Notre Dame Journal of Formal Logic
Volume 33, Number 4, Fall 1992

Refutation Calculi for Certain Intermediate Propositional Logics

TOMASZ SKURA

Abstract Using simple algebraic methods we give Łukasiewicz-style refutation calculi for the following intermediate logics: finite logics, **LC**, Yankov's logic, the logic of the weak law of excluded middle, Medvedev's logic, and certain logics without the finite model property.

0 Introduction There are at least three reasons for being interested in Łukasiewicz-style refutation calculi for propositional logics:

1. Such a formulation together with the usual axiomatic system provides a uniform complete characterization of a logic.
2. If a logic has such a refutation procedure then the set of its nontheorems is recursively enumerable (r.e.) so that the logic is decidable as long as the set of its theorems is also r.e. This method of obtaining decidability results is both simple and very general (cf. Skura [13],[14]).
3. A formulation of this kind expresses a syntactic property uniquely characterizing a logic (cf. Skura [12]). The more elegant such a formulation, the more intuitive the property.

In [6] Łukasiewicz introduced the following refutation rules:

$$(\mathbf{r}_{\mathrm{sb}}) \qquad \frac{\dashv e(\alpha)}{\dashv \alpha} \quad (e(\alpha) \text{ is a substitution instance of } \alpha),$$

$$(\mathbf{r}_{\mathrm{mp}}) \qquad \frac{\vdash \alpha \to \beta \qquad \dashv \beta}{\dashv \alpha}$$

("$\dashv\alpha$" reads "α is refutable".)

Of course $\mathbf{r}_{\mathrm{sb}}$ reverses the substitution rule, and $\mathbf{r}_{\mathrm{mp}}$ reverses the modus ponens rule.

Received November 26, 1990; revised August 1, 1991

It is easily seen that the above rules together with a fixed propositional variable p axiomatize the set of all formulas not provable in the classical propositional logic (**C**). (We also say that every nontheorem of **C** is refutable by the formula p and the rules $\mathbf{r}_{\mathrm{sb}}, \mathbf{r}_{\mathrm{mp}}$.)

In [12] it is proved that every nontheorem of the intuitionistic propositional logic (**INT**) is refutable by the formula p and the rules $\mathbf{r}_{\mathrm{sb}}, \mathbf{r}_{\mathrm{mp}}$,

$$(\mathbf{r}_n) \qquad \frac{\dashv(\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_i \ (1 \leq i \leq n)}{\dashv(\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_1 \vee \ldots \vee \alpha_n} \qquad (n \geq 2).$$

The rules $\mathbf{r}_n$ reverse the generalized disjunction property.

In this paper we give refutation calculi for some of the more important intermediate logics using simple algebraic methods.

1 Preliminaries Let For be the set of all formulas generated from the set Var of propositional variables by the connectives $\neg$, $\wedge$, $\vee$, $\to$, and let For^- be the set of all positive formulas (i.e., negation-free formulas). We write $\alpha \equiv \beta$ instead of $(\alpha \to \beta) \wedge (\beta \to \alpha)$. For any $\alpha \in \mathrm{For}$, the symbol $\mathrm{Var}(\alpha)$ will denote the set of all propositional variables occurring in α. Of course $\mathrm{Var}(X) = \bigcup\{\mathrm{Var}(\alpha) : \alpha \in X\}$ for every $X \subseteq \mathrm{For}$. By an intermediate logic we mean any set $\mathbf{INT} \subseteq \mathbf{L} \subsetneq \mathrm{For}$ closed under modus ponens and substitution, where **INT** is the set of all theorems of **INT**.

Heyting algebras (or pseudo-Boolean algebras) will be denoted by script capitals $\mathcal{A}, \mathcal{B}, \ldots$. The unit element, the zero element, the lattice ordering, and the operations of $\mathcal{A}$ are denoted by $1_{\mathcal{A}}$, $0_{\mathcal{A}}$, $\leq_{\mathcal{A}}$, and $\neg_{\mathcal{A}}$, $\wedge_{\mathcal{A}}$, $\vee_{\mathcal{A}}$, $\to_{\mathcal{A}}$ respectively. The subscript $\mathcal{A}$ is often omitted. The symbol $\mathcal{A}^-$ denotes the algebra obtained from $\mathcal{A}$ by deleting the operation $\neg_{\mathcal{A}}$. We say that $\mathcal{A}$ is strongly compact ($\mathcal{A} \in \mathrm{SC}$) iff there is a greatest element in $\mathcal{A} - \{1_{\mathcal{A}}\}$ (denoted by $*_{\mathcal{A}}$). $\mathcal{A} \in \mathrm{FSC}$ iff $\mathcal{A} \in \mathrm{SC}$ and $\mathcal{A}$ is finite. The symbol $\mathcal{A} \oplus \mathcal{B}$ denotes the Troelstra sum of $\mathcal{A}$ and $\mathcal{B}$ (i.e., $\mathcal{A} \oplus \mathcal{B}$ is the result of putting $\mathcal{B}$ on top of $\mathcal{A}$ and identifying $0_{\mathcal{B}}$ with $1_{\mathcal{A}}$). The Jaśkowski sequence is the following (cf. [3]): $\mathcal{J}_0$ is the two-element Boolean algebra, $\mathcal{J}_n = (\mathcal{J}_{n-1}^n) \oplus \mathcal{J}_0$ $(n > 0)$. We write $\mathcal{A}\oplus$, $\oplus\mathcal{A}$ instead of $\mathcal{A} \oplus \mathcal{J}_0$, $\mathcal{J}_0 \oplus \mathcal{A}$ respectively. $\Pi\{\mathcal{A}_t : t \in T\}$ is as usual the direct product of a family of algebras $\{\mathcal{A}_t : t \in T\}$.

$E(\mathcal{A})$ is the set of all tautologies of $\mathcal{A}$, i.e., $\alpha \in E(\mathcal{A})$ iff $v(\alpha) = 1_{\mathcal{A}}$ for every valuation $v : \mathrm{For} \to \mathcal{A}$ in $\mathcal{A}$. Of course $E(\mathcal{K}) = \bigcap\{E(\mathcal{A}) : \mathcal{A} \in \mathcal{K}\}$ for any family $\mathcal{K}$ of Heyting algebras.

With every countable class $\mathcal{K}$ of countable Heyting algebras we associate a one-one function $f_{\mathcal{K}}$ from $Z = \bigcup\{\mathcal{A} : \mathcal{A} \in \mathcal{K}\}$ into Var. For every $x \in Z$, $f_{\mathcal{K}}(x)$ will be denoted by p_x. For any countable $\mathcal{A}$ we define the set $D(\mathcal{A})$ as follows (cf. Yankov [18], Wroński [16]):

$$D(\mathcal{A}) = \{(p_x \otimes p_y) \equiv p_{x \otimes y} : x, y \in \mathcal{A}, \otimes \in \{\wedge, \vee, \to\}\} \cup \{\neg p_x \equiv p_{\neg x} : x \in \mathcal{A}\}.$$

If $\mathcal{A}$ is finite then $\delta_{\mathcal{A}} = \bigwedge D(\mathcal{A})$. If $\mathcal{A} \in \mathrm{FSC}$ then its characteristic formula $\chi(\mathcal{A}) = \delta_{\mathcal{A}} \to p_{*_{\mathcal{A}}}$ (cf. [18]). For any finite algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ by $\delta^1_{\mathcal{A}_1}, \ldots, \delta^n_{\mathcal{A}_n}$ we mean the results of substituting distinct variables for distinct variables in $\delta_{\mathcal{A}_1}, \ldots, \delta_{\mathcal{A}_n}$ in such a way that $\mathrm{Var}(\delta^i_{\mathcal{A}_i}) \cap \mathrm{Var}(\delta^j_{\mathcal{A}_j}) = \varnothing$ for all $i \neq j$. The variable substituted for p_x in $\delta_{\mathcal{A}_i}$ is denoted by p^i_x.

2 Some general theorems

Definition 2.1 Let $\mathfrak{A}$ be a Heyting algebra, $\psi \in \mathrm{For}$, and let F be a function from $\mathfrak{A}$ into For. We say that the pair (F, ψ) *expresses* $\mathfrak{A}$ iff the following hold true:

(i) There is a valuation v_0 in $\mathfrak{A}$ s.t. $v_0(F(x)) = x$ $(x \in \mathfrak{A})$ and $v_0(\psi) \neq 1_{\mathfrak{A}}$.
(ii) If v is a valuation in $\mathfrak{B} \in \mathrm{SC}$ s.t. $v(\psi) = *_{\mathfrak{B}}$ then the function $h: \mathfrak{A} \to \mathfrak{B}$ s.t. $h(x) = v(F(x))(x \in \mathfrak{A})$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

Example Let $\mathfrak{A} \in \mathrm{FSC}$. Then $(f_{\mathfrak{A}}, \chi(\mathfrak{A}))$ expresses $\mathfrak{A}$.

Theorem 2.2 *Let $\mathfrak{A}$ be a Heyting algebra, and let $\psi \in$ For, $F: \mathfrak{A} \to$ For be s.t. (F, ψ) expresses $\mathfrak{A}$. Then $\alpha \notin E(\mathfrak{A})$ iff $e(\alpha) \to \psi \in$ **INT** for some substitution e: For $\to$ For.*

Proof: $(\leftarrow)$ Since $\psi \notin E(\mathfrak{A})$ by Definition 2.1.

$(\rightarrow)$ Assume that $\alpha \notin E(\mathfrak{A})$. Then $v(\alpha) \neq 1_{\mathfrak{A}}$ for some valuation v in $\mathfrak{A}$. Let e be a substitution s.t. $e(p) = F(x)$ if $v(p) = x$ $(p \in \mathrm{Var})$. Now suppose that $e(\alpha) \to \psi \notin$ **INT**. Then by the well-known completeness theorem for **INT**, $w(e(\alpha) \to \psi) = *_{\mathfrak{B}}$ for some valuation w in some $\mathfrak{B} \in \mathrm{SC}$. By Definition 2.1 the function $h: \mathfrak{A} \to \mathfrak{B}$ s.t. $h(x) = w(F(x))$ $(x \in \mathfrak{A})$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$. So it is easy to verify that $w(e(\beta)) = w(F(v(\beta)))$ for any $\beta \in$ For (induction on the complexity of β). Hence $w(e(\alpha)) = w(F(v(\alpha)))$. Moreover $w(F(v(\alpha))) \neq 1_{\mathfrak{B}}$ since $v(\alpha) \neq 1_{\mathfrak{A}}$. Thus $w(e(\alpha)) \neq 1_{\mathfrak{B}}$. Contradiction.

Theorem 2.3 *Let $\mathcal{K} \subseteq$ FSC. Then $\alpha \notin E(\mathcal{K})$ iff α is refutable by the rules $\mathbf{r}_{\mathrm{sb}}$, $\mathbf{r}_{\mathrm{mp}}$, and formulas in $\{\chi(\mathfrak{A}) : \mathfrak{A} \in \mathcal{K}\}$.*

Proof: $(\leftarrow)$ Since $\chi(\mathfrak{A}) \notin E(\mathfrak{A}) \supseteq E(\mathcal{K})$ for each $\mathfrak{A} \in \mathcal{K}$.

$(\rightarrow)$ Assume that $\alpha \notin E(\mathcal{K})$. Then $\alpha \notin E(\mathfrak{A})$ for some $\mathfrak{A} \in \mathcal{K}$. Hence by Theorem 2.2 and the Example above we have $e(\alpha) \to \chi(\mathfrak{A}) \in$ **INT**. Thus $\dashv \alpha$.

Definition 2.4 Let $\mathfrak{A}$ be a countable Heyting algebra, $|\mathfrak{A}| > 1$. Then $R(\mathfrak{A}) = \{\bigwedge X \to p : X \subseteq D(\mathfrak{A}),\ |X| < \aleph_0, p \in \mathrm{Var}(X) - \{p_{1_{\mathfrak{A}}}\} \neq \varnothing\}$. Moreover if $\mathcal{K}$ is a countable class of countable algebras then $R(\mathcal{K}) = \bigcup\{R(\mathfrak{A}) : \mathfrak{A} \in \mathcal{K}\}$ provided that $\mathrm{Var}(D_{\mathfrak{A}}) \cap \mathrm{Var}(D_{\mathfrak{B}}) = \varnothing$ for all $\mathfrak{A} \neq \mathfrak{B} \in \mathcal{K}$.

Theorem 2.5 *Let $\mathcal{K}$ be a countable class of countable Heyting algebras. Then $\alpha \notin E(\mathcal{K})$ iff α is refutable by the rules $\mathbf{r}_{\mathrm{sb}}$, $\mathbf{r}_{\mathrm{mp}}$, and formulas in $R(\mathcal{K})$.*

Proof: $(\leftarrow)$ Since $R(\mathcal{K}) \cap E(\mathcal{K}) = \varnothing$.

$(\rightarrow)$ Assume that $\alpha \notin E(\mathcal{K})$. Then $\alpha \notin E(\mathfrak{A})$ for some $\mathfrak{A} \in \mathcal{K}$. Hence $e(\alpha) \to \beta \in$ **INT** for some substitution e and for some $\beta \in R(\mathfrak{A})$ by the Lemma in [13].

Corollary 2.6 *Let $\mathfrak{A}$ be a finite Heyting algebra, $|\mathfrak{A}| > 1$. Then $\alpha \notin E(\mathfrak{A})$ iff α is refutable by the rules $\mathbf{r}_{\mathrm{sb}}$, $\mathbf{r}_{\mathrm{mp}}$, and formulas in $R(\mathfrak{A})$.*

Proof: By Theorem 2.5.

Definition 2.7 Let $\mathfrak{A} = \Pi\{\mathfrak{A}_i : 1 \leq i \leq n\} \oplus$, where each $\mathfrak{A}_i$ is a finite Heyting algebra, $n \geq 1$. Then $\varphi(\mathfrak{A}) = (\gamma_{\mathfrak{A}} \to \Delta_{\mathfrak{A}}^1 \vee \ldots \vee \Delta_{\mathfrak{A}}^n) \to \gamma_{\mathfrak{A}}$, where

$$\Delta_{\mathfrak{A}}^i = \delta_{\mathfrak{A}_i}^i \wedge \bigwedge\{P_x \to p_{x_i}^i : x \in \mathfrak{A} - \{1_{\mathfrak{A}}\}\} \quad (1 \leq i \leq n),$$

$$\gamma_{\mathfrak{A}} = \bigvee\{P_x \to P_y : x \to y \neq 1_{\mathfrak{A}},\ x, y \in \mathfrak{A}\},$$

and

$$P_x = \begin{cases} p \to p & \text{if } x = 1_{\mathfrak{A}} \\ p^1_{x_1} \vee \ldots \vee p^n_{x_n} & \text{if } x = (x_1, \ldots, x_n) \end{cases} \qquad (x \in \mathfrak{A}).$$

Lemma 2.8 ([12]) *Let* $\mathfrak{A} = \Pi\{\mathfrak{A}_i : 1 \leq i \leq n\}\oplus$, *where each* $\mathfrak{A}_i$ *is a finite algebra,* $n \geq 1$, *and let* $P: \mathfrak{A} \to \mathrm{For}^-$ *be s.t.* $P(x) = P_x$ $(x \in \mathfrak{A})$. *Then*

(i) $(P, \varphi(\mathfrak{A}))$ expresses $\mathfrak{A}$.
(ii) $(P, \varphi(\mathfrak{A}^-))$ expresses $\mathfrak{A}^-$.
(iii) $\alpha \notin E(\mathfrak{A})$ iff $e(\alpha) \to \varphi_{\mathfrak{A}} \in$ **INT** for some substitution e.

Lemma 2.9 ([12]) *Let* $\mathfrak{A} = \Pi\{\mathfrak{A}_i : 1 \leq i \leq n\}\oplus$, $\mathfrak{A}_i \in \mathrm{FSC}$ $(1 \leq i \leq n)$, $n \geq 2$. *Then there are* $\alpha_1, \ldots, \alpha_n$, $\beta \in$ For *s.t.* $\varphi_{\mathfrak{A}} \equiv ((\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_1 \vee \ldots \vee \alpha_n) \in$ **INT**, *and* $(\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i \notin E(\mathfrak{A}_i)$ $(1 \leq i \leq n)$.

3 Dummett's LC

By Dummett's logic we mean the system **LC** = **INT** + $(\alpha \to \beta) \vee (\beta \to \alpha)$.

Theorem 3.1 (Dummett [1]) **LC** $= E(\mathcal{K})$, *where* $\mathcal{K}$ *is the class of finite linear Heyting algebras.*

We will use the following formulas (cf. Nagata [10]):

$$N_1 = ((p_1 \to p_0) \to p_1) \to p_1,$$
$$N_n = ((p_n \to N_{n-1}) \to p_n) \to p_n \quad (n > 1).$$

Theorem 3.2 $\alpha \notin$ **LC** *iff* α *is refutable by the rules* $\mathbf{r}_{\mathrm{sb}}$, $\mathbf{r}_{\mathrm{mp}}$, *and the formulas* $N_n (n \geq 1)$.

Proof: Simple.

The three-element algebra $\mathfrak{J}_0\oplus$ is especially important.

Lemma 3.3 $\alpha \notin E(\mathfrak{J}_0\oplus)$ *iff* $e(\alpha) \to p \vee \neg p \in$ **INT** *for some substitution e.*

Proof: By Theorem 2.2 and the fact that $\mathfrak{J}_0\oplus$ is expressed by $(F, p \vee \neg p)$, where $F(0) = \neg(p \vee \neg p)$, $F(*) = p \vee \neg p$, $F(1) = \neg\neg(p \vee \neg p)$.

The Heyting–Łukasiewicz logic **HŁ** is an extension of **INT** by the following axiom scheme: $(\neg\beta \to \alpha) \to (((\alpha \to \beta) \to \alpha) \to \alpha)$.

Theorem 3.4 (Łukasiewicz [5]) **HŁ** $= E(\mathfrak{J}_0\oplus)$.

Theorem 3.5 $\alpha \notin$ **HŁ** *iff* α *is refutable by* $\mathbf{r}_{\mathrm{sb}}$, $\mathbf{r}_{\mathrm{mp}}$, *and the formula* $p \vee \neg p$.

Proof: By Lemma 3.3.

4 Yankov's logic

By Yankov's logic we mean the system **YC** = **INT** + $\neg\neg\alpha \wedge (\alpha \to \beta) \wedge ((\beta \to \alpha) \to \alpha) \to \beta$.

Theorem 4.1 (Yankov [17], McKay [8]) **YC** $= E(\mathcal{K})$, *where* $\mathcal{K} = \{\mathfrak{J}_0^n \oplus : n \geq 1\}$.

First we define the following sequence of rules:

$$(\mathbf{y}_n) \qquad \frac{\dashv \neg\neg((\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_i)(1 \le i \le n)}{\dashv (\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_1 \vee \ldots \vee \alpha_n} \qquad (n \ge 2).$$

Theorem 4.2 *If $n \ge 2$ then the rule $\mathbf{y}_n$ is valid in* **YC**.

Proof: Assume that $\neg\neg(\gamma \to \alpha_i) \notin$ **YC** $(1 \le i \le n)$, where $\gamma = (\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n)$. Then $\neg\neg(\gamma \to \alpha_i) \notin E(\mathfrak{J}_0)$ (otherwise $\neg\neg(\gamma \to \alpha_i) \in$ **INT** $\subseteq$ **YC**). Thus it is easy enough to see that $\gamma \to \alpha_1 \vee \ldots \vee \alpha_n \notin E(\mathfrak{J}_0^n \oplus) \supseteq$ **YC** (cf. [12]).

Theorem 4.3 $\alpha \notin$ **YC** *iff α is refutable by the formula $p \vee \neg p$, and the rules* $\mathbf{r}_{\mathrm{sb}}, \mathbf{r}_{\mathrm{mp}}, \mathbf{y}_n$ $(n \ge 2)$.

Proof: $(\leftarrow)$ By Theorem 4.2.

$(\rightarrow)$ Assume that $\alpha \notin$ **YC**. Then by Theorem 4.1, $\alpha \notin E(\mathfrak{J}_0^n \oplus)$ for some $n \ge 1$. If $n = 1$ then $\dashv\alpha$ by Lemma 3.3. If $n \ge 2$ then $e(\alpha) \to \varphi(\mathfrak{J}_0^n \oplus) \in$ **INT** for some substitution e by Lemma 2.8. Moreover by Lemma 2.9 there are $\alpha_1, \ldots, \alpha_n$, $\beta \in$ For s.t. $\varphi(\mathfrak{J}_0^n \oplus) \to ((\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_1 \vee \ldots \vee \alpha_n) \in$ **INT**, and $(\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i \notin E(\mathfrak{J}_0)$ $(1 \le i \le n)$. Hence $\neg\neg((\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i) \notin E(\mathfrak{J}_0) \supseteq E(\mathfrak{J}_0 \oplus)$ $(1 \le i \le n)$, so $\dashv\neg\neg((\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i)$ $(1 \le i \le n)$ by Lemma 3.3. Applying the rule $\mathbf{y}_n$ we have $\dashv(\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_1 \vee \ldots \vee \alpha_n$. Thus $\dashv\varphi(\mathfrak{J}_0^n \oplus)$ and $\dashv\alpha$.

5 The logic of the weak law of excluded middle

The logic of the weak law of excluded middle (**WEM**) is an extension of **INT** by the axiom scheme $\neg\alpha \vee \neg\neg\alpha$.

Theorem 5.1 (Yankov [17], [19], McKay [8]) **WEM** $= E(\mathcal{K})$, *where* $\mathcal{K} = \{\oplus\mathfrak{J}_n : n \ge 0\}$.

First note the following.

Proposition *Let $\mathfrak{A}$ be a Heyting algebra. If $x, y \in \mathfrak{A}$, $\otimes \in \{\wedge, \vee, \to\}$ then* $x \otimes_{\oplus\mathfrak{A}} y = x \otimes_{\mathfrak{A}} y$.

Lemma 5.2 *Let $\mathfrak{A}$ be a Heyting algebra, and let $\psi \in$ For$^-$, $F: \mathfrak{A} \to$ For$^-$ be s.t. (F, ψ) expresses $\mathfrak{A}^-$. If $\psi' = \bigwedge\{\neg\neg F(x) : x \in \mathfrak{A}\} \to \psi$ and $F' : \oplus\, \mathfrak{A} \to$ For is s.t.*

$$F'(x) = \begin{cases} F(x) & \textit{if } x \in \mathfrak{A} \\ \neg(p \to p) & \text{otherwise} \end{cases} \qquad (x \in \oplus\mathfrak{A})$$

then

(a) (F', ψ') *expresses* $\oplus\mathfrak{A}$.

(b) $\alpha \notin E(\mathfrak{A})$ *iff* $e(\alpha) \to \psi' \in$ **INT** *for some substitution e.*

Proof: (a)

(i) (F, ψ) expresses $\mathfrak{A}^-$, so $v_0(F(x)) = x (x \in \mathfrak{A})$, $v_0(\psi) \ne 1_{\mathfrak{A}}$ for some v_0 in $\mathfrak{A}^-$. Let v_0' be a valuation in $\oplus\mathfrak{A}$ s.t. $v_0'(p) = v_0(p)$ $(p \in$ Var). Then $v_0'(\beta) = v_0(\beta)$ for every $\beta \in$ For$^-$ by the above Proposition. Hence $v_0'(\psi) = v_0(\psi)$, $v_0'(F(x)) = v_0(F(x))$ $(x \in \mathfrak{A})$. Thus $v_0'(F'(x)) = x (x \in \oplus\, \mathfrak{A})$, $v_0'(\psi') = v_0'(\psi) \ne 1_{\oplus\mathfrak{A}}$.

(ii) Assume that $v(\psi') = *_{\mathcal{B}}$, where v is a valuation in $\mathcal{B} \in \mathrm{SC}$. Then $v(\psi) = *_{\mathcal{B}}$. We have to show that $h: \oplus\ \mathcal{A} \to \mathcal{B}$ s.t. $h(x) = v(F'(x))$ $(x \in \oplus\ \mathcal{A})$ is an embedding of $\oplus\mathcal{A}$ into $\mathcal{B}$. Since (F, ψ) expresses $\mathcal{A}^-$, we have

(*) $h|_{\mathcal{A}}$ is an embedding of $\mathcal{A}^-$ into $\mathcal{B}^-$.

We show that

(1) h is a homomorphism from $\oplus\mathcal{A}$ into $\mathcal{B}$. Indeed, $h(0 \wedge x) = h(0) \wedge h(x)$, and $h(0 \vee x) = h(0) \vee h(x)$ $(x \in \oplus\mathcal{A})$. Moreover if $x \in \oplus\mathcal{A}$ then $h(0 \to x) = h(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ by (*), and $h(0) \to h(x) = 0_{\mathcal{B}} \to h(x) = 1_{\mathcal{B}}$. If $x \in \mathcal{A}$ then $h(x \to 0) = h(0)$, and $h(x) \to h(0) = v(F(x)) \to 0_{\mathcal{B}} = \neg v(F(x)) = v(\neg F(x)) = 0_{\mathcal{B}}$ since $v(\neg\neg F(x)) = 1_{\mathcal{B}}$. Also $h(\neg x) = h(x \to 0) = h(x) \to h(0) = h(x) \to 0 = \neg h(x)$ $(x \in \oplus\ \mathcal{A})$.

(2) If $x \neq y$ then $h(x) \neq h(y)$. Indeed, if $x, y \in \mathcal{A}$ then $h(x) \neq h(y)$ by (*). Assume that, say, $x = 0_{\oplus\mathcal{A}}$. Then $y \neq 0_{\oplus\mathcal{A}}$, so $h(y) = v(F(y))$. Now suppose that $h(x) = h(y)$, i.e. $v(F(y)) = 0_{\oplus\mathcal{A}}$. Then $v(\neg\neg F(y)) = 0_{\oplus\mathcal{A}}$. Contradiction.

(b) By (a) and Theorem 2.2.

Definition 5.3 Let $\mathcal{A} = \Pi\{\mathcal{A}_i : 1 \leq i \leq n\}\oplus$, where each $\mathcal{A}_i$ is a finite algebra, $n \geq 1$. Then $\epsilon(\oplus\mathcal{A}) = \bigwedge\{\neg\neg P_x : x \in A\} \to \varphi(\mathcal{A}^-)$.

Lemma 5.4 *Let* $\mathcal{A} = \Pi\{\mathcal{A}_i : 1 \leq i \leq n\}\oplus$, $\mathcal{A}_i \in \mathrm{FSC}$ $(1 \leq i \leq n)$, $n \geq 2$. *Then there are* $\alpha_1, \ldots, \alpha_n, \beta \in \mathrm{For}^-$ *s.t.* $\epsilon(\oplus\mathcal{A}) \to (\lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_1 \vee \ldots \vee \alpha_n) \in$ **WEM**, *and* $\lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i \notin E(\oplus\mathcal{A}_i)$ $(1 \leq i \leq n)$, *where* $\lambda = \bigwedge\{\neg\neg p : p \in \mathrm{Var}(\alpha_1, \ldots, \alpha_n, \beta)\}$.

Proof: Let $\beta = \Delta^1_{\mathcal{A}^-} \vee \ldots \vee \Delta^n_{\mathcal{A}^-}$,

$$\alpha_i = \bigvee\{P_x \to P_y : x_i \to y_i \neq 1_{\mathcal{A}_i}\} \cup \{P_y : y_i \neq 1_{\mathcal{A}_i}\}$$
$$\cup\ \{p^j_{1_{\mathcal{A}_j}} : 1 \leq i \neq j \leq n\}\ \ (1 \leq i \leq n).$$

Since both $(\neg\neg(p \vee q) \to r) \to (\neg\neg p \to r) \in$ **WEM** and $p \to (q \to p) \in$ **WEM**, it remains to be shown that $\lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i \notin E(\oplus\mathcal{A}_i)$ $(1 \leq i \leq n)$. Let v_i be a valuation in $\oplus\mathcal{A}_i$ s.t.

$$v_i(p^k_x) = \begin{cases} x & \text{if } k = i \\ 0_{\mathcal{A}_i} & \text{otherwise} \quad (1 \leq k,\ i \leq n,\ x \in \mathcal{A}_k). \end{cases}$$

Then it is easy to see that $v_i(P_x) = p^i_{x_i}$ $(x \in \mathcal{A} - \{1_{\mathcal{A}}\})$, and $v_i(\alpha_i) \neq 1_{\oplus\mathcal{A}_i}$ since $\oplus\mathcal{A}_i \in \mathrm{SC}$. Further $v_i(\Delta^i_{\mathcal{A}^-}) = 1_{\oplus\mathcal{A}_i}$ by the above Proposition. Hence $v_i(\beta) = 1_{\oplus\mathcal{A}_i}$. Moreover $v_i(p) \neq 0_{\oplus\mathcal{A}_i}$ for every $p \in \mathrm{Var}(\alpha_1, \ldots, \alpha_n, \beta)$, so $v_i(\neg\neg p) = 1_{\oplus\mathcal{A}_i}$. Thus $v_i(\lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i) \neq 1_{\oplus\mathcal{A}_i}$.

We define the following rules:

$$(\mathbf{w}_n)\quad \frac{\dashv\lambda \wedge (\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_i \qquad (1 \leq i \leq n)}{\dashv\lambda \wedge (\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_1 \vee \ldots \vee \alpha_n}\quad (n \geq 2),$$

where $\lambda = \bigwedge\{\neg\neg p : p \in \mathrm{Var}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)\}$, and $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are negation-free.

Theorem 5.5 *For every* $n \geq 2$ *the rule* $\mathbf{w}_n$ *is valid in* **WEM**.

Proof: Assume that $\gamma \to \alpha_i \notin \mathbf{WEM}$ $(1 \leq i \leq n)$, where

$$\gamma = \lambda \wedge (\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n),$$
$$\lambda = \bigwedge \{\neg\neg p : p \in \mathrm{Var}(\alpha_1, \ldots, \beta_n)\}, \text{ and } \alpha_1, \ldots, \beta_n \in \mathrm{For}^-.$$

Then by Theorem 5.1, $\gamma \to \alpha_i \notin E(\oplus\mathfrak{A}_i)$ for some algebra $\mathfrak{A}_i$ $(1 \leq i \leq n)$. So $v_i(\gamma \to \alpha_i) \neq 1_{\oplus\mathfrak{A}_i}$ for some valuation v_i in $\oplus\mathfrak{A}_i$. Now $v_i(p) \neq 0_{\oplus\mathfrak{A}_i}$ for each $p \in \mathrm{Var}(\alpha_1, \ldots, \beta_n)$ (otherwise $v_i(\neg\neg p) = 0_{\oplus\mathfrak{A}_i}$). Hence by the fact that $\alpha_1, \ldots, \beta_n \in \mathrm{For}^-$, $(\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_i \notin E(\mathfrak{A}_i)$ $(1 \leq i \leq n)$ (cf. Proposition). Thus by Theorem 1 in [12] we have $(\alpha_1 \to \beta_1) \wedge \ldots \wedge (\alpha_n \to \beta_n) \to \alpha_1 \vee \ldots \vee \alpha_n \notin E(\mathfrak{A})$ for some algebra $\mathfrak{A}$. By the above Proposition $\gamma \to \alpha_1 \vee \ldots \vee \alpha_n \notin E(\oplus\mathfrak{A})$.

Lemma 5.6 *$\alpha \notin E(\oplus\mathfrak{J}_1)$ iff $e(\alpha) \to N_2 \in$ **INT** for some substitution e, where N_2 is the formula defined in Section* 3.

Proof: By Theorem 2.2 and the fact that $\oplus\mathfrak{J}_1$ is expressed by (F, N_2), where $F(*_{\mathfrak{J}_1}) = p_2$, $F(0_{\mathfrak{J}_1}) = p_1$, $F(1_{\mathfrak{J}_1}) = p \to p$, $F(0_{\oplus\mathfrak{J}_1}) = \neg(p \to p)$.

Lemma 5.7 *Let $\mathfrak{A} = \Pi\{\mathfrak{A}_i : 1 \leq i \leq n\}\oplus$, where each $\mathfrak{A}_i$ is a finite algebra, $n \geq 1$. Then $(P', \epsilon(\oplus\mathfrak{A}))$ expresses $\oplus\mathfrak{A}$, where $P' : \oplus\mathfrak{A} \to$ For is s.t.*

$$P'(x) = \begin{cases} P_x & \text{if } x \in \mathfrak{A} \\ \neg(p \to p) & \text{otherwise} \end{cases} \quad (x \in \oplus\mathfrak{A}).$$

Proof: By Lemma 2.8, $(P, \varphi(\mathfrak{A}^-))$ expresses $\mathfrak{A}^-$, so by Lemma 5.2 $(P', \epsilon(\oplus\mathfrak{A}))$ expresses $\oplus\mathfrak{A}$.

Lemma 5.8 *$\alpha \notin E(\oplus\mathfrak{J}_n)$ iff $e(\alpha) \to \epsilon(\oplus\mathfrak{J}_n) \in$ **INT** for some substitution e.*

Proof: By Lemmas 5.7 and 5.2.

Theorem 5.9 *$\alpha \notin$ **WEM** iff α is refutable by the formula N_2, and the rules* $\mathbf{r}_{\mathrm{sb}}, \mathbf{r}_{\mathrm{mp}}, \mathbf{w}_n$ $(n \geq 2)$.

Proof: $(\leftarrow)$ By Theorem 5.5 and Lemma 5.6.

$(\rightarrow)$ Assume that $\alpha \notin \mathbf{WEM}$. Then by Theorem 5.1, $\alpha \notin E(\oplus\mathfrak{J}_i)$ for some $i \geq 1$. By Lemma 5.8, $e(\alpha) \to \epsilon(\oplus\mathfrak{J}_i) \in \mathbf{INT}$ for some substitution e. Let $\epsilon_n = \epsilon(\oplus\mathfrak{J}_n)$ $(n \geq 1)$. We show by induction on n that $\dashv \epsilon_n$.

(1) $n = 1$. By Lemma 5.7, $\epsilon_1 \notin E(\oplus\mathfrak{J}_1)$, so $\dashv \epsilon_1$ by Lemma 5.6.

(2) $n \geq 2$. By Lemma 5.4, there are $\alpha_1, \ldots, \alpha_n$, $\beta \in \mathrm{For}^-$ s.t. $\epsilon_n \to (\lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_1 \vee \ldots \vee \alpha_n) \in \mathbf{WEM}$, and $\lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i \notin E(\oplus\mathfrak{J}_{n-1})$ $(1 \leq i \leq n)$. Hence $\dashv \lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_i$ $(1 \leq i \leq n)$ by Lemma 5.8 and the induction hypothesis. Now applying the rule $\mathbf{w}_n$ we have $\dashv \lambda \wedge (\alpha_1 \to \beta) \wedge \ldots \wedge (\alpha_n \to \beta) \to \alpha_1 \vee \ldots \vee \alpha_n$. Thus $\dashv \epsilon_n$.

6 *Medvedev's logic*

Medvedev's logic **M** is the system $E(\mathcal{K})$, where $\mathcal{K} = \{H(\mathcal{P}) : \mathcal{P} = (2^X - \{X\}, \subseteq), X \neq \varnothing \text{ is a finite set}\}$ and $H(\mathcal{P})$ is the Heyting algebra associated with a poset $\mathcal{P}$ (cf. Medvedev [9], Gabbay [2]).

Theorem 6.1 *$\alpha \notin \mathbf{M}$ iff α is refutable by the variable p and the rules* $\mathbf{r}_{\mathrm{sb}}, \mathbf{r}'_{\mathrm{mp}}, \mathbf{r}_{\mathrm{d}}$, *where*

$$(\mathbf{r}'_{\mathrm{mp}}) \quad \frac{\vdash_{\mathbf{KP}} \alpha \to \beta \qquad \dashv\beta}{\dashv\alpha}, \qquad (\mathbf{r}_{\mathrm{d}}) \quad \frac{\dashv\alpha \qquad \dashv\beta}{\dashv\alpha \vee \beta}.$$

Remark The symbol $\vdash_{\mathbf{KP}}$ denotes provability in Kreisel–Putnam's logic $\mathbf{KP} = \mathbf{INT} + (\neg\alpha \to \beta \vee \gamma) \to (\neg\alpha \to \beta) \vee (\neg\alpha \to \gamma)$. We are using $\vdash_{\mathbf{KP}}$ rather than $\vdash_{\mathbf{M}}$ because it is not known whether $\mathbf{M}$ is recursively axiomatizable.

Proof: ($\leftarrow$) Since $\mathbf{M}$ has the disjunction property and $\mathbf{KP} \subseteq \mathbf{M}$.

($\to$) Assume that $\alpha \notin \mathbf{M}$. Then $\alpha \notin E(\mathcal{B})$ for some $\mathcal{B} \in \mathcal{K}$ whence $e(\alpha) \to \chi(\mathcal{B}) \in \mathbf{INT}$ for some substitution e. From the proof of Theorem 5 in Maksimova [7] it follows that $s(\chi_{\mathcal{B}}) \to \neg\alpha_1 \vee \ldots \vee \neg\alpha_m \in \mathbf{KP}$ for some substitution s, and $\neg\alpha_i$ is not a classical tautology $(1 \leq i \leq m)$. Thus $\dashv\chi(\mathcal{B})$, so $\dashv\alpha$.

7 *Logics without the finite model property*

Theorem 7.1 (Wroński [15], cf. also Kuznetsov and Gerchiu [4]) *Let $\mathcal{A} \in$ FSC be s.t. $0_{\mathcal{A}} = x_1 \wedge x_2 \wedge x_3$ for some pairwise incomparable $x_1, x_2, x_3 \in \mathcal{A}$, and let $\mathcal{R}$ be the Rieger–Nishimura lattice (cf. [11]). Then the logic $E(\mathcal{R} \oplus \mathcal{A})$ does not have the finite model property.*

A refutation calculus for each such logic can be obtained by Theorem 2.5, but we give here much simpler formulations for these logics.

Theorem 7.2 *Let $\mathcal{A}$ and $\mathcal{R}$ be as in Theorem 7.1. Then $\alpha \notin E(\mathcal{R} \oplus \mathcal{A})$ iff α is refutable by the rules* $\mathbf{r}_{\mathrm{sb}}$, $\mathbf{r}_{\mathrm{mp}}$, *and the formula* $(\neg\neg p \vee (\neg\neg p \to p) \to \chi_{\mathcal{A}^-}) \to \chi_{\mathcal{A}^-}$, *where* $p \notin \mathrm{Var}(\chi_{\mathcal{A}^-})$.

Proof: By the proof of Lemma 12 in [15].

REFERENCES

[1] Dummett, M., "A propositional calculus with denumerable matrix," *The Journal of Symbolic Logic*, vol. 24 (1959), pp. 97–106.

[2] Gabbay, D. M., *Semantical Investigations in Heyting's Intuitionistic Logic*, Reidel, Dordrecht, 1981.

[3] Jaśkowski, S., "Recherches sur le systeme de la logique intuitioniste," pp. 58–61 in *Actes du Congres International de Philosophie Scientifique 6*, Paris, 1936.

[4] Kuznetsov, A. V. and V. Y. Gerchiu, "On superintuitionistic logics and finite approximability," *Soviet Mathematics Doklady*, vol. 11 (1970), pp. 1614–1619.

[5] Łukasiewicz, J., "Die Logik und das Grundlagenproblem," pp. 82–100 in *Les Entretiens de Zurich sur les Fondements et la Methode des Sciences Mathematiques 1938*, Zurich, 1941.

[6] Łukasiewicz, J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Clarendon Press, Oxford, 1951.

[7] Maksimova, L. L., "On maximal intermediate logics with the disjunction property," *Studia Logica*, vol. 45 (1986), pp. 69–75.

[8] McKay, C. G., "Some completeness results for intermediate propositional logics," *Notre Dame Journal of Formal Logic*, vol. 8 (1967), pp. 191–194.

[9] Medvedev, T., "Finite problems" (in Russian), *Doklady AN SSSR*, vol. 142 (1962), pp. 1015–1018.

[10] Nagata, S., "A series of successive modifications of Peirce's rule," *Proceedings of the Japan Academy*, vol. 42 (1966), pp. 859–861.

[11] Nishimura, I., "On formulas in one variable in intuitionistic propositional calculus," *The Journal of Symbolic Logic*, vol. 25 (1960), pp. 327–331.

[12] Skura, T., "A complete syntactical characterization of the intuitionistic logic," *Reports on Mathematical Logic*, vol. 23 (1989), pp. 75–80.

[13] Skura, T., "A new criterion of decidability for intermediate logics," *Bulletin of the Section of Logic*, vol. 19 (1990), pp. 10–14.

[14] Skura, T., "On decision procedures for sentential logics," *Studia Logica*, vol. 50 (1991), pp. 173–179.

[15] Wroński, A., "Intermediate logics and the disjunction property," *Reports on Mathematical Logic*, vol. 1 (1973), pp. 39–51.

[16] Wroński, A., "On cardinalities of matrices strongly adequate for the intuitionistic propositional logic," *Reports on Mathematical Logic*, vol. 3 (1974), pp. 67–72.

[17] Yankov, V. A., "Some superconstructive propositional calculi," *Soviet Mathematics Doklady*, vol. 4 (1963), pp. 1103–1105.

[18] Yankov, V. A., "On the relationship between deducibility in the intuitionistic propositional calculus and finite implicational structures," *Soviet Mathematics Doklady*, vol. 8 (1963), pp. 1203–1204.

[19] Yankov, V. A., "The calculus of the weak law of excluded middle," *Mathematics USSR Izvestija*, vol. 2 (1968), pp. 997–1004.

Department of Logic
Uniwersytet Wrocławski
Szewska 36
50-139 Wrocław, Poland