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A Structurally Complete Fragment of Relevant Logic

JOHN K. SLANEY and ROBERT K. MEYER

This note contains a proof that the implication-conjunction fragment of the relevant logic **R** is *structurally complete*: in it, every admissible rule is derivable. The analogous result for the same fragment of intuitionist logic has been known at least since the early 1970's, and indeed the generalization to all fragments of intuitionist propositional logic not containing both implication and disjunction is reported in Mints [4]. The extension to relevant logic, however, is nontrivial and so far has been secured only for the one fragment.[1] In order to make this paper self-contained, and because the argument in the relevant case follows it closely, we begin by exhibiting the proof for intuitionist logic.[2]

By a *rule*, for the purposes of this note, is meant an object

$$A_1, \ldots, A_n \Rightarrow B$$

where the A_i and B are formulas in the vocabulary of a logic. Such a rule is *admissible* iff for every substitution σ of formulas for atoms, if each $\sigma(A_i)$ is a theorem of the logic, so is $\sigma(B)$. A rule is *derivable* in the sense appropriate to this note iff the special theory obtained by taking all of the A_i as proper axioms has B as a theorem. In the context of intuitionist logic this amounts to the logical provability of the formula

$$(A_1 \wedge \ldots \wedge A_n) \rightarrow B.$$

Note that for present purposes, such transformations as uniform substitution itself do not count as rules, so there are no trivial counterexamples to the conjecture that admissibility in our sense and derivability in our sense coincide. What has to be shown is that there are no nontrivial ones either.

The result for $\mathbf{J}_{\&}$, the implication-conjunction fragment of intuitionist logic, is very easy. Since the premises of a rule might as well be conjoined, we lose no generality in restricting attention to single premise rules. Thus, let

$$A \Rightarrow B$$

be admissible. Let σ be the substitution which replaces each atom p by the corresponding formula $A \rightarrow p$.

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Lemma 1 *Let B be any formula of $\mathbf{J}_{\&}$. Then $\sigma(B) \dashv\vdash A \rightarrow B$.*

Proof: Proof is by induction on the complexity of B. We note that if B is $C \wedge D$ then, by the induction hypothesis, $\sigma(B)$ is equivalent to $(A \rightarrow C) \wedge (A \rightarrow D)$ which is equivalent to $A \rightarrow (C \wedge D)$; and if B is $C \rightarrow D$ then $\sigma(B)$ is equivalent to $(A \rightarrow C) \rightarrow (A \rightarrow D)$ which is equivalent to $A \rightarrow (C \rightarrow D)$.

Theorem 2 (Mints) *Let $A \Rightarrow B$ be an admissible rule of $\mathbf{J}_{\&}$. Then $A \rightarrow B$ is a theorem of $\mathbf{J}_{\&}$.*

Proof: For proof, consider the substitution σ defined above. Since all instances of the rule are admissible, if $\vdash \sigma(A)$ then $\vdash \sigma(B)$. But by Lemma 1, $\sigma(A)$ is equivalent to $A \rightarrow A$ and so is a theorem, whereas $\sigma(B)$ is equivalent to $A \rightarrow B$.

So much for the proof with respect to intuitionist logic **J**. As usual the analogous result for relevant logic is a little more delicate. The problem is the relevant invalidity of the principle K

$$A \vdash B \rightarrow A$$

which is crucially involved in the proof. It is needed, in fact, to establish the equivalence

$$A \rightarrow (B \rightarrow C) \dashv\vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$$

or, more specifically, the right-to-left half of it. A typical proof of that might proceed

$$\dfrac{\dfrac{B \vdash A \rightarrow B}{(A \rightarrow B) \rightarrow (A \rightarrow C) \vdash B \rightarrow (A \rightarrow C)}\ \text{suffixing}}{(A \rightarrow B) \rightarrow (A \rightarrow C) \vdash A \rightarrow (B \rightarrow C)}\ \text{permuting}$$

The moves of suffixing and permutation are relevantly acceptable, but the axiom is not. To address this problem, we look again at our working definition of derivability of rules. To give the conjecture a chance, we take a rule $A \Rightarrow B$ to be derivable in the relevant logic $\mathbf{R}_{\&}$ not just when B is in the $\mathbf{R}_{\&}$ theory whose proper axiom is A (for that would require that $A \rightarrow B$ be an $\mathbf{R}_{\&}$ theorem, which would be too strong a condition), but when B is in the *regular* $\mathbf{R}_{\&}$ theory whose proper axiom is A. Recall (from Meyer and Slaney [7], for example) that an $\mathbf{R}_{\&}$ theory is regular iff it contains all the theorems of $\mathbf{R}_{\&}$. What derivability now amounts to in terms given by deduction theorems is that the enthymematic implication $(t \wedge A) \rightarrow B$ is provable. The sentential constant t is a formula such that for all A

$$t \rightarrow A \dashv\vdash A$$

where the turnstiles represent provability in $\mathbf{R}_{\&}$ of the corresponding relevant conditionals.

The constant t is not officially in the vocabulary of $\mathbf{R}_{\&}$, nor can it be added without upsetting the proof to be given below; but it can be simulated for our purposes. A given rule, involving formulas A and B, will be in a particular finite vocabulary. That is, the atomic formulas occurring in A or B will be a finite set $\{p_1, \ldots, p_m\}$. A suitable t for formulas in this vocabulary will be

$$(p_1 \to p_1) \wedge \ldots \wedge (p_{m+1} \to p_{m+1})$$

so while A and B are fixed for the purposes of the theorem let us so define t. For a discussion of t surrogates of this kind, see Meyer [3]. For more on sentential constants generally, see Slaney [5] and [6]. We follow our usual convention of defining an enthymematic hook

$$C \supset D =_{df} (t \wedge C) \to D$$

where this time, however, t is the surrogate generated from the atoms in the chosen admissible rule, not the all-purpose constant. Now it is well known that $\supset$ is very close to intuitionist implication (see, for example, Anderson and Belnap [1] and Meyer [2]). Specifically, the positive fragment of intuitionist logic is exactly the fragment of **R** with conjunction, disjunction, and the hook as connectives. The pure implication and the implication-conjunction fragments of intuitionist logic correspond similarly to enthymematic parts of **R**.

The system $\mathbf{R}_{\&}$ is that fragment of **R** whose connectives are $\to$ and &. For the record, we take its axioms to be the instances of

Axiom 1 $A \to A$
Axiom 2 $(A \to B) \to ((B \to C) \to (A \to C))$
Axiom 3 $A \to ((A \to B) \to B)$
Axiom 4 $(A \to (A \to B)) \to (A \to B)$
Axiom 5 $(A \wedge B) \to A$
Axiom 6 $(A \wedge B) \to B$
Axiom 7 $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$.

The primitive rules are detachment and adjunction

(DET) $\quad A \to B, A \Rightarrow B$

(ADJ) $\quad A, B \Rightarrow A \wedge B$.

As in the case of $\mathbf{J}_{\&}$, any finitary rule can be considered to have a single premise just by conjoining; so the one-premise case is general. Let $A \Rightarrow B$ be such a rule, let t and $\supset$ be defined as above, and let τ be the substitution which puts in $A \supset p$ for each p.

Lemma 3 *Let C be any subformula of A or of B. Then both $t \to C \vdash C$ and $C \vdash t \to C$ in* $\mathbf{R}_{\&}$.

Proof: That $t \to C \vdash C$ is obvious given $\vdash t$ and Axiom 3. The converse is proved by induction on the complexity of C. It is easier to think of proving the permuted form $t \vdash C \to C$. The case in which C is atomic is trivial, and the induction steps hardly less so. If C is a conjunction $D \wedge E$ where $t \vdash D \to D$ and $t \vdash E \to E$, we apply a little lattice logic and the $\mathbf{R}_{\&}$ fact

$$(D \to D) \wedge (E \to E) \vdash (D \wedge E) \to (D \wedge E).$$

Finally, if C is an implication $D \to E$, where $t \vdash D \to D$, we simply apply Axiom 2 with $D \to D$ as antecedent.

Lemma 4 *Let C be any subformula of A or of B. Then* $\tau(C) \dashv\vdash A \supset C$.

Proof: Proof is by induction on complexity as before. We should, however, check out the troublesome induction case in which C is an implication $D \to E$. We need to show

$$(A \supset (D \to E)) \dashv\vdash (A \supset D) \to (A \supset E).$$

From left to right this is obvious. From right to left it comes down to proving the R theorem

$$((A \supset D) \to (A \supset E)) \to (A \supset (D \to E)).$$

Lemma 3 starts us off:

$$\dfrac{\dfrac{D \to (t \to D)}{D \to (A \supset D)}\ \text{strengthening an antecedent}}{\dfrac{((A \supset D) \to (A \supset E)) \to (D \to (A \supset E))}{((A \supset D) \to (A \supset E)) \to (A \supset (D \to E))}\ \text{permuting antecedents}}\ \text{suffixing}$$

The rest of the proof follows that of Lemma 1.

Theorem 5 *Let* $A \Rightarrow B$ *be an admissible rule of* $\mathbf{R}_{\&}$. *Then* $A \supset B$ *is a theorem of* $\mathbf{R}_{\&}$.

Proof: The proof is as for Theorem 2. If the rule is admissible, then if $\tau(A)$ is a theorem of $\mathbf{R}_{\&}$ so is $\tau(B)$. But $\tau(A)$ *is* a theorem, and $\tau(B)$ is equivalent to $A \supset B$.

Theorems about the admissibility of rules are notoriously sensitive to choice of vocabulary and to the exact formulation of the logic in question. On the latter point we are inclined to go somewhat against modern trends by taking the *theorems* of a logic to be the guide to the meaning of 'derivable'. For a rule to be derivable is for the logic to tell us that there is an available inference from its premises to its conclusion, which is what implication connectives are for. Hence, we look to the theorems for a guide to the valid implications and to the deduction theorem to translate these into derivable rules. In relevant logic, as in such systems as linear logic, Łukasiewicz many-valued logics, and even modal logics, complexities arise from the fact that there are several types of implicative formula and several different forms of deduction theorem to go with them. To get a worthwhile theorem for the purposes of this paper we had to choose the "right" implication, which for $\mathbf{R}_{\&}$ was a vocabulary-relative enthymematic one.

The issue of which connectives and other particles are in the language is critical. Both intuitionistically and relevantly, the induction to the main lemma breaks down in the presence of disjunction, since in neither system can we prove $A \to (B \vee C) \to ((A \to B) \vee (A \to C))$. We have hopes of a similar theorem for the pure implication fragment of $\mathbf{R}$, though the technique of this paper—of looking for an appropriate implication connective and working syntactically—is not applicable relevantly in such a poor vocabulary so we expect to have to resort to algebraic methods. Most annoying of all is the failure of the argument when the real sentential constant t, as opposed to its local surrogate, is in the language. The problem is that the base case of the induction fails as stated, for $\tau(t)$ is t, which is not equivalent to $A \supset t$. It is enthymematically equivalent, but were we to replace the condition to be proved in Lemma 4 with the alternative that $\tau(C) \supset (A \supset C)$ and $(A \supset C) \supset \tau(C)$ be theorems, the induction would break down on the $\rightarrow$ case. We conjecture that any admissible rule in the implication-conjunction-t fragment of **R** remains admissible when the local t-surrogate is substituted for t. If this is so, the addition of t to the language does not destroy the results of this paper, since by the argument above the rule with t-surrogates in place of t is derivable whence the original rule containing t is derivable.[3] We further add that we do not see how our conjecture can fail to be true, but failure to see how p does not, alas, entail not-p.

We have, then, observed a satisfyingly simple proof that every rule admissible in $\mathbf{J}_{\&}$ is derivable. However, the extension of the same result to $\mathbf{R}_{\&}$ is not so trivial; and it is certainly new with this paper. We like the clear example of enthymematic implication paying its way in relevant logic, and are pleased to see the device of defining a local surrogate for the constant t being put to an essential use. The questions of what further fragments of what related systems show the same behavior we leave open.

Acknowledgment This note originated as a solution to what had been conveyed to us as a conjecture concerning intuitionist logic. We wish to thank Lloyd Humberstone and the anonymous referee for drawing our attention to the pertinent work of Mints, Prucnal, Pogorzelski, and others.

NOTES

1. That is, only for one fragment of **R** itself. There are structural completeness results in Tokarz [8] for certain extensions of the semi-relevant logic **RM**.
2. The proof is essentially that of [4], where a related result is attributed to Prucnal. The same was also obtained recently (by a different argument) by Wil Dekkers. We take this opportunity to thank Dekkers for illuminating conversations on this and related topics.
3. It would be in the spirit of results such as those of this paper to allow substitution for t, in which case there would be no bar to adding it to the language. Such a move is rather unsatisfactory, however, unless again it can be shown that no admissible rules are lost by such an apparent strengthening of the criterion for admissibility.

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Automated Reasoning Project
Australian National University
Canberra, ACT 26 01
Australia