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# The Minimal System $L_0'$

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**Abstract** In this paper, I consider a tense logic of the class of arbitrary orders which has $H, G, H', G'$ as its tense operators and has the usual formal semantics. For the tense logic, a sound and complete axiomatic system $L_0'$ is established, using the method of deductively closed sets and maximal consistent sets.

***1 Introduction*** The purpose of this paper is to present a sound and complete axiomatic system $L_0'$ for the $H', G'$-tense logic of the class of arbitrary orders, which is a minimal system and can be seen as an intermediate system in a sense. The completeness proof here will use the method of deductively closed sets (DCSs) and maximal consistent sets (MCSs), first developed by Burgess [1] for the $S, U$-tense logics of linear orders, and then applied by Ming Xu [3] to the $S, U$-tense logics of nonlinear orders. The concerns of this paper are primarily technical, although I think there are other motives for developing such tense logics.

***2 Formal syntax*** The language for $H', G'$-tense logic takes as propositional variables $p_0, p_1, p_2, \ldots$. The (well-formed) formulas are built up from the variables using connectives $\sim$ (negation) and $\wedge$ (conjunction), and unary tense operators $H$ (strong past), $G$ (strong future), $H'$ (uninterruptedly past), and $G'$ (uninterruptedly future), where $H$, $G$, $H'$, and $G'$ can be read respectively as 'it has always been the case that', 'it is always going to be the case that', 'it has for some time been uninterruptedly the case that', and 'it is for some time going to be uninterruptedly the case that'. Connectives $\vee$ (disjunction), $\rightarrow$ (material conditional) and $\leftrightarrow$ (material biconditional), and constants T (truth) and $\perp$ (falsity) are introduced as abbreviations in the usual way. Unary tense operators $P$ (weak past), $F$ (weak future), $P'$ (recently), and $F'$ (soon) are introduced as abbreviations as follows: $P$ for $\sim H \sim$, $F$ for $\sim G \sim$, $P'$ for $\sim H' \sim$, and $F'$ for $\sim G' \sim$, and can be read respectively as 'it was the case that', 'it will be the case that', 'it has arbitrarily recently been the case that', and 'it will arbitrarily soon be the case that'. The mirror image of a formula is the result of replacing each occurrence

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of $H$ and $H'$ in the formula with $G$ and $G'$ respectively, and vice versa. We use $\alpha, \beta, \gamma, \delta$ to range over formulas, and $A, B, C, D$ (without or with superscriptions) to range over sets of formulas.

***3 Formal semantics*** A frame or order is a pair $F = (X,R)$ consisting of a nonempty set and a binary relation on it. A valuation in a frame $F$ is a function $V$ assigning each propositional variable a subset of $X$. We extend $V$ inductively to a function (still called $V$) defined on all formulas as follows:

$$\begin{aligned}
V(\sim\alpha) &= X - V(\alpha)\\
V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta)\\
V(H\alpha) &= \{x \in X : \forall y \in X(yRx \to y \in V(\alpha))\}\\
V(G\alpha) &= \{x \in X : \forall y \in X(xRy \to y \in V(\alpha))\}\\
V(H'\alpha) &= \{x \in X : \exists y \in X(yRx \wedge \forall z \in X(yRz \wedge zRx \to z \in V(\alpha)))\}\\
V(G'\alpha) &= \{x \in X : \exists y \in X(xRy \wedge \forall z \in X(xRz \wedge zRy \to z \in V(\alpha)))\}.
\end{aligned}$$

A formula $\alpha$ is valid in a frame $F$, in symbols, $F \models \alpha$ if $V(\alpha) = X$ for every valuation $V$ in $F$, and satisfiable in $F$ if $V(\alpha) \neq \varnothing$ for some valuation $V$ in $F$. Further, $\alpha$ is valid over a class $K$ of frames if it is valid in every $F \in K$, and satisfiable over $K$ if it is satisfiable in some $F \in K$. We use $Th(K)$ to denote the set of all formulas valid over $K$.

In this paper, what we are interested in is the class $K_0$ of all frames.

***4 Axiomatic system*** Our axiomatic system $L_0'$ takes as axioms all truth-functional tautologies, called nontense axioms. In addition, $L_0'$ takes as axioms the following list of extra schemas together with their mirror images (the latter being labeled A1b–A7b)

**(A1a)** $G(\alpha \to \beta) \to (G\alpha \to G\beta)$
**(A2a)** $\alpha \to GP\alpha$
**(A3a)** $G\alpha \wedge G'\beta \to G'(\alpha \wedge \beta)$
**(A4a)** $G\alpha \wedge F'\beta \to F'(\alpha \wedge \beta)$
**(A5a)** $G\alpha \to GH'\alpha$
**(A6a)** $F'\alpha \to GP\alpha$
**(A7a)** $G'\alpha \wedge F'\gamma_1 \wedge \ldots \wedge F'\gamma_n \to F(H'\alpha \wedge P(\alpha \wedge \gamma_1) \wedge \ldots \wedge P(\alpha \wedge \gamma_n))$. $(n = 0,1,2,\ldots)$

As rules of inference we take modus ponens (MP) plus temporal generalization (TG): From $\alpha$ to infer $G\alpha$ and $H\alpha$, and (TM'): from $\alpha \to \beta$ to infer $G'\alpha \to G'\beta$ and $H'\alpha \to H'\beta$.

Basic notions (relative to $L_0'$) such as thesishood, consequence, and consistency can be defined in the usual way. For any formula $\alpha$ and set $A$ of formulas, we write $\vdash \alpha$ to indicate that $\alpha$ is a thesis of $L_0'$, $A \vdash \alpha$ to indicate that $\alpha$ is a consequence in $L_0'$ of $A$, and $Cn(A)$ to denote the set of all consequences in $L_0'$ of $A$.

***5 Axiomatizability*** The result of this paper is the following theorem, which says that the $H', G'$-tense logic of $K_0$ can be axiomatized by $L_0'$.

**Theorem 5.1** (Axiomatizability) $Th(K_0) = Cn(\varnothing)$.

*Proof:* It is easy to show by induction that $Cn(\varnothing) \subseteq Th(K_0)$. In order to show the opposite inclusion $Th(K_0) \subseteq Cn(\varnothing)$ (i.e., $L_0'$ is complete for $K_0$), we must show that every consistent formula $\alpha_0$ is satisfiable. For this we need several preliminary lemmas and definitions.

**Derived Rules 5.2** The following rules of inference preserve thesishood:

(a) from $\alpha_1, \alpha_2, \ldots, \alpha_n$ to infer any truth-functional consequence $\beta$
(b) from $\alpha \to \beta$ to infer $F\alpha \to F\beta$ and $P\alpha \to P\beta$
(c) from $\alpha \leftrightarrow \beta$ and $\theta(\alpha/p)$ to infer $\theta(\beta/p)$
(d) from $\alpha$ to infer its mirror image.

*Proof:* Omitted.

**Theses 5.3** We list some theses for future reference, but omit their deductions in $L_0'$ (see Burgess [2]).

(a) $G\alpha \wedge F\beta \to F(\alpha \wedge \beta)$
(b) $G\alpha \wedge G\beta \leftrightarrow G(\alpha \wedge \beta)$
(c) $PG\alpha \to \alpha$

**Consistency Criterion 5.4** For any MCS $A$, DCS $B$, and formula $\alpha$ we have:

(a) any consistent set of formulas can be extended to a maximal consistent set.
(b) if $\alpha \notin B$, then there exists an MCS $D$ such that $B \subseteq D$ and $\alpha \notin D$.
(c) if $F\alpha \in A$, then $\alpha$ is consistent.
(d) if $P\alpha \in A$, then $\alpha$ is consistent.

*Proof:* (a) is Lindenbaum's Lemma. (b) follows from (a). For (c), if $\alpha$ is not consistent, then $\sim\alpha$ is a thesis; so $G \sim \alpha = \sim F \sim\sim \alpha$ is a thesis by TG; so $\sim F\alpha$ is a thesis by 5.2(c); so $F\alpha$ is not consistent and therefore cannot belong to the MCS $A$. (d) is the mirror image of (c), and so can be proved in the same way.

**Definition 5.5** For any MCSs $A$, $C$, and formula $\beta$, we let:

(a) $r'(A,\beta,C)$ iff whenever $H\alpha \in C$, $\alpha \wedge G'(\beta \wedge \alpha) \in A$, and whenever $P'\gamma \in C$, $F(\beta \wedge \gamma) \in A$;
(b) $r'(A,B,C)$ iff $B$ is a DCS and $r'(A,\beta,C)$ for every $\beta \in B$;
(c) $R'(A,B,C)$ iff $r'(A,B,C)$ and $r'(A,B',C)$ never holds for any proper extension $B'$ of $B$.

**Lemma 5.6** *Let $A, C$ be MCSs, $B$ a DCS, $\beta, \delta$ any formulas.*
(a) *If $r'(A,\beta,C)$ holds, then there exists a DCS $B'$ such that $r'(A,B',C)$ holds and $\beta \in B'$.*
(b) *If $r'(A,B,C)$ holds, then there exists a DCS $B'$ such that $R'(A,B',C)$ holds and $B \subseteq B'$.*
(c) *If $R'(A,B,C)$ holds and $\delta \notin B$, then there exists a $\beta_0 \in B$ such that $r'(A,\beta_0 \wedge \delta,C)$ does not hold.*

*Proof:* (a) Let $B' = Cn(\beta)$; then $B'$ is required. For (b), a required set can be found using Zorn's Lemma. For (c), let $B^* = Cn(B \cup \{\delta\})$. $B^*$ is a DCS and a proper extension of $B$. If for every $\beta \in B$, $r'(A, \beta \wedge \delta, C)$ holds, then since for every $\beta^* \in B^*$ there is a $\beta \in B$ with $\vdash \beta \wedge \delta \to \beta^*$, $r'(A, B^*, C)$ must hold, contrary to the maximality of $B$ with respect to the property $r'(A, -, C)$.

**Lemma 5.7** *Let $A, C$ be MCSs. The following are equivalent for any formula $\beta$:*
(a) *$r'(A, \beta, C)$ holds,*
(b) *whenever $G\alpha \in A$, $\alpha \wedge H'(\beta \wedge \alpha) \in C$, and whenever $F'\gamma \in A$, $P(\beta \wedge \gamma) \in C$.*

*Proof:* We only show that (a) implies (b). Assume (a). Let $G\alpha \in A$. If $\alpha \notin C$, then $\sim\alpha \in C$. By 5.3(c), we get $\sim PG\alpha = H \sim G\alpha \in C$. By (a), we then get $\sim G\alpha \in A$, a contradiction. If $H'(\beta \wedge \alpha) \notin C$, then $\sim H'(\beta \wedge \alpha) = P' \sim (\beta \wedge \alpha) \in C$. By (a), we get $F(\beta \wedge \sim (\beta \wedge \alpha)) \in A$. By 5.2(a) and (b), we then get $F \sim \alpha = \sim G\alpha \in A$, a contradiction again. So we have $\alpha \wedge H'(\beta \wedge \alpha) \in C$.

Now let $F'\gamma \in A$. If $P(\beta \wedge \gamma) \notin C$, then $\sim P(\beta \wedge \gamma) = H \sim (\beta \wedge \gamma) \in C$. By (a), we get $G'(\beta \wedge \sim (\beta \wedge \gamma)) \in A$. By TM′, we then get $G' \sim \gamma = \sim F'\gamma \in A$, a contradiction.

**Lemma 5.8** *Let $A, C$ be MCSs, and let $\beta$ be any formula.*
(a) If $G'\beta \in A$, *then there exist an MCS $D$ and a DCS $B$ such that $R'(A, B, D)$ and $\beta \in B$.*
(b) *If $H'\beta \in C$, then there exist an MCS $D$ and a DCS $B$ such that $R'(D, B, C)$ and $\beta \in B$.*

*Proof:* We only treat (a). As for (b), since it is the mirror image of (a), the proof is similar throughout.

Let $D_0 = \{\alpha \wedge H'(\beta \wedge \alpha) : G\alpha \in A\} \cup \{P(\beta \wedge \gamma) : F'\gamma \in A\}$. We claim that $D_0$ is consistent. Clearly, it will suffice to show that whenever $G\alpha \in A$, $F'\gamma_1, \ldots, F'\gamma_n \in A$, the following formula is consistent:

$$\delta = \alpha \wedge H'(\beta \wedge \alpha) \wedge P(\beta \wedge \gamma_1) \wedge \ldots \wedge P(\beta \wedge \gamma_n).$$

For this, by 5.4(c), it will suffice to show that $F\delta \in A$. When $G\alpha \in A$, $F'\gamma_1, \ldots, F'\gamma_n \in A$, since $G'\beta \in A$ by hypothesis, (A3a) yields $G'(\beta \wedge \alpha) \in A$, and (A7a) then yields

$$F(H'(\beta \wedge \alpha) \wedge P(\beta \wedge \alpha \wedge \gamma_1) \wedge \ldots \wedge P(\beta \wedge \alpha \wedge \gamma_n)) \in A.$$

By 5.2(a) and (b) it follows that

$$F(H'(\beta \wedge \alpha) \wedge P(\beta \wedge \gamma_1) \wedge \ldots \wedge P(\beta \wedge \gamma_n)) \in A.$$

And then by 5.3(a) it follows that $F\delta \in A$, proving our claim.

Now we can get an MCS $D$ extending $D_0$ by 5.4(a), and we have $r'(A, \beta, D)$ by construction using 5.7. Using 5.6(a) and (b) we then get a DCS $B$ such that $R'(A, B, D)$ and $\beta \in B$, completing the proof.

**Lemma 5.9** *Suppose we have $R'(A, B, C)$. Then we have:*
(a) *$\alpha \in B$ for every $G\alpha \in A$,*
(b) *$\gamma \in B$ for every $H\gamma \in C$.*

*Proof:* We only treat (a); (b) can be treated in a completely similar way.

Let $G\alpha \in A$. If $\alpha \notin B$, then by 5.6(c) and 5.7 there exist $\beta_0 \in B$ and $G\alpha_0 \in A$ with $\alpha_0 \wedge H'(\beta_0 \wedge \alpha \wedge \alpha_0) \notin C$, or there exist $\beta_0 \in B$ and $F'\gamma_0 \in A$ with $P(\beta_0 \wedge \alpha \wedge \gamma_0) \notin C$. But in the former case, since we have $G(\alpha \wedge \alpha_0) \in A$ by

5.3(b) and $\beta_0 \in B$, by 5.7, we then have $\alpha \wedge \alpha_0 \wedge H'(\beta_0 \wedge \alpha \wedge \alpha_0) \in C$, from which a contradiction follows. In the latter case, since we have $F'(\alpha \wedge \gamma_0) \in A$ by (A4a) and $\beta_0 \in B$, we then have $P(\beta_0 \wedge \alpha \wedge \gamma_0) \in C$ by 5.7, a contradiction again.

**Lemma 5.10** *For any MCSs $A, C$, we have:*
(a) *If $\alpha \in C$ for every $G\alpha \in A$, then $r'(A,T,C)$ holds.*
(b) *If $\alpha \in A$ for every $H\alpha \in C$, then $r'(A,T,C)$ holds.*

*Proof:* We only treat (a); (b) can be treated in a completely similar way. By 5.7 we must show that whenever $G\alpha \in A$, $\alpha \wedge H'\alpha \in C$, and whenever $F'\gamma \in A$, $P\gamma \in C$. If $G\alpha \in A$, we have $\alpha \in C$ by hypothesis, and $H'\alpha \in C$ by (A5a) and hypothesis; so we do have $\alpha \wedge H'\alpha \in C$. If $F'\gamma \in A$, we also have $P\gamma \in C$ by (A6a) and hypothesis, completing the proof.

**Lemma 5.11** *Suppose we have $r'(A,B,C)$. If $G'\beta \notin A$ or $H'\beta \notin C$, then there exist an MCS $D$ and DCSs $B', B''$ such that $\beta \notin D$, $B \subseteq D$ and $R'(A,B',D)$, $R'(D,B'',C)$.*

*Proof:* By 5.6(b) we can extend $B$ to a maximal DCS $B^*$ such that $R'(A,B^*,C)$. If $G'\beta \notin A$ or $H'\beta \notin C$, then we have $\beta \notin B^*$ by 5.5 or 5.7. Using 5.4(b) we can get an MCS $D$ such that $B^* \subseteq D$ and $\beta \notin D$, from which it follows that $r'(A,T,D), r'(D,T,C)$ by 5.9 and 5.10. Now using 5.6(a) and (b) we can get DCSs $B', B''$ such that $R'(A,B',D), R'(D,B'',C)$, completing the proof.

**Lemma 5.12** *Let $A, C$ be MCSs; let $\gamma$ be any formula.*
(a) *If $F\gamma \in A$, then there exist an MCS $D$ and a DCS $B$ such that $R'(A,B,D)$ and $\gamma \in D$.*
(b) *If $P\gamma \in C$, then there exist an MCS $D$ and a DCS $B$ such that $R'(D,B,C)$ and $\gamma \in D$.*

*Proof:* We only treat (a).

Let $D_0 = \{\gamma\} \cup \{\alpha : G\alpha \in A\}$. It is easy to verify that $D_0$ is consistent, so there exists an MCS $D$ extending $D_0$. By 5.10 we have $r'(A,T,D)$. By 5.6 there exists a DCS $B$ with $R'(A,B,D)$, completing the proof.

**Definition 5.13** A *chronicle* on a frame $(X,R)$ is a pair $(f,g)$ satisfying:

(C0) $f$ is a function from $X$ to the set of all MCSs.
(C1) $g$ is a function from $\{(x,y) : x,y \in X \wedge xRy\}$ to the set of all DCSs.
(C2) Whenever $xRy$, then $r'(f(x), g(x,y), f(y))$ holds.
(C3) Whenever $xRy$, $xRz$, and $zRy$, then $g(x,y) \subseteq f(z)$.

A chronicle $(f,g)$ on a frame $(X,R)$ is said to be *perfect* if it satisfies the following additional conditions, as well as their mirror images (C4b, C5b, and C6b respectively):

(C4a) Whenever $G'\beta \in f(x)$, there is some $y \in X$ with $xRy$ and $\beta \in g(x,y)$.
(C5a) Whenever $xRy$ and $G'\beta \notin f(x)$, there is some $z \in X$ with $xRz$, $zRy$, and $\beta \notin f(z)$.
(C6a) Whenever $F\gamma \in f(x)$, there is some $y \in X$ with $xRy$ and $\gamma \in f(y)$.

**Definition 5.14** Fix a denumerably infinite set $W$. Let $M$ be the set of all quadruples $(X,R,f,g)$ such that:

(a) $X$ is a nonempty finite subset of $W$
(b) $R$ is an antisymmetric binary relation on $X$
(c) $(f, g)$ is a chronicle on $(X, R)$.

**Chronicle Lemma 5.15** *For any perfect chronicle* $(f, g)$ on $(X, R)$, define a valuation $V$ in $(X, R)$ by letting

(*) $V(p_i) = \{x \in X : p_i \in f(x)\}$.

Then (*) in fact holds for all formulas $\alpha$.

*Proof:* By induction on the complexity of $\alpha$. Details are omitted.

By Lemma 5.15, in order to show that every consistent formula $\alpha_0$ is satisfiable, it will suffice to construct a perfect chronicle $(f, g)$ on some frame $(X, R)$ such that $\alpha_0 \in f(x_0)$ for some $x_0 \in X$. Generally speaking, a chronicle $(f, g)$ on a frame $(X, R)$ with $(X, R, f, g) \in M$ is not a perfect chronicle; but we have the following.

**Counterexample Lemma 5.16** *Let* $\mu = (X, R, f, g) \in M$ *and suppose* $x, \beta$ *(respectively,* $y, \beta$*) constitute a counterexample to* (C4a) *(respectively,* (C4b)*) for* $\mu$. *Then there exists an extension* $\mu' = (X', R', f', g') \in M$ *of* $\mu$ *for which* $x, \beta$ $(y, \beta)$ *do not constitute a counterexample to* (C4a) ((C4b))

*Proof:* The desired extension can be found by applying 5.8(a) to $A = f(x)$ (respectively, 5.8(b) to $C = f(y)$).

**Counterexample Lemma 5.17** *Let* $\mu = (X, R, f, g) \in M$ *and suppose* $x, y, \beta$ *constitute a counterexample to* (C5a) *(respectively* (C5b)*) for* $\mu$. *Then there exists an extension* $\mu' = (X', R', f', g') \in M$ *of* $\mu$ *for which* $x, y, \beta$ *do not constitute a counterexample to* (C5a) ((C5b)).

*Proof:* In either case, the desired extension can be found by applying 5.11 to $A = f(x)$, $B = g(x, y)$, and $C = f(y)$.

**Counterexample Lemma 5.18** *Let* $\mu = (X, R, f, g) \in M$ *and suppose* $x, \gamma$ *(respectively,* $y, \gamma$*) constitute a counterexample to* (C6a) *(respectively* (C6b)*) for* $\mu$. *Then there exists an extension* $\mu' = (X', R', f', g') \in M$ *of* $\mu$ *for which* $x, \gamma$ $(y, \gamma)$ *do not constitute a counterexample to* (C6a) ((C6b)).

*Proof:* The desired extension can be found by applying 5.12(a) to $A = f(x)$ (respectively, 5.12(b) to $C = f(y)$).

*Proof of the Completeness of* $L_0'$ *for* $K_0$*:* We will construct a perfect chronicle $(f, g)$ on some frame $(X, R)$ containing an $x_0$ with $\alpha_0 \in f(x_0)$ for any consistent formula $\alpha_0$ to complete the proof. Fix an enumeration $x_0, x_1, x_2, \ldots$ of $W$ and an enumeration $\alpha_0, \alpha_1, \alpha_2, \ldots$ of all formulas. We assign code numbers to the conditions (C4a ~ C6b) as follows:

| | |
|---|---|
| $2 \cdot 11^i \cdot 13^i \cdot 17^k$ | to (C4a) with $x = x_i$, $\beta = \alpha_k$ |
| $3 \cdot 11^j \cdot 13^j \cdot 17^k$ | to (C4b) with $y = x_j$, $\beta = \alpha_k$ |
| $4 \cdot 11^i \cdot 13^j \cdot 17^k$ | to (C5a) with $x = x_i$, $y = x_j$, $\beta = \alpha_k$ |
| $5 \cdot 11^i \cdot 13^j \cdot 17^k$ | to (C5b) with $x = x_i$, $y = x_j$, $\beta = \alpha_k$ |
| $6 \cdot 11^i \cdot 13^i \cdot 17^k$ | to (C6a) with $x = x_i$, $\gamma = \alpha_k$ |
| $7 \cdot 11^j \cdot 13^j \cdot 17^k$ | to (C6b) with $y = x_j$, $\gamma = \alpha_k$. |

Fix an MCS $C_0$ with $\alpha_0 \in C_0$ and let $\mu_0 = (X_0, R_0, f_0, g_0)$ where $X_0 = \{x_0\}$, $R = \emptyset$, $f = \{(x_0, C_0)\}$, $g = \emptyset$. If $\mu_n$ is defined, consider the condition, which among all those which are counterexamples for $\mu_n$, has the least code number (otherwise, $(f_n, g_n)$ would be the desired perfect chronicle on $(X_n, R_n)$). Let $\mu_{n+1}$ be an extension of $\mu_n$ for which that condition is no longer a counterexample, as provided by the Counterexample Lemmas above. Let $(X, R, f, g)$ be the union of the $\mu_n = (X_n, R_n, f_n, g_n)$. It is readily verified that $(f, g)$ and $(X, R)$ are desired.

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