(\pi)$ such that $K(\pi_w)$ contains a minimal set of $r(\pi)$ voters for $r(\pi_w) = r(\pi)$. More precisely, $r(\pi_w) = r(\pi)$, and there is no proper subset $w'$ of $w$ such that $r(\pi_{w'}) = r(\pi)$.

Let $\pi' = \pi_w$ for any such minimal set $w$. Then $r(\pi') = r(\pi)$.

*Part 2B:* Let $\pi''$ be a voting pattern that defeats $\pi'$. We are going to prove that $r(\pi'') \neq r(\pi')$. Suppose that $\pi''$ defeats $\pi'$ and that $r(\pi'') = r(\pi')$.

Let $i \in \mathit{diff}(\pi', \pi'')$. Suppose that $v(i, \pi') \in B(i)$. Since $i \in \mathit{diff}(\pi', \pi'')$ and $\pi''$ defeats $\pi'$, it follows by Definition 2.4 that $[r(\pi''), v(i, \pi'')] P_i [r(\pi'), v(i, \pi')]$. This, however, is impossible (by Definition 3.5) since $r(\pi'') = r(\pi')$ and $v(i, \pi') \in B(i)$.

It can be concluded that if $i \in \mathit{diff}(\pi', \pi'')$, then $v(i, \pi') \notin B(i)$. It follows by the construction of $\pi'$ that $r(\pi'') \neq r(\pi')$.

*Part 2C:* We now have to prove that for all $\pi''$, if $\pi''$ defeats $\pi'$, then there is a voting pattern $\pi^*$ such that $r(\pi^*) = r(\pi'')$ and that $\pi^*$ defeats $\pi$.

Let $\pi''$ be any voting pattern that defeats $\pi'$. We construct $\pi^*$ as follows:

If $v(i,\pi'') = r(\pi'')$, then $v(i,\pi^*) = r(\pi'')$. Otherwise, $v(i,\pi^*) = v(i,\pi)$. It follows by nonobliquity that $r(\pi^*) = r(\pi'')$.

Let $i$ be any element of *diff*$(\pi,\pi^*)$. Then $v(i,\pi^*) = v(i,\pi'') = r(\pi^*) = r(\pi'')$. Furthermore, since $\pi''$ defeats $\pi'$, either $v(i,\pi'') = v(i,\pi')$ or $[r(\pi''), v(i,\pi'')]P_i[r(\pi'),v(i,\pi')]$. These will be treated as separate cases.

Case (a): $v(i,\pi'') = v(i,\pi')$. Then $v(i,\pi') = r(\pi'')$. By the construction of $\pi'$, for all $i$ either $v(i,\pi') \in B(i)$ or $v(i,\pi') = r(\pi')$. In the latter case, we would have $r(\pi') = v(i,\pi') = v(i,\pi'') = r(\pi'')$, which is impossible according to Part 2B of the present proof. Thus $v(i,\pi') \in B(i)$. Since $r(\pi^*) = r(\pi'') = v(i,\pi')$, it follows that $r(\pi^*) \in B(i)$.

Suppose that $v(i,\pi) \in B(i)$. Then, by the construction of $\pi'$ it follows that $v(i,\pi') = v(i,\pi)$. Thus $v(i,\pi) = v(i,\pi'') = v(i,\pi^*)$, contrary to $i \in$ *diff*$(\pi,\pi^*)$. It follows that $v(i,\pi) \notin B(i)$.

From $r(\pi^*) \in B(i)$, $v(i,\pi^*) = r(\pi^*)$ and $v(i,\pi) \notin B(i)$ it follows that $[r(\pi^*),v(i,\pi^*)]P_i[r(\pi),v(i,\pi)]$.

Case (b): $[r(\pi''),v(i,\pi'')]P_i[r(\pi'),v(i,\pi')]$.

We first observe that by the construction of $\pi'$, if $v(i,\pi') \neq v(i,\pi)$, then $v(i,\pi') = b(i)$. Therefore, $v(i,\pi') \geq_i v(i,\pi)$ for all $i$. From this and $r(\pi') = r(\pi)$ it follows that for all $i$, $[r(\pi'),v(i,\pi')]R_i[r(\pi),v(i,\pi)]$.

From this and $[r(\pi''),v(i,\pi'')]P_i[r(\pi'),v(i,\pi')]$, it follows that $[r(\pi''), v(i,\pi'')]P_i[r(\pi),v(i,\pi)]$ or, equivalently, $[r(\pi^*),v(i,\pi^*)]P_i[r(\pi),v(i,\pi)]$.

Summarizing cases (a) and (b), if $i \in$ *diff*$(\pi,\pi^*)$, then $[r(\pi^*),v(i,\pi^*)] P_i[r(\pi),v(i,\pi)]$. Thus $\pi^*$ defeats $\pi$. This concludes the proof.

*Proof of Theorem 3.8:* In the proof we will assume that $R_i$ is a preference relation that conforms to an outcome ordering $\geq_i$ and for which it is known, for all $x$ and $y$, whether or not $[x,x]R_i[y,b(i)]$ and whether or not $[y,b(i)] R_i[x,x]$. It will be shown that this information is sufficient to determine which voting patterns are ($A$-)stable.

*Part 1:* A voting pattern $\pi$ will be called a *candidate* (for stability) iff for all $i$, either $v(i,\pi) = r(\pi)$ or $v(i,\pi) \in B(i)$.

Let $\pi$ be a voting pattern that is not a candidate, and let $\pi'$ be such that for all $i$: If $v(i,\pi) = r(\pi)$ or $v(i,\pi) \in B(i)$, then $v(i,\pi') = v(i,\pi)$. Otherwise, $v(i,\pi') = b(i)$.

If $r(\pi) = \lambda$ then by tie avoidance $r(\pi') \geq_i r(\pi)$ for all $i \in I$. If $r(\pi) \neq \lambda$ then by nonobliquity $r(\pi') = r(\pi)$. Thus in both cases, $r(\pi') \geq_i r(\pi)$ for all $i \in I$.

Let $i \in$ *diff*$(\pi,\pi')$. Then from $r(\pi') \geq_i r(\pi)$ and $v(i,\pi') >_i v(i,\pi)$ it follows that $[r(\pi'),v(i,\pi')]P_i[r(\pi),v(i,\pi)]$. Thus $\pi'$ defeats $\pi$.

Since all noncandidates can be defeated, in our search for stable voting patterns we need only consider the candidates.

*Part 2:* Let $\pi$ be a candidate. Suppose that it can be defeated by a voting pattern $\pi'$ such that $r(\pi') = \lambda$. Furthermore let $\pi''$ be a voting pattern such that for all $i$, if $i \in$ *diff*$(\pi',\pi)$, then $v(i,\pi'') = b(i)$ and that otherwise $v(i,\pi'') = v(i,\pi)$.

Let $i \in$ *diff*$(\pi'',\pi)$. Then $i \in$ *diff*$(\pi',\pi)$, and therefore, since $\pi'$ defeats $\pi$, $[r(\pi'),v(i,\pi')]P_i[r(\pi),v(i,\pi)]$. By tie avoidance and $r(\pi') = \lambda$ it follows that $r(\pi'') \geq_i r(\pi')$. From $v(i,\pi'') \in B(i)$, it follows that $v(i,\pi'') \geq_i v(i,\pi')$. From $r(\pi'') \geq_i r(\pi')$ and $v(i,\pi'') \geq_i v(i,\pi')$, it follows that $[r(\pi''),v(i,\pi'')]$

$R_i[r(\pi'), v(i,\pi')]$. From this and $[r(\pi'), v(i,\pi')]P_i[r(\pi), v(i,\pi)]$, it follows that $[r(\pi''), v(i,\pi'')]P_i[r(\pi), v(i,\pi)]$.

Thus if $\pi$ can be defeated by a voting pattern with the outcome $\lambda$, it can be defeated by a voting pattern $\pi''$ such that if $i \in \mathit{diff}(\pi'', \pi)$, then $v(i,\pi'') \in B(i)$.

*Part 3:* Let $\pi$ be a candidate. Suppose that it can be defeated by a voting pattern $\pi'$ such that $r(\pi') \neq \lambda$. Furthermore let $\pi''$ be a voting pattern such that for all $i \in I$, if $v(i,\pi') = r(\pi')$ then $v(i,\pi'') = r(\pi')$, and if $v(i,\pi') \neq r(\pi')$ then $v(i,\pi'') = v(i,\pi)$. By nonobliquity, $r(\pi'') = r(\pi')$.

Let $i \in \mathit{diff}(\pi, \pi'')$. Then $v(i,\pi') = r(\pi') = v(i,\pi'') = r(\pi'')$. Thus $i \in \mathit{diff}(\pi, \pi')$. Since $\pi'$ defeats $\pi$ it follows that $[r(\pi'), v(i,\pi')]P_i[r(\pi), v(i,\pi)]$, or, equivalently, that $[r(\pi''), v(i,\pi'')]P_i[r(\pi), v(i,\pi)]$. Thus $\pi''$ defeats $\pi$.

It can be concluded that if a voting pattern $\pi$ can be defeated by a voting pattern with a nontie outcome, then it can be defeated by a voting pattern $\pi''$ with that outcome and with the property that if $i \in \mathit{diff}(\pi, \pi'')$, then $v(i,\pi'') = r(\pi'')$.

*Part 4:* It follows from Parts 2 and 3 that to determine whether a candidate $\pi$ is stable or not, it is sufficient to determine whether it is defeated by any voting pattern $\pi''$ such that if $i \in \mathit{diff}(\pi, \pi'')$, then either $v(i,\pi'') \in B(i)$ or $v(i,\pi'') = r(\pi'')$. From the fact that $\pi$ is a candidate, it follows that either $v(i,\pi) = r(\pi)$ or $v(i,\pi) \in B(i)$.

We may therefore conclude that comparisons between expressions of the forms $[x,x]$ and $[y, b(i)]$ are sufficient to determine the stability of outcomes. The theorem follows straightforwardly from this result.

*Proof of Theorem 4.2:* This follows directly from Theorem 3.8 and Definition 4.1.

*Proof of Part (1) of Theorem 4.3:* We will assume that $\langle u,c \rangle$ is a difference model, and construct from it an outcome ordering $\geq_i$ and a preference ordering $R_i$. We will then prove that $R_i$ conforms with $\geq_i$ (so that it is single-based) and that it conforms with $\langle u,c \rangle$. To prove this we need, according to Definitions 3.5 and 4.1, to prove the following:

(i) For all $x, x' \in A \cup \{\lambda\}$ and $y \in A$:
$[x,y]R_i[x',y]$ iff $x \geq_i x'$

(ii) For all $x, \in A \cup \{\lambda\}$ and $y, y' \in A$:
$[x,y]R_i[x,y']$ iff $y \geq_i y'$

(iii) For all $x, y \in A \cup \{\lambda\}$:
$x \geq_i y$ iff $u(x) \geq u(y)$

(iv) For all $x \in A \backslash B(i)$ and $y \in A \cup \{\lambda\}$:
$[x,x]R_i[y, b(i)]$ iff $u(x) - u(y) \geq c$, and

(v) For all $x \in A \backslash B(i)$ and $y \in A \cup \{\lambda\}$:
$[y, b(i)]R_i[x,x]$ iff $u(x) - u(y) \leq c$.

The *construction* will be as follows:

$$x \geq_i y \text{ holds iff } u(x) \geq u(y).$$

To define $R_i$, the following series of definitions will be used:

$s(x) = u(b(i)) - u(x)$

If $x = b(i)$, then $t(x) = 0$. Otherwise, $t(x) = \sqrt{(c^2 + 2 \cdot c \cdot s(x))}$.

$m(x,y) = \sqrt{((s(x)^2) + (t(y)^2))}$

$[x,y]R_i[z,w]$ iff $m(x,y) \leq m(z,w)$.

*Verification:*

(i) The construction yields that $[x,y]R_i[x',y]$ iff $m(x',y) \leq m(x,y)$, iff $\sqrt{s(x')^2 + t(y)^2} \leq \sqrt{s(x)^2 + t(y)^2}$, iff $s(x') \leq s(x)$, iff $u(x) \geq u(x')$, iff $x \geq_i x'$.

(ii) The construction yields that $[x,y]R_i[x,y']$ iff $m(x,y') \leq m(x,y)$, iff $\sqrt{s(x)^2 + t(y')^2} \leq \sqrt{s(x)^2 + t(y)^2}$, iff $t(y') \leq t(y)$, iff $s(y') \leq s(y)$, iff $u(y) \geq u(y')$, iff $y \geq_i y'$.

(iii) Holds by definition.

(iv) By the construction, $[x,x]R_i[y,b(i)]$ iff $m(x,x) \leq m(y,b(i))$, iff $\sqrt{s(x)^2 + t(x)^2} \leq \sqrt{s(y)^2 + t(b(i))^2}$, iff $s(x)^2 + c^2 + 2 \cdot c \cdot s(x) \leq s(y)^2$, iff $s(x) + c \leq s(y)$, iff $u(x) - u(y) \geq c$.

(v) This is proved in the same way as (iv).

*Proof of Part (2) of Theorem 4.3:* We will assume that $R_i$ is a preference relation that conforms to an outcome ordering $\geq_i$. We will construct a difference model $\langle u,c \rangle$ such that for all $x$ and $y$:

(a) $x \geq_i y$ iff $u(x) \geq u(y)$,

(b) $[x,x]R_i[y,b(i)]$ iff $u(x) - u(y) \geq c$, and

(c) $[x,x]I_i[y,b(i)]$ iff $u(x) - u(y) = c$.

Suppose that $\geq_i$ is not strict, i.e., that there are two elements $x$ and $x'$ such that $x \equiv_i x'$. Then according to (a), $u(x) = u(y)$. To simplify the proof we will assume that $\geq_i$ is a strict preference relation. Then there is a unique best element $b(i)$ in the $\geq_i$-ordering.

*Construction:*

The elements are assumed to be numbered in the $\geq_i$-ordering: $x_1, x_2, \ldots x_n$ with $x_1 = b(i)$.

We will use 1 (arbitrarily) as a difference limit. Furthermore, we will have use for a very small number, $\delta$. Let $\delta = (\frac{1}{2})^{n+1}$.

The values of $u(x_m)$ will be determined recursively, from $x_1$ downward.

$u(x_1)$ is assigned an arbitrary number.

For each $x_m$ in order, $u(x_m)$ is assigned a value according to whichever of the following three procedures is applicable:

(1) If there is an $x_k$ such that $x_k >_i x_m$ and $[x_k,x_k]I_i[x_m,b(i)]$, then let $u(x_m) = u(x_k) - 1$.

(2) If there are $x_k$ and $x_{k+1}$ such that $x_k >_i x_{k+1} >_i x_m$ and that $[x_k,x_k] P_i[x_m,b(i)]P_i[x_{k+1},x_{k+1}]$, then let $u(x_m) = u(x_{k+1}) + (u(x_k) - u(x_{k+1})) \cdot ((n + 1 - m)/(n + 1)) - 1$.

(3) If $[x_k,x_k]P_i[x_m,b(i)]$ for all $x_k >_i x_m$ then let $u(x_m) = u(x_{m-1}) - (1 + \delta)$.

*Verification:*

We are going to prove by induction that the construction satisfies the desired properties (a)–(c). It follows straightforwardly that (a)–(c) hold for $\{x_1,x_2\}$. For the induction we will assume that the three properties hold for $\{x_1,, \ldots x_{m-1}\}$.

*Case (1):* $u(x_m)$ was assigned a value according to Procedure 1 in the construction. Then there is an $x_k$ such that $x_k >_i x_m$, $[x_k,x_k]I_i[x_m,b(i)]$, and $u(x_m) = u(x_k) - 1$.

To prove *property* $(a)$ it is sufficient, by the induction hypothesis, to prove that $u(x_m) < u(x_{m-1})$. Suppose not. Then from $u(x_m) = u(x_k) - 1$ and $u(x_{m-1}) \leq u(x_m)$ we have $u(x_k) - u(x_{m-1}) \geq 1$. Thus, by the induction hypothesis, $[x_k,x_k]R_i[x_{m-1},b(i)]$. By this and $[x_k,x_k]I_i[x_m,b(i)]$ it follows that $[x_m,b(i)]R_i[x_{m-1},b(i)]$, which is impossible since $x_{m-1} >_i x_m$.

To prove *property* $(b)$, it is sufficient by the induction hypothesis to show that for all $x'$ with as yet assigned values, $[x',x']R_i[x_m,b(i)]$ iff $u(x') - u(x_m) \geq 1$.

From $[x_k,x_k]I_i[x_m,b(i)]$ it follows that $[x',x']R_i[x_m,b(i)]$ iff $[x',x']R_i[x_k,x_k]$, thus iff $x' \geq_i x_k$ and, by the induction hypothesis, iff $u(x') \geq u(x_k)$, i.e., iff $u(x') - u(x_m) \geq 1$.

To prove *property* $(c)$, it is sufficient by the induction hypothesis to show that for all $x'$ with as yet assigned values, $[x',x']I_i[x_m,b(i)]$ iff $u(x') - u(x_m) = 1$. Since $\geq_i$ is a strict ordering, there can be no $x'$ other than $x_k$ such that $[x',x']I_i[x_m,b(i)]$. By the induction hypothesis, $u(x') = u(x_k)$ iff $x' = x_k$, thus $u(x') - u(x_m) = 1$ iff $x' = x_k$.

*Case 2:* $u(x_m)$ was assigned a value according to Procedure (2) in the construction. Then there is an $x_k$ such that $x_k >_i x_{k+1} >_i x_m$, $[x_k,x_k]P_i[x_m,b(i)]P_i[x_{k+1},x_{k+1}]$, and $u(x_m) = u(x_{k+1}) + (u(x_k) - u(x_{k+1})) \cdot (n+1-m/n+1) - 1$.

To prove *property* $(a)$, it is again sufficient, by the induction hypothesis, to prove that $u(x_m) < u(x_{m-1})$. Suppose not. Then from $u(x_{m-1}) \leq u(x_m)$, $u(x_k) > u(x_{k+1})$ and $u(x_m) = u(x_{k+1}) + (u(x_k) - u(x_{k+1})) \cdot (n+1-m/n+1) - 1$ it follows that $u(x_k) - u(x_{m-1}) > 1$. Thus, by the induction hypothesis, $[x_k,x_k]P_i[x_{m-1},b(i)]$.

$u(x_{m-1})$ was assigned a value according to one of the three construction procedures. This will give rise to three subcases.

*Subcase* $(\alpha)$: $u(x_{m-1})$ was assigned a value according to Procedure 1. Then there is an $x_r$ such that $[x_r,x_r]I_i[x_{m-1},b(i)]$ and $u(x_{m-1}) = u(x_r) - 1$. Then by the induction hypothesis and $[x_k,x_k]P_i[x_{m-1},b(i)]$, it follows that $[x_k,x_k]P_i[x_r,x_r]$, so that $u(x_r) < u(x_k)$, thus $u(x_r) \leq u(x_{k+1})$. From this and $u(x_{m-1}) = u(x_r) - 1$, it follows that $u(x_{k+1}) - u(x_{m-1}) \geq 1$. Thus, $[x_{k+1},x_{k+1}]R_i[x_{m-1},b(i)]$. Then $[x_{k+1},x_{k+1}]R_i[x_m,b(i)]$, contrary to the conditions.

*Subcase* $(\beta)$: $u(x_{m-1})$ was assigned a value according to Procedure 2. Then there is an $x_r$ such that $x_r >_i x_{r+1} >_i x_{m-1}$, $[x_r,x_r]P_i[x_{m-1},b(i)]P_i[x_{r+1},x_{r+1}]$, and $u(x_{m-1}) = u(x_{r+1}) + (u(x_r) - u(x_{r+1})) \cdot (n+1-(m-1)/n+1) - 1$.

Suppose that $x_r >_i x_k$. Then $x_{r+1} \geq_i x_k$, so $u(x_{r+1}) \geq u(x_k)$. Since $u(x_{m-1}) > u(x_{r+1}) - 1$ and $u(x_m) < u(x_k) - 1$, it can then be concluded that $u(x_m) < u(x_{m-1})$, contrary to our assumptions.

Suppose that $x_r \equiv_i x_k$. Then by the strictness of $\geq_i, x_r = x_k$. It follows directly from the expressions defining $u(x_m)$ and $u(x_{m-1})$ that $u(x_{m-1}) - u(x_m) = (u(x_k) - u(x_{k+1})) \cdot (1/n+1)$. Since $u(x_k) - u(x_{k+1}) > 0$, it can again be concluded that $u(x_m) < u(x_{m-1})$, contrary to what we have supposed.

Next suppose that $x_k >_i x_r$. Then $x_{k+1} \geq_i x_r$, thus $[x_{k+1},x_{k+1}]R_i[x_r,x_r]$. We also know that $[x_m,b(i)]P_i[x_{k+1},x_{k+1}]$ and that $[x_r,x_r]P_i[x_{m-1},b(i)]$. It follows from these three conditions that $[x_m,b(i)]P_i[x_{m-1},b(i)]$. Thus $x_m \geq_i x_{m-1}$, contrary to the conditions.

*Subcase* $(\gamma)$: $u(x_{m-1})$ was assigned a value according to Procedure 3. Then $[x_r,x_r]P_i[x_{m-1},b(i)]$ for all $x_r$ such that $x_r >_i x_{m-1}$. Thus $[x_r,x_r]P_i[x_m,b(i)]$ for all such $x_r$. From this and $[x_k,x_k]P_i[x_m,b(i)]P_i[x_{k+1},x_{k+1}]$, it follows that $x_{k+1} = x_{m-1}$ and $x_k = x_{m-2}$. According to the construction of $u(x_{m-1})$, it follows that $u(x_{m-1}) = u(x_{m-2}) - (1+\delta)$. According to the construction of $u(x_m)$, it follows that $u(x_m) = u(x_{k+1}) + (u(x_k) - u(x_{k+1}))\cdot(n+1-m/n+1) - 1 = u(x_{m-1}) + (u(x_{m-2}) - u(x_{m-1}))\cdot(n+1-m/n+1) - 1 = u(x_{m-1}) + (1+\delta)\cdot(n+1-m/n+1) - 1$. Since $(1+\delta)\cdot(n+1-m/n+1) < 1$, it can be concluded that $u(x_m) < u(x_{m-1})$.

Thus in all three subcases, property (a) is satisfied.

To prove *property* $(b)$, it is sufficient by the induction hypothesis to show that for all $x'$ with as yet assigned values, $[x',x']R_i[x_m,b(i)]$ iff $u(x') - u(x_m) \geq 1$.

From $[x_k,x_k]P_i[x_m,b(i)]P_i[x_{k+1},x_{k+1}]$, it follows that $[x',x']R_i[x_m,b(i)]$ iff $x' \geq_i x_k$. By the construction, $x' \geq_i x_k$ iff $u(x') - u(x_m) \geq 1$.

To prove *property* $(c)$, it is sufficient by the induction hypothesis to show that for all $x'$ with as yet assigned values, $[x',x']I_i[x_m,b(i)]$ iff $u(x') - u(x_m) = 1$. Since in the construction, Procedure 2 was chosen for assigning $u(x_m)$ a value, there is no $x'$ such that $[x',x']I_i[x_m,b(i)]$. Furthermore, since $u(x_k) > u(x_{m+1}) > u(x_{k+1})$, there can be no $x'$ such that $u(x') - u(x_m) = 1$. Thus property (c) holds.

*Case 3:* $u(x_m)$ was assigned a value according to Procedure 3 in the construction. Then it is the case that $[x_k,x_k]P_i[x_m,b(i)]$ for all $x_k >_i x_m$ and that $u(x_m) = u(x_{m-1}) - (1+\delta)$.

To prove *property* $(a)$ it is sufficient, by the induction hypothesis, to prove that $u(x_m) < u(x_{m-1})$. This follows directly by the construction.

To prove *property* $(b)$ it is sufficient by the induction hypothesis to show that for all $x'$ with as yet assigned values, $[x',x']R_i[x_m,b(i)]$ iff $u(x') - u(x_m) \geq 1$.

By the conditions for Case 3, $[x',x']R_i[x_m,b(i)]$ iff $x' >_i x_m$. By the construction, $u(x') - u(x_m) \geq 1$ iff $x' >_i x_m$. Thus property (b) holds.

To prove *property* $(c)$, it is sufficient by the induction hypothesis to show that for all $x'$ with as yet assigned values, $[x',x']I_i[x_m,b(i)]$ iff $u(x') - u(x_m) = 1$. According to the conditions of Case 3, there is no $x'$ such that $[x',x']I_i[x_m,b(i)]$. According to the construction, there is also no $x'$ such that $u(x') - u(x_m) = 1$. This is sufficient to prove that property (c) holds in this case, thereby concluding the proof.

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