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A Functional Partial Semantics for Intensional Logic

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Abstract In this paper a partial semantics for the higher order modal language of Intensional Logic is suggested. Partial semantic values of functional types are defined as monotone functions on partially ordered sets; it is shown that this characterization is materially adequate for representing partial values and that it overcomes the difficulties that arise when we attempt to introduce one-place partial functions in the hierarchy of types. Partial values of any type are related to classical values of the same type by means of a relation of approximation. This allows us to compare partial models with classical models. Classical semantics then appears to be a part of partial semantics to the extent that there exists a bijective mapping from classical models onto totally defined partial models. This also allows us to define, according to the partial semantics, a notion of entailment which is coextensive with the classical notion.

1 Introduction Even though much work has been done in partial semantics for propositional and quantified first order languages, little has been said about partial semantics for higher order languages. The first attempt to introduce partiality in Type Theory of which we are aware goes back to Tichý [11]. There is also the interesting work of Muskens [9], which suggests a partialized version of Montague's semantics. Musken's semantics is relational, in the sense that partial semantical objects are defined as partial relations, not as partial functions. This strategy is in large part justified by the apparent impossibility, discussed in this paper, of coding partial relations or partial many-place functions by one-place functions, as Schönfinkel's theorem might suggest. Note that the same problem has motivated Tichý's own strategy, which consists of considering only many-place partial functions. For our point, we do not claim that Tichý's and Musken's strategies are inadequate, nor do we think they are uninteresting. Our claim is merely that it is possible, using only one-place functions, to construct a partial semantics for the higher-order modal language of Intensional Logic.

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Our strategy, which will be explained in detail in Section 3.1, is in fact inspired by another interesting attempt to introduce partiality in the semantics of Intensional Logic, that of Lepage [4] (and also in [5], [6], and [7]). However, in our opinion, many notions used by Lepage were not precise enough to serve the foundations for a partial semantics for Intensional Logic (Lepage's purpose was not to suggest a foundational theory of partiality in Type Theory, but rather to suggest a formal analysis of knowledge and belief-sentences by using the notion of incomplete knowledge of semantic values; of course, these two enterprises are not mutually exclusive). In particular, Lepage's notion of a *good representation of a classical semantic value*, or equivalently, of an *approximation of a classical semantic value*, was not analytical enough to make precise the idea that a partial value which is total behaves exactly as one and only one classical value. Consequently, the notion of a *partial model which is total* was not precise enough either. In this paper we shall fill these gaps by providing a unifying view of both partial semantics and classical semantics. In particular, we shall give a precise account of the idea that classical models are limit cases of partial models. This may seem trivial at first glance, but in fact it is not, for the class of classical models is disjoint from the class of partial models.

Let us first present our formal framework: the language of Intensional Logic and its standard classical semantics.

2 The language of Intensional Logic and its standard classical semantics

The formal system described in this section is very close to the one described by Gallin in [3], as a version of Montague's Intensional Logic [8].

Let e, t, and s be three distinct objects. *The modal hierarchy of types* is the smallest set T such that:

(i) $e, t \in T$;
(ii) if $\alpha \in T$ and $\beta \in T$, then $(\alpha, \beta) \in T$;
(iii) if $\alpha \in T$, then $(s, \alpha) \in T$.

When no confusion arises, (α, β) and (s, α) will be abbreviated by $\alpha\beta$ and $s\alpha$ respectively. We shall use the Greek letters α, β, and σ as variables of types.

The language of *Intensional Logic* (IL) has the following resources:

(1) The improper expressions: [,], $\equiv$, $^\wedge$, $^\vee$, λ.
(2) For every $\alpha \in T$, an almost denumerable set Con_α of *constants* of types α and a denumerable set Var_α of *variables* of type α.
(3) For every $\alpha \in T$, a set of *terms* of type α, recursively defined as the smallest set Trm_α such that:
   (i) $\mathrm{Con}_\alpha \cup \mathrm{Var}_\alpha \subseteq \mathrm{Trm}_\alpha$;
   (ii) if $A \in \mathrm{Trm}_{\alpha\beta}$ and $B \in \mathrm{Trm}_\alpha$, then $[AB] \in \mathrm{Trm}_\beta$;
   (iii) if $A \in \mathrm{Trm}_\alpha$, then $^\wedge A \in \mathrm{Trm}_{s\alpha}$;
   (iv) if $A \in \mathrm{Trm}_{s\alpha}$, then $^\vee A \in \mathrm{Trm}_\alpha$;
   (v) if $x \in \mathrm{Var}_\alpha$ and $A \in \mathrm{Trm}_\beta$, then $\lambda x A \in \mathrm{Trm}_{\alpha\beta}$;
   (vi) if $A, B \in \mathrm{Trm}_\alpha$, then $[A \equiv B] \in \mathrm{Trm}_t$.

Henceforth, for any type α, we shall use the symbols A_α, B_α, C_α, etc., as schemata of terms of type α. We shall use more particularly the symbols x_α, y_α, z_α, etc., as schemata of variables of type α and the symbols c_α, d_α, e_α, etc., as schemata of constants of type α. When no confusion arises, subscripts will be dropped. Here are some abbreviations:

$$\mathsf{T} := [\lambda x_t x \equiv \lambda x_t x]$$
$$\mathsf{F} := [\lambda x_t x \equiv \lambda x_t \mathsf{T}]$$
$$\neg A_t := [A \equiv \mathsf{F}]$$
$$\wedge_{t(tt)} := \lambda x_t \lambda y_t [\lambda z_{t(tt)}[[zx]y] \equiv \lambda z_{t(tt)}[[z\mathsf{T}]\mathsf{T}]]$$
$$[A_t \wedge B_t] := [[\wedge_{t(tt)} A]B]^1$$
$$[A_t \vee B_t] := \neg[\neg A \wedge \neg B]$$
$$[A_t \to B_t] := [\neg A \vee B]$$
$$\forall x_\alpha A_t := [\lambda x A \equiv \lambda x \mathsf{T}]$$
$$\exists x_\alpha A_t := \neg \forall x \neg A$$
$$[A_\alpha \equiv B_\alpha] := [{}^\wedge A \equiv {}^\wedge B]$$
$$\Box A_t := [A \equiv \mathsf{T}]$$
$$\Diamond A_t := \neg \Box \neg A.$$

All these abbreviations are from [3], except "$\wedge_{t(tt)}$", which is borrowed from Andrews [1]. The motivation for this choice will be explained in Remark 25, Section 3.4.

Let E and I be two nonempty and disjoint sets. The *standard system of classical objects based on E and I* is the indexed family $\{M_\alpha\}_{\alpha \in T}$ of sets, such that:

(i) $M_e = E$
(ii) $M_t = \{0,1\}$
(iii) $M_{\alpha\beta} = M_\beta^{M_\alpha}$
(iv) $M_{s\alpha} = M_\alpha^I$.

A *standard classical model based on E and I* is an ordered pair $M = \langle \{M_\alpha\}_{\alpha \in T}, m \rangle$, where $\{M_\alpha\}_{\alpha \in T}$ is the standard system of classical objects based on E and I, and m is a function from all constants such that for every constant c_α, $m(c_\alpha) \in M_{s\alpha}$. We denote by $\mathrm{As}(M)$ the *set of assignments over M*, that is, the set of all functions $\mathbf{a}$ from all variables such that for every variable x_α, $\mathbf{a}(x_\alpha) \in M_\alpha$. For every $\mathbf{a} \in \mathrm{As}(M)$, every variable x_α and every $z \in M_\alpha$, $\mathbf{a}(x_\alpha/z)$ is that assignment in $\mathrm{As}(M)$ such that $\mathbf{a}(x_\alpha/z)(x_\alpha) = z$ and for every variable $x_\beta \neq x_\alpha$, $\mathbf{a}(x_\alpha/z)(x_\beta) = \mathbf{a}(x_\beta)$. We recursively define the *classical value* $[\![A_\alpha]\!]^M_{\mathbf{a},i}$ in M of a term A_α *according to* an assignment $\mathbf{a} \in \mathrm{As}(M)$ and an $i \in I$ as follows (in what follows we shall sometimes omit the superscript "M"):

(i) $[\![c_\alpha]\!]_{\mathbf{a},i} = (m(c_\alpha))(i)$;
(ii) $[\![x_\alpha]\!]_{\mathbf{a},i} = \mathbf{a}(x_\alpha)$;
(iii) $[\![A_{\alpha\beta} B_\alpha]\!]_{\mathbf{a},i} = [\![A_{\alpha\beta}]\!]_{\mathbf{a},i}([\![B_\alpha]\!]_{\mathbf{a},i})$;
(iv) $[\![{}^\wedge A_\alpha]\!]_{\mathbf{a},i}$ = the function f from I such that for every $j \in I$, $f(j) = [\![A_\alpha]\!]_{\mathbf{a},j}$;
(v) $[\![{}^\vee A_{s\alpha}]\!]_{\mathbf{a},i} = [\![A_{s\alpha}]\!]_{\mathbf{a},i}(i)$;
(vi) $[\![\lambda x_\alpha A_\beta]\!]_{\mathbf{a},i}$ = the function f from M_α such that for every $z \in M_\alpha$, $f(z) = [\![A_\beta]\!]_{\mathbf{a}',i}$, where $\mathbf{a}' = \mathbf{a}(x_\alpha/z)$;
(vii) $[\![[A_\alpha \equiv B_\alpha]]\!]_{\mathbf{a},i} = 1$ if $[\![A_\alpha]\!]_{\mathbf{a},i} = [\![B_\alpha]\!]_{\mathbf{a},i}$, and 0 otherwise.

It is easy to check by induction that for every $\alpha \in T$ and every term A_α, $[\![A_\alpha]\!]_{\mathbf{a},i} \in M_\alpha$.

A formula of IL is a term of type t. Let A_t be a formula, M a standard classical model, $\mathbf{a} \in \mathrm{As}(M)$, and let $i \in I$. A_t is *satisfied in M according to* **a** *and i*, formally: $\models_{M,\mathbf{a},i} A_t$, iff $[\![A_t]\!]^M_{\mathbf{a},i} = 1$. A_t is *not satisfied in M according to* **a** *and i*, formally: $\not\models_{M,\mathbf{a},i} A_t$, iff $[\![A_t]\!]^M_{\mathbf{a},i} = 0$. If Γ is a set of formulas, then Γ is *satisfied in M according to* **a** *and i*, formally: $\models_{M,\mathbf{a},i}\Gamma$, iff $\models_{M,\mathbf{a},i} A_t$ for every $A_t \in \Gamma$. A formula A_t is *true in M* iff $\models_{M,\mathbf{a},i} A_t$ for every $\mathbf{a} \in \mathrm{As}(M)$ and every $i \in I$. A set Γ of formulas *classically entails* a formula A_t, formally: $\Gamma \models A_t$, iff for every standard classical model M, every $\mathbf{a} \in \mathrm{As}(M)$ and every $i \in I$, $\models_{M,\mathbf{a},i}\Gamma$ only if $\models_{M,\mathbf{a},i} A_t$. Finally, A_t is *classically valid*, formally: $\models A_t$, iff $\varnothing \models A_t$, that is to say, iff A_t is true in every standard classical model.

3 *The standard partial semantics for the language of IL*

3.1 Introduction Our only intuition for the construction of a partial semantics for **IL** is the following: partial values must be *approximations* of classical values. How can we formally define partial values in order to meet this intuitive requirement? Since in the classical semantics of **IL** the value of a given functional type (type $\alpha\beta$ or $s\alpha$) is a *total function*, it seems appropriate to define the partial value of a given functional type as a *partial function* approximating a total function of the same type. This raises some questions, however.

What is a partial function? Under what conditions is a partial function an approximation of a given total function? First, it is usual to define formally, and in its full generality, the notion of a function as follows. Let X and Y be two nonempty sets. A *function from X* (starting set) *into Y* (target set) is a relation $f \subset X \times Y$ such that for (x,y), $(x',y') \in f$, $x = x'$ only if $y = y'$. Let f be a function from X into Y. The *domain* of f is the set $D(f) \subseteq X$ such that for every $x \in X$, $x \in D(f)$ iff there exists $y \in Y$ such that $(x,y) \in f$. Henceforth, we shall say that f is *total* if $D(f) = X$, *nontotal* if $D(f) \subset X$, and *partial* if $D(f) \subseteq X$. If $x \in X$, f is partial and $x \notin D(f)$, then we shall write "$f(x)$ is undefined" (or "f is not defined for x"). Let f and f' be two partial functions from X into Y. We shall say that f is *an approximation of* f' (or that f' is *at least as defined as* f), and we shall write: "$f \leq f'$", iff $D(f) \subseteq D(f')$ and for every $x \in D(f)$, $f(x) = f'(x)$. Let $P(Y^X)$ be the set of all partial functions from X into Y. Clearly, the relation of approximation defined on $P(Y^X)$ is a partial order. In fact, $(P(Y^X), \leq)$ is a meet-semilattice, whose smallest element is the least defined function (that is to say, the function f such that $D(f) = \varnothing$). For instance, if we represent each $f \in P(\{0,1\}^{\{0,1\}})$ by the (ordered) image of $\{0,1\}$ under f:

$$f = \langle f(0), f(1) \rangle$$

and if we use the asterisk $*$ to indicate that f is undefined for a particular argument, then we can represent the whole semilattice $P(\{0,1\}^{\{0,1\}})$ as shown in Figure 1 (underlined numerals are names).

Are these concepts rich enough to be of use in developing an adequate theory of partial functions in Type Theory? More specifically, are they sufficient to characterize the partial values of any given functional type as partial functions which approximate classical values of the same type? Unfortunately not, for we are faced with three major difficulties.

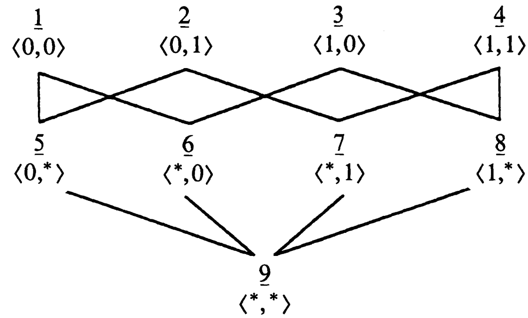


Figure 1. The approximating semilattice $P(\{0,1\}^{\{0,1\}})$.

Difficulty 1 It would be certainly possible to use these notions to define the partial values of types (t,t), (e,e), (t,e), (e,t), (s,t), and (s,e), and then to compare, in terms of approximation, the partial values of each of these types with the classical values of the same type. But in the case of higher-order types, things would become much more complicated, since partial functions of these types can take partial functions as arguments. For instance, the domain of a partial function of type $((t,t)(t,t))$, if nonempty, may contain total or nontotal functions in $P(\{0,1\}^{\{0,1\}})$ and the values of such functions can also be total or nontotal functions in $P(\{0,1\}^{\{0,1\}})$. The problem we encounter is thus that we would have to compare, in terms of approximation, partial functions of a given higher-order type with the classical functions of the same type, but the starting and target sets of the partial functions would be different from the starting and target sets of the classical functions. The conditions of inclusion of domains and of identity of values would thus not be directly applicable.

Difficulty 2 A second difficulty concerns the reiteration of functional applications and raises a related question about the status of the *undefined* in Type Theory. By way of example, let f be a classical function of type (t,t), that is, let $f \in \{0,1\}^{\{0,1\}}$. So, for every $x \in \{0,1\}$, $f(x)$ is of type t, and therefore, $f(f(x))$ is also of type t. We believe that such reiteration of functional applications should be possible in the universe of partial functions. But consider a partial function g of type (t,t), that is, a function $g \in P(\{0,1\}^{\{0,1\}})$, such that $g(1)$ is *undefined*. Now try to apply g to $g(1)$. Strictly speaking, this application makes no sense, because $g(1)$ is not an argument at all. But on the other hand, if g is the value of an expression A_{tt} of type (t,t) and 1 is the value of an expression B_t of type t, then according to the principle of compositionality, $g(g(1))$ *must be* the value of the expression $[A[AB]]$ of type t. But again, one cannot see what $g(g(1))$ could be.

Difficulty 3 A third problem was pointed out by Pavel Tichý [11]. Schönfinkel's famous theorem:

$$X^{Y \times Z} \approx (X^Y)^Z$$

which is fundamental in lambda calculus, appears to be invalid in the universe of partial functions. Indeed, Tichý offered the following counterexample. Let the function f from $\{0,1\} \times \{0,1\}$ into $\{0,1\}$ be such that:

$$(1) \qquad f(x,y) = \begin{cases} y & \text{if } x = 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

There are *two* distinct partial functions from $\{0,1\}$ which correspond to f: one assigns the identity function to 0 and is undefined for 1; the other assigns the identity function to 0 and to 1 the function which is undefined both for 0 and 1. The idea here is that if one of these two functions corresponds to f, the other one must also correspond to f if it corresponds to anything.

We believe that these difficulties can be simultaneously removed by applying the following three measures: (i) give the status of object to the undefined at the level of types e and t, and for each functional type, identify the undefined with the least defined function; (ii) on each domain of partial objects, define a partial order relation, interpreted as a relation of approximation between these objects; and (iii) restrict the function spaces to functions which are monotone with respect to the relation of approximation. Let us see how the application of these measures can solve our problems.

In what follows we shall adopt the following notational convention. If X and Y are two nonempty sets (not necessarily distinct) and O is an operation such that when it is applied to an element $x \in X$, it gives exactly one element $O(x) \in Y$, then by $\lambda x \in X.O(x)$ we mean the function $f: X \to Y$ such that $f(x) = O(x)$ for every $x \in X$. When there is no chance of confusion, we shall simply write "$\lambda x.O(x)$".

Consider first Tichý's difficulty. Tichý's counterexample reveals the fact that for any nonempty and finite sets X, Y, and Z:

$$P(X^{Y \times Z}) \not\approx P(P(X^Y)^Z).$$

Indeed, writing $|X|$ for the cardinality of a set X, it is the case that for any nonempty and finite sets X, Y, and Z:

$$|P(X^{Y\times Z})| = (|X|+1)^{|Y\times Z|} = ((|X|+1)^{|Y|})^{|Z|} < (((|X|+1)^{|Y|})+1)^{|Z|} = |P(P(X^Y)^Z)|.$$

However, since $(|X|+1)^{|Y\times Z|} = ((|X|+1)^{|Y|})^{|Z|} = |P(X^Y)|^{|Z|}$, we have:

$$P(X^{Y\times Z}) \approx P(X^Y)^Z.$$

This means that we can move toward solving the problem if we identify, in a given class of partial functions, the undefined with the least defined function in that class. Accordingly, to the function f defined by (1) corresponds an unique function, which is the function f' from $\{0,1\}$ into $P(\{0,1\}^{\{0,1\}})$ such that:

$$(2) \qquad f'(x) = \begin{cases} \lambda y.y & \text{if } x = 0 \\ \lambda y.\ \text{undefined} & \text{otherwise.} \end{cases}$$

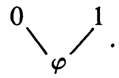
Of course, this raises another problem, which is reminiscent of our second difficulty, which is that the application of f' so defined to $(f'(1))(1)$ is meaningless.

The application of f' to $(f'(1))(1)$ would make a sense, however, if the notion of undefined were admitted among the possible arguments of f'. Intuitively, the only constraint necessary would be that the *result* of applying f' to the undefined should itself be undefined.

Let us represent the undefined of type t by φ, and define the set of *partial objects of type t* as the set $PM_t := \{0, 1, \varphi\}$. As usual, 0 and 1 can be considered as the truth values *false* and *true*, respectively; therefore $\varphi \in PM_t$ can be considered as a truth value which is not defined. It seems natural to define on PM_t a relation of approximation, say "$\leq$", in this way: for $x, y \in PM_t$

$$x \leq y \quad \text{iff } x = \varphi \text{ or } x = y.$$

Under this relation, PM_t is a (flat) meet-semilattice which can be pictured as follows:



(Note that PM_t is reminiscent of Dana Scott's BOOL, minus the top.) We can define the set PM_e of partial objects of type e in a similar manner. Given a nonempty set E of individuals, let $PM_e := E \cup \{\varphi\}$ (we can of course distinguish between $\varphi \in PM_e$ and $\varphi \in PM_t$ by using e and t as subscripts of φ). Then a relation of approximation can be defined on PM_e in the same way as for PM_t.

It is possible to formally identify each $f \in PM_t^{PM_t}$ with the function $\rho(f) \in P(\{0,1\}^{\{0,1\}})$, where $\rho : PM_t^{PM_t} \to P(\{0,1\}^{\{0,1\}})$ is the surjective function defined as follows:

$$\rho(f) = \lambda x \in \{0,1\} \cdot \begin{cases} f(x) & \text{if } f(x) \neq \varphi \\ \text{undefined} & \text{otherwise.} \end{cases}$$

From this point of view, a function $f \in PM_t^{PM_t}$ is *total* if $f(x) \neq \varphi$ for every $x \neq \varphi$, and *nontotal* if $f(x) = \varphi$ for at least one $x \neq \varphi$. So it seems that only elements in $\{0,1\}$ are relevant arguments for the functions in $PM_t^{PM_t}$. Consequently, we may think that only *strict functions* in $PM_t^{PM_t}$ represent suitably partial functions from $\{0,1\}$ into $\{0,1\}$, where by a strict function in $PM_t^{PM_t}$ we mean any function f in this set such that $f(\varphi) = \varphi$. But we must be careful, for it is possible that a total function might be defined for an undefined argument of the right type.

For instance, consider the function $f = \lambda x.\ 0 \in \{0,1\}^{\{0,1\}}$ and let $y = 1$ or $y = 0$. The function f is thus of type (t, t) and the object y, being of type t, is a possible argument of f. But this y is not defined precisely: it is either 0 or 1. However, we can infer the value of f for it: in any case, $f(y) = 0$. From this point of view, the function $f' = \lambda x.0 \in PM_t^{PM_t}$ suitably represents the function f, since $f'(\varphi) = 0$. On the other hand, given $g \in \{0,1\}^{\{0,1\}}$ such that $g(0) = 1$ and $g(1) = 0$, it is impossible to infer the value of g for y; we must therefore let $g(y)$ go undefined. So the *strict* function $g' \in PM_t^{PM_t}$ such that $g'(1) = 0$ and $g'(0) = 1$ suitably represents g.

To conclude the argument, we must remove from the set $PM_t^{PM_t}$ all nontotal functions in $PM_t^{PM_t}$ which are not strict, and also all nonstrict and total functions $f \in PM_t^{PM_t}$ such that $f(x) \neq f(\varphi)$ for at least one $x \neq \varphi$. All the remainders suitably represent partial functions of type (t, t).

The set PM_{tt} of all partial functions of type (t, t) can be precisely defined by means of a monotonicity constraint. Let $X(=(X,\leq))$ and $Y(=(Y,\leq))$ be two partially ordered sets; we say that a function $f \in Y^X$ is *$\leq$-monotone* (henceforth, *monotone*) iff for any $x, x' \in X$, $x \leq x'$ only if $f(x) \leq f(x')$. Moreover, define:

$$(X \rightarrow Y) := \{f \in Y^X \mid f \text{ is monotone}\}.$$

The set $(X \rightarrow Y)$ is ordered pointwise: for any $f, g \in (X \rightarrow Y)$, $f \leq g$ iff for every $x \in X$, $f(x) \leq g(x)$. Now, define:

$$PM_{tt} := (PM_t \rightarrow PM_t).$$

Again $(PM_{tt},\leq)$ is a meet-semilattice. If we represent each $f \in PM_{tt}$ by the image of PM_t under f:

$$f = \begin{matrix} f(0) & & f(1) \\ & f(\varphi) & \end{matrix}$$

then we may represent PM_{tt} by Figure 2.

If we compare this figure with Figure 1, we immediately see that each nontotal function $\underline{n}$ in $P(\{0,1\}^{\{0,1\}})$ can be identified with the nontotal function $\underline{n}'$ in PM_{tt}. Moreover, the total functions $\underline{2}$ and $\underline{3}$ in $P(\{0,1\}^{\{0,1\}})$ can naturally be identified with the total functions $\underline{2}'$ and $\underline{3}'$ in PM_{tt}, respectively. Concerning the total and constant functions $\underline{1}$ and $\underline{4}$ in $P(\{0,1\}^{\{0,1\}})$, the above considerations lead us to identify them with the functions $\underline{1}''$ and $\underline{4}''$ in PM_{tt}, respectively.

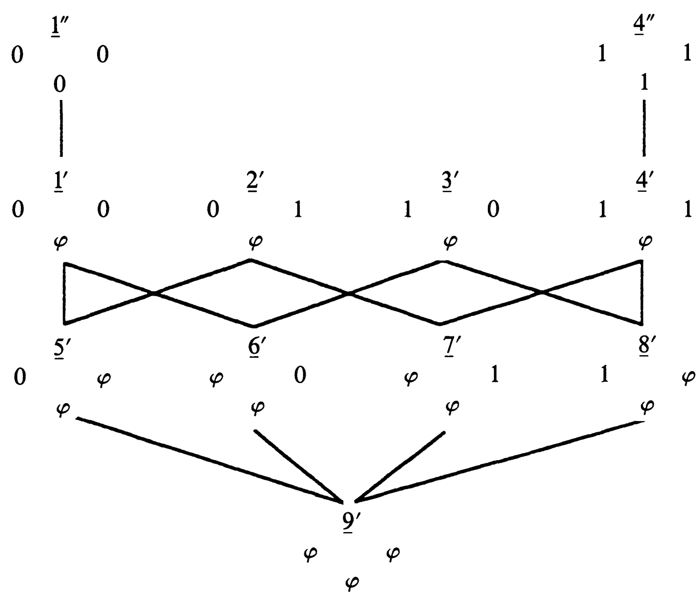


Figure 2. The approximating semilattice PM_{tt}.

Consider again the function f' defined by (2). Looking at Figure 1, we see that the functions $f'(0)$ and $f'(1)$ are, respectively, the functions $\underset{\sim}{2}$ and $\underset{\sim}{9}$ in $P(\{0,1\}^{\{0,1\}})$. Looking at Figure 2, we see that these functions correspond respectively to the functions $\underset{\sim}{2}'$ and $\underset{\sim}{9}'$ in PM_{tt}. So the function f' can be identified with the function $g': PM_t \to PM_{tt}$ such that:

$$g'(x) = \begin{cases} \lambda y.y & \text{if } x = 0 \\ \lambda y.\varphi & \text{otherwise} \end{cases}$$

and we see that unlike the application of f' to $(f'(1))(1)$, the application of g' to $(g'(1))(1)$ makes sense. Indeed, $g'((g'(1))(1)) = g'(\lambda y.\varphi(1)) = g'(\varphi) = \lambda y.\varphi$. Note that according to our convention, the function $\lambda y.\varphi$ stands for the undefined of type (t,t); hence from this point of view $g'((g'(1))(1))$ is undefined. This identification is no doubt artificial, but it is nevertheless adequate for our purpose. Moreover, it is easy to verify that g' is monotone relative to the partial order on PM_t, so it belongs to the set:

$$PM_{t(tt)} := (PM_t \to (PM_t \to PM_t)).$$

Now if we define a partial order on the product $PM_t \times PM_t$ in the standard way, that is to say as follows: for (x,y), $(x',y') \in PM_t \times PM_t$, $(x,y) \leq (x',y')$ iff $x \leq x'$ and $y \leq y'$, then we can easily verify that $(PM_t \times PM_t \to PM_t)$ is isomorphic to $PM_{t(tt)}$. Hence not only monotonicity allows us to define suitably partial functions; the sets of functions restricted to monotone functions also satisfy Schönfinkel's theorem.

Consider again Figure 2. According to the terminology we shall adopt in the following sections, the functions $\underset{\sim}{1}'$, $\underset{\sim}{1}''$, $\underset{\sim}{2}'$, $\underset{\sim}{3}'$, $\underset{\sim}{4}'$ and $\underset{\sim}{4}''$ in PM_{tt} are *(partial) total objects*. However, only functions $\underset{\sim}{1}''\underset{\sim}{2}'$, $\underset{\sim}{3}'$ and $\underset{\sim}{4}''$ are *maximal approximations* of classical objects in $\{0,1\}^{\{0,1\}}$. By determining the maximal approximation of each classical object, we can compare, in terms of approximation, partial objects with classical objects and vice versa. For instance, function $\underset{\sim}{1}''$ in PM_{tt} maximally approximates function $\lambda x.0 \in \{0,1\}^{\{0,1\}}$. Hence every function $f \in PM_{tt}$ such that $f \leq \underset{\sim}{1}''$ can be seen as an approximation of $\lambda x.0 \in \{0,1\}^{\{0,1\}}$.

This approach can be applied to Modal Type Theory.

3.2 The domains of partial objects

Notational convention 1 In what follows, the symbols $x, y, z, \ldots, x', y', z', \ldots$ are used as metavariables of semantical objects. The symbols $f, g, h, \ldots, f', g'$, $h', \ldots$ are more specifically used as metavariables of semantical objects of functional types.

Definition 2 Let T be the modal hierarchy of types and let E, I be two nonempty and disjoint sets. *The standard system of partial objects based on E and I* is the indexed family $\{PM_\alpha\}_{\alpha \in T}$ of partially ordered sets, such that:

(i) $PM_e = E \cup \{\varphi\}$, where for $x, y \in PM_e: x \leq y$ iff $x = \varphi$ or $x = y$
(ii) $PM_t = \{0, 1, \varphi\}$, where for $x, y \in PM_t: x \leq y$ iff $x = \varphi$ or $x = y$
(iii) $PM_{\alpha\beta} = (PM_\alpha \to PM_\beta)$, where for $f, g \in PM_{\alpha\beta}$:

$$f \leq g \text{ iff for every } x \in PM_\alpha, f(x) \leq g(x)$$

(iv) $PM_{s\alpha} = PM_{\alpha}^{I}$, where for $f, g \in PM_{s\alpha}$:

$$f \leq g \text{ iff for every } i \in I, f(i) \leq g(i).$$

Remark 3 It is needless to define a partial order relation on the set I, since there is no function having I as its target set. Every domain PM_α stands for the set of Lepage's *good representations* of classical objects in M_α (see [4]), provided that the systems $\{M_\alpha\}_{\alpha \in T}$ and $\{PM_\alpha\}_{\alpha \in T}$ are both based on the same sets E and I.

The next proposition is standard and can be easily proven.

Proposition 4 *For every $\sigma \in T$, PM_σ is a meet-semilattice, where for $x, y \in PM_\sigma$, the infimum of x and y, denoted by $x \wedge y$, is inductively given as follows:*
(i) *for $\sigma = e$ or $t : x \wedge y = x$ if $x = y$, otherwise $x \wedge y = \varphi$;*
(ii) *for $\sigma = \alpha\beta : f \wedge g = \lambda x \in PM_\alpha . f(x) \wedge g(x)$;*
(iii) *for $\sigma = s\alpha : f \wedge g = \lambda i \in I . f(i) \wedge g(i)$.*

Moreover, if the supremum of x and y, denoted by $x \vee y$, exists, then:
(i) *for $\sigma = e$ or $t : x \vee y = y$ if $x = \varphi$,*
$\qquad = x$ *if $y = \varphi$ or $x = y$,*
(ii) *for $\sigma = \alpha\beta : f \vee g = \lambda x \in PM_\alpha . f(x) \vee g(x)$;*
(iii) *for $\sigma = s\alpha : f \vee g = \lambda i \in I . f(i) \vee g(i)$.*

The notation defined below is very important. We borrow it from [5].

Definition 5 For every $\sigma \in T$, we inductively define the *strong difference* between objects in PM_σ (formally $x \neq^* y$) as follows:

(i) for $\sigma = e$ or $t : x \neq^* y$ iff $x \neq \varphi$, $y \neq \varphi$ and $x \neq y$;
(ii) for $\sigma = \alpha\beta : f \neq^* g$ iff there exists $x \in PM_\alpha$ such that $f(x) \neq^* g(x)$;
(iii) for $\sigma = s\alpha : f \neq^* g$ iff there exists $i \in I$ such that $f(i) \neq^* g(i)$.

Intuitively, strong difference means *incompatibility*: two distinct partial objects are not necessarily incompatible. Hence, the next proposition says that incompatibility is itself monotone.

Proposition 6 *For every $\sigma \in T$ and for $x, y, x', y' \in PM_\sigma$: if $x \neq^* y$, $x \leq x'$ and $y \leq y'$, then $x' \neq^* y'$.*

Proof: We proceed by induction on the basis of types e and t.

(*i*) $\sigma = e$ or t. If $x \neq^* y$, then by definition $x \neq \varphi$, $y \neq \varphi$ and $x \neq y$. So by definition again, if $x \leq x'$ and $y \leq y'$, then $x = x'$ and $y = y'$. Therefore $x' \neq^* y'$.

(ii) $\sigma = \alpha\beta$. If $f \neq^* g$, then by definition, there exists $x \in PM_\alpha$ such that $f(x) \neq^* g(x)$, so that if $f \leq f'$ and $g \leq g'$, then $f(x) \leq f'(x)$ and $g(x) \leq g'(x)$, which implies $f'(x) \neq^* g'(x)$ by the induction hypothesis. Therefore $f' \neq^* g'$.

(iii) $\sigma = s\alpha$. As in (ii).

According to the next proposition, compatible objects have a join.

Proposition 7 *For every $\sigma \in T$, let C_σ be the set of all nonempty subsets of PM_σ which contain only objects which are not pairwise strongly different (formally we write $\neg(x \neq^* y)$ to express that x and y are not strongly different):*

$$C_\sigma := \{X \subseteq PM_\sigma \mid X \neq \varnothing \ \&\ \forall x, y \in PM_\sigma : x, y \in X \Rightarrow \neg(x \neq^* y)\};$$

then for every $X \in C_\sigma$, $\bigvee X$ exists in PM_σ.

Proof: We proceed by induction on the basis of type e and t.

(i) $\sigma = e$ or t. This is straightforward.

(ii) $\sigma = \alpha\beta$. Let $F \in C_\sigma$; so for every $x \in PM_\alpha$, $\{f(x) \mid f \in F\} \in C_\beta$. Indeed, suppose the contrary. Then there exists $x \in PM_\alpha$ and there exists $f, g \in F$ such that $f(x) \neq^* g(x)$. But this implies that $f \neq^* g$, which contradicts that $f, g \in F$.

Since for every $x \in PM_\alpha$, $\{f(x) \mid f \in F\} \in C_\beta$, then by the induction hypothesis, for every $x \in PM_\alpha$, $\bigvee\{f(x) \mid f \in F\}$ exists in PM_β. Therefore, the function:

$$\lambda x \in PM_\alpha . \bigvee\{f(x) \mid f \in F\}$$

is well defined, exists in PM_σ, and is exactly $\bigvee F$.

(iii) $\sigma = s\alpha$. As in (ii).

Now we can begin comparing partial objects with classical objects.

3.3 Comparing partial and classical objects Henceforth, given two nonempty sets E and I, we shall consider $\{M_\alpha\}_{\alpha \in T}$ to be the standard system of classical objects based on E and I and $\{PM_\alpha\}_{\alpha \in T}$ to be the standard system of partial objects based on the same E and I. No confusion should arise.

At first glance, comparison in terms of approximation of partials and classical objects is easily done. Consider the following definition, suggested by Lepage in [6]. First, inductively define, for every $\sigma \in T$, a relation $\leq^\circ \subseteq PM_\sigma \times M_\sigma$ as follows ($x \leq^\circ y$ means "x is an approximation of y"):

(i) for $\sigma = e$ or $t : x \leq^\circ y$ iff $x = \varphi$ or $x = y$;
(ii) for $\sigma = \alpha\beta : f \leq^\circ g$ iff for every $x \in PM_\alpha$ and every $z \in M_\alpha$ such that $x \leq^\circ z$, $f(x) \leq^\circ g(z)$;
(iii) for $\sigma = s\alpha : f \leq^\circ g$ iff for every $i \in I$, $f(i) \leq^\circ g(i)$.

Surely this definition is intuitively suitable. Moreover, every classical object $z \in M_\sigma$ is naturally identifiable with the partial object $x \in PM_\sigma$ which maximally approximates z, x being simply $\bigvee\{x' \in PM_\sigma \mid x' \leq^\circ z\}$. Unfortunately, it appears far from obvious how to demonstrate the existence of such an object, and at any rate the proof, were it possible, would require far too many lemmas. But we think that there is a simpler, more general, and more elegant way to obtain the same result.

To begin with, we shall inductively define, for every $\sigma \in T$, the property of *being a total object in PM_σ*. Intuitively, to be a total object in PM_σ is to be an approximation of exactly one classical object in M_σ.

Definition 8 We inductively define, for every $\sigma \in T$, the set $PT_\sigma (\subseteq PM_\sigma)$ of *total objects* in PM_σ as follows:

(i) for $\sigma = e$ or $t : PT_\sigma := \{x \in PM_\sigma \mid x \neq \varphi\}$;
(ii) for $\sigma = \alpha\beta : PT_\sigma := \{f \in PM_\sigma \mid \forall x \in PT_\alpha : f(x) \in PT_\beta\}$;
(iii) for $\sigma = s\alpha : PT_\sigma := \{f \in PM_\sigma \mid \forall i \in I : f(i) \in PT_\alpha\}$.

Remark 9 Obviously, $PT_e = M_e = E$ and $PT_t = M_t = \{0, 1\}$. However, for every functional type $\alpha\beta$, it is not the case that $M_{\alpha\beta} \approx PT_{\alpha\beta}$. For instance, M_{tt}

contains exactly four objects, whereas PT_{tt} contains six objects—if one takes a look at Figure 2 of Section 3.1, one sees that $PT_{tt} = \{\underline{1}', \underline{2}', \underline{3}', \underline{4}', \underline{1}'', \underline{4}''\}$. But an equivalence relation can be defined on a set of total objects. For instance, the functions $\underline{1}'$ and $\underline{1}''$ in PT_{tt}, *restricted* to PT_t, are identical; so we may consider that both are very good approximations of the function $\lambda x.0$ in M_{tt}. Similarly, the functions $\underline{4}'$ and $\underline{4}''$ in PT_{tt} may both be thought of as very good approximations of the function $\lambda x.1$ in M_{tt}. From this point of view, we can consider $\underline{1}'$ and $\underline{1}''$ (and $\underline{4}'$ and $\underline{4}''$) as equivalent total objects. The next definition generalizes this view to all types.

Definition 10 For every $\sigma \in T$, we define inductively an equivalence relation between objects in PT_σ (formally, $x <> y$) as follows:

(i) for $\sigma = e$ or t and $x, y \in PT_\sigma : x <> y$ iff $x = y$;
(ii) for $\sigma = \alpha\beta$ and $f, g \in PT_\sigma : f <> g$ iff for every $x \in PT_\alpha$, $f(x) <> g(x)$;
(iii) for $\sigma = s\alpha$ and $f, g \in PT_\sigma : f <> g$ iff for every $i \in I$, $f(i) <> g(i)$.

The next proposition says that a partial object z, which is at least as defined as a total object x, is total and equivalent to that x.

Proposition 11 *For every $\sigma \in T$, every $x \in PT_\sigma$, and every $z \in PM_\sigma$: if $x \leq z$, then $z \in PT_\sigma$ and $x <> z$.*

Proof: Left to the reader (proceed by induction on the basis of types e and t).

The next proposition says that nonequivalent total objects are strongly different (we write $\neg(x <> y)$ to express that x, y are nonequivalent total objects).

Proposition 12 *For every $\sigma \in T$ and for $x, y \in PT_\sigma$: if $\neg(x <> y)$, then $x \neq^* y$.*

Proof: Left to the reader (proceed by induction on the basis of types e and t).

Definition 13 For every $\sigma \in T$ and every $x \in PT_\sigma$, define $\langle x \rangle := \{z \in PT_\sigma \mid z <> x\}$ and $\Pi_\sigma := \{\langle x \rangle \mid x \in PT_\sigma\}$. Informally, $\langle x \rangle$ is the set of total objects equivalent to x and Π_σ is the partition of PT_σ generated by $<>$.

The next proposition is very important, for it provides the key to comparing partial and classical objects (see Remark 19).

Proposition 14 *Let $x \in PM_\sigma$. If $x \in PT_\sigma$, then $\bigvee \langle x \rangle$ exists and belongs to $\langle x \rangle$ and for every $z \in PT_\sigma$ such that $x \neq^* z$, $\neg(x <> z)$. If $x \notin PT_\sigma$, then there exists $y \in PT_\sigma$ such that $x \leq y$.*

Proof: We proceed by induction on the basis of types e and t.

(i) $\sigma = e$ or t. This is straightforward.

(ii) $\sigma = \alpha\beta$. Assume that this property holds for types α and β. Let $f \in PT_\sigma$. Suppose that $\bigvee \langle f \rangle$ does not exist. Then by Proposition 7, $\langle f \rangle \notin C_\sigma$. So there exists $g, h \in \langle f \rangle$ such that $g \neq^* h$ and thus, by definition, there exists $y \in PM_\alpha$ such that $g(y) \neq^* h(y)$. But $g, h \in PT_\sigma$; thus, if $y \in PT_\alpha$, then by the induction hypothesis, $\neg(g(y) <> h(y))$, and thus by definition, $\neg(g <> h)$, which contradicts that $g, h \in \langle f \rangle$. On the other hand, if $y \notin PT_\alpha$, then by the induction hypothesis, there exists $z \in PT_\alpha$ such that $y \leq z$, and hence, since g and h are monotone, such that $g(y) \leq g(z)$ and $h(y) \leq h(z)$. But $g(y) \neq^* h(y)$; so $g(z) \neq^* h(z)$, by Proposition 6. So by the induction hypothesis, $\neg(g(z) <>$

$h(z))$, and thus $\neg(f<>g)$ by definition, which contradicts again that $g, h \in \langle f \rangle$. Therefore, $\bigvee \langle f \rangle$ does exist. A similar argument shows that for every $g \in PT_\sigma$ such that $f \neq^* g$, $\neg(f<>g)$. Moreover, since $f \leq \bigvee \langle f \rangle$, it is the case that $f<> \bigvee \langle f \rangle$, by Proposition 11. So by definition, $\bigvee \langle f \rangle \in \langle f \rangle$.

Secondly, suppose that $f \notin PT_\sigma$. We shall show that there exists $g \in PT_\sigma$ such that $f \leq g$. Indeed, if $f \notin PT_\sigma$, then by definition, there exists $x \in PT_\alpha$ such that $f(x) \notin PT_\beta$. But by the induction hypothesis, there exists $y \in PT_\beta$ such that $f(x) \leq y$, and since $\bigvee \langle x \rangle$ exists and belongs to $\langle x \rangle$, this assures that for every $x \in PT_\alpha$, the set:

$$S_{f(\bigvee \langle x \rangle)} := \{ y \in PT_\beta \,|\, f(\bigvee \langle x \rangle) \leq y \},$$

is not empty. Define

$$U_f := \{ S_{f(\bigvee \langle x \rangle)} \,|\, x \in PT_\alpha \},$$

and let χ be any function from U_f into PT_β such that for every $X \in U_f$, $\chi(X) \in X$. Now consider $g: PM_\alpha \to PM_\beta$ such that:

$$g(x) = \begin{cases} f(x) & \text{if } x \notin PT_\alpha \\ f(\bigvee \langle x \rangle) & \text{if } x \in PT_\alpha \text{ and } f(\bigvee \langle x \rangle) \in PT_\beta \\ \chi(S_{f(\bigvee \langle x \rangle)}) & \text{otherwise.} \end{cases}$$

One easily verifies that if g is monotone, then $g \in PT_\sigma$ and $f \leq g$. So let us check whether g is monotone.

Let $x, x' \in PM_\alpha$ such that $x \leq x'$. A priori, there are four possible cases for x, x':

(1) $x, x' \notin PT_\alpha$;
(2) $x \notin PT_\alpha$ and $x' \in PT_\alpha$;
(3) $x \in PT_\alpha$ and $x' \notin PT_\alpha$;
(4) $x, x' \in PT_\alpha$.

Case (3) is excluded by Proposition 11. Let us inspect the other cases.

In Case (1), $g(x) = f(x) \leq f(x') = g(x')$. Hence $g(x) \leq g(x')$.

In Case (2), $g(x) = f(x) \leq f(x') \leq f(\bigvee \langle x' \rangle) = g(x')$, if $f(\bigvee \langle x' \rangle) \in PT_\beta$. If not, $g(x) = f(x) \leq f(x') \leq \chi(S_{f(\bigvee \langle x' \rangle)}) = g(x')$. So in both cases, $g(x) \leq g(x')$.

In Case (4), we cannot have $f(\bigvee \langle x \rangle) \notin PT_\beta$ and $f(\bigvee \langle x' \rangle) \in PT_\beta$, or vice versa, because $\langle x \rangle = \langle x' \rangle$ by Proposition 11. So $\bigvee \langle x \rangle = \bigvee \langle x' \rangle$. Thus, if $f(\bigvee \langle x \rangle) \in PT_\beta$, then $g(x) = f(\bigvee \langle x \rangle) = f(\bigvee \langle x' \rangle) = g(x')$. On the other hand, if $f(\bigvee \langle z \rangle) \notin PT_\beta$, then $g(x) = \chi(S_{f(\bigvee \langle x \rangle)}) = \chi(S_{f(\bigvee \langle x' \rangle)}) = g(x')$. So in both cases, $g(x) \leq g(x')$.

So, g is monotone.

(iii) $\sigma = s\alpha$. As in (ii), but simpler.

The next proposition is analogous to Leibniz's law. Its proof requires the monotonicity of functions of type $\alpha\beta$.

Proposition 15 *For every $\alpha\beta \in T$, every $f \in PT_{\alpha\beta}$ and for $x, y \in PT_\alpha$: if $x<>y$, then $f(x)<>f(y)$.*

Proof: If $x <> y$, then $\neg(x \neq^* y)$ by Proposition 14. Hence by Proposition 7, there exists $z \in PM_\alpha$ such that $z = x \vee y$ and so such that $x \leq z$ and $y \leq z$. Therefore, by the monotonicity of f, $f(x) \leq f(z)$ and $f(y) \leq f(z)$. But $f(x), f(y) \in PT_\beta$; so by Proposition 11, $f(z) \in PT_\beta$, $f(x) <> f(z)$ and $f(y) <> f(z)$. Therefore $f(x) <> f(y)$.

Proposition 16 *(i) Let $f \in PT_{\alpha\beta}$, $x \in PT_\alpha$ and $y \in PT_\beta$ such that $(\bigvee\langle f\rangle)(\bigvee\langle x\rangle) \in \langle y\rangle$; then $(\bigvee\langle f\rangle)(\bigvee\langle x\rangle) = \bigvee\langle y\rangle$. (ii) Let $f \in PT_{s\alpha}$, $i \in I$ and $y \in PT_\alpha$ such that $(\bigvee\langle f\rangle)(i) \in \langle y\rangle$; then $(\bigvee\langle f\rangle)(i) = \bigvee\langle y\rangle$.*

Proof: (i) By Proposition 14, $\bigvee\langle f\rangle \in \langle f\rangle$, so that:

(*) For every $g \in \langle f\rangle$, $\bigvee\langle f\rangle \leq g$ only if $\bigvee\langle f\rangle = g$.

Suppose then that $(\bigvee\langle f\rangle)(\bigvee\langle x\rangle) \neq \bigvee\langle y\rangle$. Since $(\bigvee\langle f\rangle)(\bigvee\langle x\rangle) \in \langle y\rangle$, this means that there exists $z \in \langle y\rangle$ such that $\bigvee\langle f\rangle(\bigvee\langle x\rangle) = z \neq \bigvee\langle y\rangle$. Now consider the function $g: PM_\alpha \to PM_\beta$ such that for every $z' \in PM_\alpha$:

$$g(z') = \begin{cases} \bigvee\langle y\rangle & \text{if } z' = \bigvee\langle x\rangle \\ (\bigvee\langle f\rangle)(z') & \text{otherwise.} \end{cases}$$

One can easily verify that if g exists, then $g \in PT_{\alpha\beta}$, $\bigvee\langle f\rangle \leq g$ and $g \neq \bigvee\langle f\rangle$. But by Proposition 11, this implies that $g \in \langle f\rangle$, contradicting (*) above. So we conclude that $(\bigvee\langle f\rangle)(\bigvee\langle x\rangle) = \bigvee\langle y\rangle$.

(ii) As in (i), but simpler.

We would like to associate with every classical object $x \in M_\sigma$ the equivalence class $X \in \Pi_\sigma$ of objects in PT_σ which approximates x. Inversely, we would like to associate with each equivalence class $X \in \Pi_\sigma$ the classical object $x \in M_\sigma$ which is approximated by the objects in X. This is the purpose of the next definition.

Definition 17 We define inductively two functions:

$$\Phi: \bigcup_{\sigma\in T} M_\sigma \to \bigcup_{\sigma\in T} \Pi_\sigma$$

$$\Theta: \bigcup_{\sigma\in T} \Pi_\sigma \to \bigcup_{\sigma\in T} M_\sigma$$

as follows:

(i) for $\sigma = e$ or t: $\Phi(x) = \{x\}$ and $\Theta(\{x\}) = x$;

(ii) for $\sigma = \alpha\beta$:

$$\Phi(f) = \{f' \in PT_\sigma \mid \forall x \in M_\alpha : \forall y \in \Phi(x) : f'(y) \in \Phi(f(x))\};$$

$$\Theta(F) = \lambda x \in M_\alpha . \Theta(\{z \in PT_\beta \mid \forall g \in F : \forall y \in \Phi(x) : z <> g(y)\});$$

(iii) for $\sigma = s\alpha$:

$$\Phi(f) = \{f' \in PT_\sigma \mid \forall i \in I : f'(i) \in \Phi(f(i))\};$$

$$\Theta(F) = \lambda i \in I . \Theta(\{z \in PT_\alpha \mid \forall g \in F : z <> g(i)\}).$$

Proposition 18 *Φ is bijective and $\Phi^{-1} = \Theta$.*

Proof: It is sufficient to show that $\Theta(\Phi(x)) = x$ and $\Phi(\Theta(X)) = X$. One proceeds by induction on the basis of types e and t.

(i) $\sigma = e$ or t. This is straightforward.

(ii) $\sigma = \alpha\beta$. Suppose that this property holds for types α and β. Let $f \in M_\sigma$ and $F = \Phi(f)$. So $\Theta(\Phi(f)) = \Theta(F) =$

(1) $$\lambda x \in M_\alpha . \Theta(\{z \in PT_\beta \mid \forall g \in F : \forall y \in \Phi(x) : z <> g(y)\}).$$

By definition, $g(y) \in \Phi(f(x))$ for every $g \in F$, every $x \in M_\alpha$ and every $y \in \Phi(x)$. Therefore, **(1)** is equal to:

$$\lambda x \in M_\alpha . \Theta(\{z \in PT_\beta \mid z \in \Phi(f(x))\})$$

and this is of course equal to:

(2) $$\lambda x \in M_\alpha . \Theta(\Phi(f(x))).$$

Therefore **(2)** is equal to $\lambda x \in M_\alpha . f(x)$, that is f.

On the other hand, let $F \in \Pi_\sigma$ and $f = \Theta(F)$. So $\Phi(\Theta(F)) = \Phi(f) =$

(1′) $$\{f' \in PT_\sigma \mid \forall x \in M_\alpha : \forall y \in \Phi(x) : f'(y) \in \Phi(f(x))\}.$$

But by definition, for every $x \in M_\alpha$:

$$f(x) = \Theta(\{z \in PT_\beta \mid \forall g \in F : \forall y \in \Phi(x) : z <> g(y)\}).$$

Therefore **(1′)** is the set:

$$\{f' \in PT_\sigma \mid \forall x \in M_\alpha : \forall y \in \Phi(x) : f'(y) \in \{z \in PT_\beta \mid \forall g \in F : \forall y \in \Phi(x) : z <> g(y)\}\},$$

which is equal to the set:

$$F' = \{f' \in PT_\sigma \mid \forall x \in M_\alpha : \forall y \in \Phi(x) : \forall g \in F : \forall z \in \Phi(x) : f'(y) <> g(z)\}$$

Now, $F \subseteq F'$. Indeed, suppose there exists $f \in F$ such that $f \notin F'$. Therefore:

(2′) $$\exists x \in M_\alpha : \exists y \in \Phi(x) : \exists g \in F : \exists z \in \Phi(x) : \neg(f(y) <> g(z)).$$

But for every $x \in M_\alpha$ and for $y, z \in \Phi(x)$, $y, z \in PT_\alpha$ and $y <> z$. Hence **(2′)** implies:

(3′) $$\exists g \in F : \exists y, z \in PT_\alpha : y <> z \ \&\ \neg(f(y) <> g(z)).$$

But since $f \in F$, $f(x) <> g(x)$ for every $g \in F$ and every $x \in PT_\alpha$. This and Proposition 15 imply that for any $y, z \in PT_\alpha$ such that $y <> z$, $f(y) <> f(z) <> g(z)$, which implies that $f(y) <> g(z)$, and this clearly contradicts **(3′)**.

On the other hand, $F' \subseteq F$. Indeed, suppose there is $f' \in F'$ such that $f' \notin F$. This means that $\neg(f' <> g)$ for every $g \in F$, and so:

(4′) $$\exists z \in PT_\alpha : \neg(f'(z) <> g(z)).$$

But since $f' \in F'$:

(5′) $$\forall x \in M_\alpha : \forall y \in \Phi(x) : f'(y) <> g(y).$$

So **(4′)**, **(5′)** and Proposition 14 imply:

(6′) $$\exists z \in PT_\alpha : \forall x \in M_\alpha : \forall y \in \Phi(x) : \neg(z <> y).$$

But $PT_\alpha = \{y \in \Phi(x) \mid x \in M_\alpha\}$. Therefore (6′) is equivalent to:

$$\exists z \in PT_\alpha : \forall y \in PT_\alpha : \neg(z <> y),$$

and this implies that there exists $z \in PT_\alpha$ such that $\neg(z <> z)$, which is absurd.

(iii) $\sigma = s\alpha$. Assuming that this property holds for type α, the proof is as in (ii), but simpler.

Remark 19 Now we know that for every $\sigma \in T$, $M_\sigma \approx \Pi_\sigma$. Moreover, Proposition 14 assures that for every classical object $x \in M_\sigma$, the object $\bigvee \Phi(x) \in PT_\sigma$ is the unique partial total object which maximally approximates x. Therefore we can identify each classical object $x \in M_\sigma$ with its maximal approximation $\bigvee \Phi(x)$, and then consider any partial object $y \in PM_\sigma$ such that $y \leq \bigvee \Phi(x)$, as an approximation of x.

Notational convention 20 For every $\sigma \in T$ and every $x \in M_\sigma$, we denote $\bigvee \Phi(x)$ (the maximal approximation of x) by "ma(x)".

Proposition 21 *(i) Let $f \in M_{\alpha\beta}$ and $x \in M_\alpha$; then* $(\mathrm{ma}(f))(\mathrm{ma}(x)) = \mathrm{ma}(f(x))$. (ii) *Let $f \in M_{s\alpha}$; then for every $i \in I$,* $(\mathrm{ma}(f))(i) = \mathrm{ma}(f(i))$.

Proof: (i) By Definition 17(ii) and Proposition 14, $(\mathrm{ma}(f))(\mathrm{ma}(x)) \in \Phi(f(x))$. Therefore by Proposition 16(i), $(\mathrm{ma}(f))(\mathrm{ma}(x)) = \mathrm{ma}(f(x))$. (ii) By definition 17(iii) and Proposition 14, $(\mathrm{ma}(f))(i) \in \Phi(f(i))$. Therefore, by Proposition 16(ii), $(\mathrm{ma}(f))(i) = \mathrm{ma}(f(i))$.

Remark 22 The partial functions of type (t, t) or $(t, (t, t))$ of our system which correspond to the truth functions of Kleene's strong three-valued logic (**KSL**) are exactly the maximal approximations of the classical truth functions. Indeed, to the function of negation according to **KSL** corresponds the function $\underline{3}'$ in PM_{tt} (see Figure 2, Section 3.1) and this function is obviously the maximal approximation of the classical truth function of negation. Moreover, to each binary truth function of **KSL** corresponds a function f in $PM_{t(tt)}$ which can be represented by the image of PM_t under f, as follows:

Conjunction

0 0 0 1
0 φ
0 φ
φ

Disjunction

0 1 1 1
φ 1
φ 1
φ

Conditional

1 1 0 1
1 φ
φ 1
φ

Biconditional

1 0 0 1
φ φ
φ φ
φ

It is thus easy to check that each of these functions belongs to $PT_{t(tt)}$, is not dominated by another function, and so is the maximal approximation of its classical analogue.

This ends the description of the domains of partial objects. We can now define the notion of a standard partial model for the language of **IL**.

3.4. Definition of the notion of a standard partial model Let E and I be two nonempty and disjoint sets. A *standard partial model based on E and I* is an ordered pair $PM = \langle\{PM_\alpha\}_{\alpha\in T}, pm\rangle$, where $\{PM_\alpha\}_{\alpha\in T}$ is the standard system of partial objects based on E and I, and pm is a function from all constants such that for every constant $c_\alpha, pm(c_\alpha) \in PM_{s\alpha}$. We denote by $\mathrm{As}(PM)$ the *set of assignments over PM*, that is the set of all functions **pa** from all variables such that for every variable x_α, $\mathbf{pa}(x_\alpha) \in PM_\alpha$. For every $\mathbf{pa} \in \mathrm{As}(PM)$, every variable x_α and every $z \in M_\alpha$, $\mathbf{pa}(x_\alpha/z)$ is that assignment in $\mathrm{As}(PM)$ such that $\mathbf{pa}(x_\alpha/z)(x_\alpha) = z$ and for every variable $x_\beta \neq x_\alpha$, $\mathbf{pa}(x_\alpha/z)(x_\beta) = \mathbf{pa}(x_\beta)$. We recursively define the *partial value* $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}$ *in PM* of a term A_α *according to* an assignment $\mathbf{pa} \in \mathrm{As}(PM)$ and an $i \in I$ as follows (in what follows we shall sometimes omit the superscript "PM"):

(i) $[\![c_\alpha]\!]_{\mathbf{pa},i} = (pm(c_\alpha))(i)$;
(ii) $[\![x_\alpha]\!]_{\mathbf{pa},i} = \mathbf{pa}(x_\alpha)$;
(iii) $[\![A_{\alpha\beta}B_\alpha]\!]_{\mathbf{pa},i} = [\![A_{\alpha\beta}]\!]_{\mathbf{pa},i}([\![B_\alpha]\!]_{\mathbf{pa},i})$;
(iv) $[\![{}^\wedge A_\alpha]\!]_{\mathbf{pa},i}$ = the function f from I such that for every $j \in I$, $f(j) = [\![A_\alpha]\!]_{\mathbf{pa},j}$;
(v) $[\![{}^\vee A_{s\alpha}]\!]_{\mathbf{pa},i} = [\![A_{s\alpha}]\!]_{\mathbf{pa},i}(i)$;
(vi) $[\![\lambda x_\alpha A_\beta]\!]_{\mathbf{pa},i}$ = the function f from PM_α such that for every $z \in PM_\alpha$, $f(z) = [\![A_\beta]\!]_{\mathbf{pa}',i}$ where $\mathbf{pa}' = \mathbf{pa}(x_\alpha/z)$;
(vii) $$[\![[A_\alpha \equiv B_\alpha]]\!]_{\mathbf{pa},i} = \begin{cases} 1 & \text{if } [\![A_\alpha]\!]_{\mathbf{pa},i}, [\![B_\alpha]\!]_{\mathbf{pa},i} \in PT_\alpha \\ & \quad \text{and } [\![A_\alpha]\!]_{\mathbf{pa},i} <> [\![B_\alpha]\!]_{\mathbf{pa},i} \\ 0 & \text{if } [\![A_\alpha]\!]_{\mathbf{pa},i} \neq^* [\![B_\alpha]\!]_{\mathbf{pa},i} \\ \varphi & \text{otherwise.} \end{cases}$$

Remark 23 The rule (vii) of identity makes indiscernible, in the object language, all total equivalent objects, and by Propositions 12 and 14 (which imply that $x \neq^* y$ iff $\neg(x <> y)$) it makes discernible all nonequivalent total objects. So identity between total objects has the same behavior as standard identity between classical objects. This expresses the idea that if the *known denotations* of two expressions are total objects, then one can safely determine whether or not the two expressions *in fact* denote the same thing. On the other hand, it is sufficient that the known denotations of two expressions be incompatible for one to be able to infer that the denotations of the expressions are not in fact the same. However, if the known denotations of two expressions are not total but are compatible, one cannot determine whether or not the two expressions in fact denote the same thing. This is the case even when the known denotations of both expressions are equally partially defined.

Proposition 24 *Let* $PM = \langle\{PM_\alpha\}_{\alpha\in T}, pm\rangle$ *and* $PM' = \langle\{PM\}_{\alpha\in T}, pm'\rangle$ *be two partial models such that for every* $\alpha \in T$ *and every constant* $c_\alpha, pm(c_\alpha) \leq$

$pm'(c_\alpha)$; moreover, let $\mathbf{pa}, \mathbf{pa}' \in \mathrm{As}(PM)$ *be two assignments such that for every* $\alpha \in T$ *and every variable* x_α, $\mathbf{pa}(x_\alpha) \leq \mathbf{pa}'(x_\alpha)$. *For every* $\alpha \in T$, *every term* A_α *and every* $i \in I$:

(i) $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}, [\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i} \in PM_\alpha$;

(ii) $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$.

Proof: This is immediately verified in the case where A_α is a constant or a variable—this follows from the definitions of pm, pm', $\mathbf{pa}$, and $\mathbf{pa}'$. The other cases are as follows.

- Consider a term $[A_{\alpha\beta}B_\alpha]$. (i) By the induction hypothesis, $[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i} \in PM_{\alpha\beta}$ and $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \in PM_\alpha$, which implies that $[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i}([\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}) = [\![[A_{\alpha\beta}B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \in PM_\beta$. Similarly, $[\![[A_{\alpha\beta}B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i} \in PM_\beta$. (ii) By the induction hypothesis, $[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_{\alpha\beta}]\!]^{PM'}_{\mathbf{pa}',i}$ and $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![B_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$. By the definition of the partial order on $PM_{\alpha\beta}$ and by the monotonicity of the objects in $PM_{\alpha\beta}$, this implies that $[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i}([\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}) = [\![[A_{\alpha\beta}B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_{\alpha\beta}]\!]^{PM'}_{\mathbf{pa}',i}([\![B_\alpha]\!]^{PM'}_{\mathbf{pa}',i}) = [\![[A_{\alpha\beta}B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i}$.
- Consider a term $^\wedge A_\alpha$. (i) By the induction hypothesis, for every $j \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},j} \in PM_\alpha$. But by definition, $[\![^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i}$ is the function f from I such that for every $j \in I$, $f(j) = [\![A_\alpha]\!]^{PM}_{\mathbf{pa},j}$, and so, $[\![^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i} \in PM^I_\alpha = PM_{s\alpha}$. Similarly, $[\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa}',i} \in PM_{s\alpha}$. (ii) By definition, for every $j \in I$, $[\![^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i}(j) = [\![A_\alpha]\!]^{PM}_{\mathbf{pa},j}$ and $[\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa}',i}(j) = [\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa}',j}$. By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},j} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',j}$ for every $j \in I$. By the definition of the partial order on $PM_{s\alpha}$, this means that $[\![^\wedge A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![^\wedge A_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$.
- Consider a term $^\vee A_{s\alpha}$. (i) By the induction hypothesis, for every $i \in I$, $[\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} \in PM^I_\alpha = PM_{s\alpha}$ and by definition, $[\![^\vee A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} = [\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i}(i) \in PM_\alpha$. Similarly, $[\![^\vee A_{s\alpha}]\!]^{PM'}_{\mathbf{pa}',i} \in PM_\alpha$. (ii) By definition, for every $i \in I$, $[\![^\vee A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} = [\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i}(i)$ and $[\![^\vee A_{s\alpha}]\!]^{PM'}_{\mathbf{pa}',i} = [\![A_{s\alpha}]\!]^{PM'}_{pa',i}(i)$. By the induction hypothesis, $[\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_{s\alpha}]\!]^{PM'}_{\mathbf{pa}',i}$ for every $i \in I$. Therefore $[\![A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i}(i) \leq [\![A_{s\alpha}]\!]^{PM'}_{\mathbf{pa}',i}(i)$. This means that $[\![^\vee A_{s\alpha}]\!]^{PM}_{\mathbf{pa},i} \leq [\![^\vee A_{s\alpha}]\!]^{PM'}_{\mathbf{pa}',i}$.
- Consider a term $\lambda x_\alpha A_\beta$. (i) Let $z, z' \in PM_\alpha$ such that $z \leq z'$ and let $\mathbf{pb} = \mathbf{pa}(x_\alpha/z)$ and $\mathbf{pb}' = \mathbf{pa}(x_\alpha/z')$. So, for any $\sigma \in T$ and every variable x_σ, $\mathbf{pb}(x_\sigma) \leq \mathbf{pb}'(x_\sigma)$. So, by the induction hypothesis, $[\![A_\beta]\!]^{PM}_{\mathbf{pb},i'}$, $[\![A_\beta]\!]^{PM}_{\mathbf{pb}',i} \in PM_\beta$ and $[\![A_\beta]\!]^{PM}_{\mathbf{pb},i} \leq [\![A_\beta]\!]^{PM}_{\mathbf{pb}',i}$. But by definition, $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(z) = [\![A_\beta]\!]^{PM}_{\mathbf{pb},i}$ and $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(z') = [\![A_\beta]\!]^{PM}_{\mathbf{pb}',i}$. Therefore, $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i} \in PM_{\alpha\beta}$. Similarly, $[\![\lambda x_\alpha A_\beta]\!]^{PM'}_{\mathbf{pa}',i} \in PM_{\alpha\beta}$. (ii) Again, by definition, for every $z \in PM_\alpha$, $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i}(z) = [\![A_\beta]\!]^{PM}_{\mathbf{pb},i}$, where $\mathbf{pb} = \mathbf{pa}(x_\alpha/z)$, and $[\![\lambda x_\alpha A_\beta]\!]^{PM'}_{\mathbf{pa}',i}(z) = [\![A_\beta]\!]^{PM'}_{\mathbf{pb}',i}$, where $\mathbf{pb}' = \mathbf{pa}'(x_\alpha/z)$. But by the induction hypothesis, $[\![A_\beta]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\beta]\!]^{PM'}_{\mathbf{pa}',i}$ and of course, $\mathbf{pa}(x_\alpha/z)(x_\alpha) \leq \mathbf{pa}'(x_\alpha/z)(x_\alpha)$ for every $z \in PM_\alpha$. This means that $[\![\lambda x_\alpha A_\beta]\!]^{PM}_{\mathbf{pa},i} \leq [\![\lambda x_\alpha A_\beta]\!]^{PM'}_{\mathbf{pa}',i}$.
- Consider a term $[A_\alpha \equiv B_\alpha]$. (i) By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}$, $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \in PM_\alpha$. So since $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i}$ is either 0, or 1, or φ, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \in PM_t$. Similarly, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i} \in PM_t$. (ii) By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$ and $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![B_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$ and there are only three possible cases for $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i}$:

(1) $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = 1$;
(2) $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = 0$;
(3) $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = \varphi$.

In case (1), $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}$, $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \in PT_\alpha$ and $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} <> [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}$. So by Proposition 11, $[\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$, $[\![B_\alpha]\!]^{PM'}_{\mathbf{pa}',i} \in PT_\alpha$ and $[\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i} <> [\![B_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$, and so, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i} = 1$. Therefore, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \leq [\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i}$.

In case (2), $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \neq^* [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}$. So by Proposition 6, $[\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i} \neq^* [\![B_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$, which implies that $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i} = 0$. Therefore, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i} \leq [\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i}$.

In case (3), this is trivial, since $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i}$ is either 0, or 1, or φ, so that in any case, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} \leq [\![[A_\alpha \equiv B_\alpha]]\!]^{PM'}_{\mathbf{pa}',i}$.

Remark 25 Consider the abbreviations given in Section 2. On the basis of the definition of identity between partial objects, it is easy to verify that "T" denotes 1, "F" denotes 0 and that the definition of "$\neg$" is equivalent to the definition of the negation according to **KSL**. The definition of "$\wedge$" induces the following truth conditions (we consider that the values are relative to some fixed assignment **pa** and some fixed $i \in I$):

TC.$\wedge$
$$[\![[A_t \wedge B_t]]\!] = \begin{cases} 1 & \text{if } [\![A]\!] = [\![B]\!] = 1 \\ 0 & \text{if } [\![A]\!] = 0 \text{ or } [\![B]\!] = 0 \\ \varphi & \text{otherwise.} \end{cases}$$

These truth conditions are exactly those of the conjunction according to **KSL**. Indeed, if $[\![A]\!] = [\![B]\!] = 1$, then $[\![[\lambda z_{t(tt)}[[zA]B] \equiv \lambda z_{t(tt)}[[z\mathsf{T}]\mathsf{T}]]]\!] = 1$, according to Proposition 15. On the other hand, the falsity condition of $[A \wedge B]$ is:

There exists $f \in PM_{t(tt)}$ such that $f([\![A]\!])([\![B]\!]) \neq^* f(1)(1)$.

Clearly, $[\![A]\!]$ and $[\![B]\!]$ cannot both be 1; nor could one of them be φ when the other is either 1 or φ, by f's monotonicity. On the other hand, if either $[\![A]\!]$ or $[\![B]\!]$ (or both) is 0, then the condition is satisfied by taking, as a witness of f, the truth function of conjunction in **KSL**.

Notice that in Gallin's presentation of **IL**, conjunction is introduced by the following definition:

G.$\wedge$
$$\wedge := \lambda x_t \lambda y_t [\lambda z_{tt}[[zx] \equiv y] \equiv \lambda z_{tt}[z\mathsf{T}]].$$

In the classical semantics, G.$\wedge$ induces the classical truth conditions. But in the partial semantics, G.$\wedge$ induces the following clause:

$$[\![[A \wedge B]]\!] = \varphi \text{ if } [\![A]\!] = 0 \text{ and } [\![B]\!] = \varphi.$$

In all other cases, however, the clauses agree with TC.$\wedge$. So with G.$\wedge$, we obtain a noncommutative conjunction in the partial semantics, which is, of course, undesirable.

Since "$\wedge$" behaves exactly like the conjunction in **KSL**, the definitions of "$\vee$" and "$\rightarrow$" are respectively equivalent to the definitions of the disjunction and the conditional in **KSL**. Notice that the biconditional is the sign "$\equiv$" restricted to Trm_t and it is easy to check that "$\equiv$" so restricted corresponds to the biconditional according to **KSL**. The definition of the universal quantifier "$\forall$" induces interesting truth conditions, for instance, one easily sees that:

$$[\![\forall x_e[A_{et}x_e]]\!] = \begin{cases} 1 & \text{if for every } x \in PM_e \text{ such that } x \neq \varphi,\ [\![A_{et}]\!](x) = 1 \\ 0 & \text{if there exists } x \in PM_e \text{ such that } [\![A_{et}]\!](x) = 0 \\ \varphi & \text{otherwise.} \end{cases}$$

At first glance, this result may appear to be inconsistent. Indeed, one could imagine that A_{et} is such that $[\![A_{et}]\!](\varphi) = 0$ and $[\![A_{et}]\!](x) = 1$ for every $x \in PM_e$ such that $x \neq \varphi$. Clearly the result would be that $[\![\forall x_e[A_{et}x_e]]\!] = 1$ and 0. But in fact no predicate may have this behavior, for this behavior is not monotone. Monotonicity implies that if there is an $x \in PM_e$ such that $[\![A_{et}]\!](x) = 0$, then $[\![A_{et}]\!](z) = 0$ for every $z \in PM_e$ such that $x \leq z$. One can verify that the definition of the existential quantifier "$\exists$" induces the following truth conditions:

$$[\![\exists x_e[A_{et}x_e]]\!] = \begin{cases} 1 & \text{if there exists } x \in PM_e \text{ such that } [\![A_{et}]\!](x) = 1 \\ 0 & \text{if for every } x \in PM_e \text{ such that } x \neq \varphi,\ [\![A_{et}]\!](x) = 0 \\ \varphi & \text{otherwise.} \end{cases}$$

Finally, one may easily see that the definitions of the modal operators induce the following truth conditions:

$$[\![\Box A_t]\!]_{\mathbf{pa},i} = \begin{cases} 1 & \text{if for every } j \in I,\ [\![A_t]\!]_{\mathbf{pa},j} = 1 \\ 0 & \text{if there exists } j \in I \text{ such that } [\![A_t]\!]_{\mathbf{pa},j} = 0 \\ \varphi & \text{otherwise.} \end{cases}$$

$$[\![\Diamond A_t]\!]_{\mathbf{pa},i} = \begin{cases} 1 & \text{if there exists } j \in I \text{ such that } [\![A_t]\!]_{\mathbf{pa},j} = 1 \\ 0 & \text{if for every } j \in I,\ [\![A_t]\!]_{\mathbf{pa},j} = 0 \\ \varphi & \text{otherwise.} \end{cases}$$

Definition 26 Let $PM = \langle\{PM_\alpha\}_{\alpha\in T}, pm\rangle$ be a standard partial model and let $\mathbf{pa} \in \mathrm{As}(PM)$. An *extension* of PM is a standard partial model $PM' = \langle\{PM_\alpha\}_{\alpha\in T}, pm'\rangle$ such that for every $\alpha \in T$ and every constant c_α, $pm(c_\alpha) \leq pm'(c_\alpha)$. An *extension of* $\mathbf{pa}$ is an assignment $\mathbf{pa}' \in \mathrm{As}(PM)$ such that for every $\alpha \in T$ and every variable x_α, $\mathbf{pa}(x_\alpha) \leq \mathbf{pa}'(x_\alpha)$. A *maximal extension* of PM is an extension $PM' = \langle\{PM_\alpha\}_{\alpha\in T}, pm'\rangle$ of PM such that for every $\alpha \in T$ and every constant c_α, $pm'(c_\alpha) = \bigvee\langle x\rangle$ for some $x \in PT_{s\alpha}$. A *maximal extension* of $\mathbf{pa}$ is an extension $\mathbf{pa}'$ of $\mathbf{pa}$ such that for every $\alpha \in T$ and every variable x_α, $\mathbf{pa}'(x_\alpha) = \bigvee\langle x\rangle$ for some $x \in PT_\alpha$. Finally, a standard partial model is *total* if it is a maximal extension of some standard partial model.

Proposition 27 *Let PM be a standard partial model and let* $\mathbf{pa} \in \mathrm{As}(PM)$; *then*: (i) *PM has a maximal extension* (*not necessarily unique*); (ii) $\mathbf{pa}$ *has a maximal extension* (*not necessarily unique*).

Proof: (i) Let $PM = \langle\{PM_\alpha\}_{\alpha\in T}, pm\rangle$ be a standard partial model. Obviously, for every $\alpha \in T$ and every constant c_α, either $pm(c_\alpha) \in PT_{s\alpha}$ or $pm(c_\alpha) \notin PT_{s\alpha}$. In the latter case, Proposition 14 assures that there exists an $x \in PT_{s\alpha}$ such that $pm(c_\alpha) \leq x$. Let $PM' = \langle \{PM_\alpha\}_{\alpha \in T}, pm' \rangle$ be any standard partial model such that for every $\alpha \in T$ and every constant $c_\alpha, pm'(c_\alpha) = \bigvee \langle pm(c_\alpha) \rangle$ if $pm(c_\alpha) \in PT_{s\alpha}$; otherwise, $pm'(c_\alpha) = \bigvee \langle x \rangle$ for some $x \in PT_{s\alpha}$ such that $pm(c_\alpha) \leq x$. It is clear that PM' is an extension of PM which, moreover, is maximal. (ii) As in (i), we are considering **pa** $\in$ As(PM) instead of pm.

3.5 The notions of entailment and validity in the partial sense Most of the so-called partial logics are weakened logics, lacking many fundamental laws of classical logic such as the excluded middle. But we are somewhat ill at ease with such an exclusion. What is troublesome is not the thesis that there are sentences which are neither true nor false (there are indeed many good reasons to believe that). The problem is that on this basis, one concludes that the law of excluded middle (this is just an example) is *not valid.* But this conclusion rests on a particular notion of validity: to be valid is to be true according to every model, or equivalently, it is to be true under every substitution of terms for nonlogical constituents. In classical logic, of course, this notion is equivalent to the notion of being *not false* according to every model, or equivalently, to the notion of being *not false* under every substitution of terms for nonlogical constituents. This follows obviously from bivalence. In partial logics, however, this equivalence no longer holds: the class of valid formulas according to the first notion of validity is generally smaller than the class of valid formulas according to the second. For instance, Kleene's strong three-valued propositional logic does not have any valid formulas according to the first notion because given any formula A, it is always possible to construct a model in which A is not true. But Rescher [10] showed that for the same logic (and others like it) the class of valid formulas according to the second notion is exactly the class of valid formulas in classical logic. A more general result can be obtained. Indeed, the language of propositional logic interpreted by the coherent partial situation semantics—whose meaning postulates for logical connectives are equivalent to those of Kleene's strong three-valued logic—has been provided with a notion of entailment (by van Benthem in [2]) which turns out to be coextensive with the classical notion. This notion (called "weak consequence") superficially appears identical with the classical notion: a set Γ of formulae entails a formula A if and only if there is no model in which all formulae in Γ are true and A is false. In the spirit of partiality, however, this amounts to saying that a deductively valid argument whose conclusion is a sentence B is a sequence $A_1, \ldots, A_n, B$ of formulas such that, necessarily if $A_1, \ldots, A_n$ are *true*, then B is *not false.* We think that this is the essential property of a valid argument, for though it may appear much too weak, it is both uncontroversial (in the sense that nobody can seriously think it false), and it leads to a class of valid formulas which is identical with the class of valid formulas according to the classical definition. Let us apply this notion of entailment to our system.

Let A_t be a formula of **IL**, PM a standard partial model, **pa** $\in$ As(PM), and let $i \in I$. A_t is *satisfied in PM according to* **pa** and i, formally: $\models_{PM,\mathbf{pa},i} A_t$, iff $[\![A_t]\!]^{PM}_{\mathbf{pa},i} = 1$. A_t is *not satisfied* (or is *unsatisfied*) *in PM according to* **pa** and i, formally: $=\!\!|_{PM,\mathbf{pa},i} A_t$, iff $[\![A_t]\!]^{PM}_{\mathbf{pa},i} = 0$, A_t is *not unsatisfied in PM according to* **pa** and i, formally: $\not=\!\!|_{PM,\mathbf{pa},i} A_t$, iff $[\![A_t]\!]^{PM}_{\mathbf{pa},i} = 1$ or φ. If Γ is a set of formulas, then Γ is *satisfied in PM according to* **pa** and i, formally: $\models_{PM\mathbf{pa},i}\Gamma$, iff $\models_{PM,\mathbf{pa},i}A_t$ for every $A_t \in \Gamma$. A formula A_t is *true or undefined in M* iff $\not\dashv_{PM,\mathbf{pa},i}A_t$ for every $\mathbf{pa} \in \mathrm{As}(M)$ and every $i \in I$. A set Γ of formulas *entails* a formula A_t (in the partial sense), formally: $\Gamma \not\dashv A_t$, iff for every standard partial model *PM*, every $\mathbf{pa} \in \mathrm{As}(PM)$ and every $i \in I$, $\models_{PM,\mathbf{pa},i}\Gamma$ only if $\not\dashv_{PM,\mathbf{pa},i}A_t$. Finally, a formula A_t is *valid* (in the partial sense), formally: $\not\dashv A_t$, iff $\varnothing \not\dashv A_t$, that is to say, iff A_t is true or undefined in every standard partial model.

The notion of entailment in the partial sense is equivalent to the notion of classical entailment (see Proposition 32, Section 3.6). This is due to the fact that partial models which are total can be identified with classical models. Of course, this presupposes that we can compare partial models with classical models.

3.6 Comparing partial and classical models

Proposition 28 *Let PM be a standard partial model,* $\mathbf{pa} \in \mathrm{As}(PM)$, *PM′ a maximal extension of PM, and let* $\mathbf{pa}' \in \mathrm{As}(PM) = \mathrm{As}(PM')$ *be a maximal extension of* **pa**. *For every* $\alpha \in T$, *every term* A_α *and every* $i \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \leq [\![A_\alpha]\!]^{PM'}_{\mathbf{pa}',i}$.

Corollary *Let PM be a standard partial model,* $\mathbf{pa} \in \mathrm{As}(PM)$, *PM′ a maximal extension of PM, and let* $\mathbf{pa}' \in \mathrm{As}(PM) = \mathrm{As}(PM')$ *be a maximal extension of* **pa**. *For any formula* A_t *and every* $i \in I$: $\models_{PM,\mathbf{pa},i}A_t$ *implies* $\models_{PM',\mathbf{pa}',i}A_t$ and $\dashv_{PM,\mathbf{pa},i}A_t$ *implies* $\dashv_{PM',\mathbf{pa}',i}A_t$.

Proof: By Definition 26 and Proposition 24(ii).

Definition 29 Let $M = \langle\{M_\alpha\}_{\alpha\in T}, m\rangle$ be a standard classical model and $\mathbf{a} \in \mathrm{As}(M)$. The *partial replica* of M is the standard partial model $\mathrm{PR}(M) = \langle\{PM_\alpha\}_{\alpha\in T}, pm\rangle$ such that for every $\alpha \in T$ and every constant $c_\alpha, pm(c_\alpha) = \mathbf{ma}(m(c_\alpha))$. The *partial replica* of **a** is the assignment $\mathrm{PR}(\mathbf{a}) = \mathbf{pa} \in \mathrm{As}(\mathrm{PR}(M))$ such that for every $\alpha \in T$ and every variable $x_\alpha, \mathbf{pa}(x_\alpha) = \mathbf{ma}(\mathbf{a}(x_\alpha))$. Clearly, the partial replica of a classical model is a partial model which is total.

Proposition 30 *Let M be a standard classical model,* $\mathbf{a} \in \mathrm{As}(M)$, $PM = \mathrm{PR}(M)$, *and let* $\mathbf{pa} = \mathrm{PR}(\mathbf{a})$. *For every* $\alpha \in T$, *every term* A_α *and every* $i \in I$, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} = \mathbf{ma}([\![A_\alpha]\!]^{M}_{\mathbf{a},i})$.

Corollary *Let M be a standard classical model,* $\mathbf{a} \in \mathrm{As}(M)$, $PM = \mathrm{PR}(M)$, *and let* $\mathbf{pa} = \mathrm{PR}(\mathbf{a})$. *For any formula* A_t *and for every* $i \in I$: $\models_{PM,\mathbf{pa},i}A_t$ iff $\vdash_{M,\mathbf{a},i}A_t$ *and* $\dashv_{PM,\mathbf{pa},i}A_t$ *iff* $\nvdash_{M,\mathbf{a},i}A_t$.

Proof: This is immediately verified for variables. This is also straightforward for constants, because provided that $M = \langle\{M_\alpha\}_{\alpha\in T}, m\rangle$, $[\![c_\alpha]\!]^{PM}_{\mathbf{pa},i} = (\mathbf{ma}(m(c_\alpha)))(i) = \mathbf{ma}((m(c_\alpha))(i))$ (by Proposition 2.1-(ii)) $= \mathbf{ma}([\![c_\alpha]\!]^{M}_{\mathbf{a},i})$. The other cases are as follows.

- Consider a term $[A_{\alpha\beta}B_\alpha]$. By the induction hypothesis:

$$[\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i} = \mathbf{ma}([\![A_{\alpha\beta}]\!]^{M}_{\mathbf{a},i}) \text{ and } [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} = \mathbf{ma}([\![B_\alpha]\!]^{M}_{\mathbf{a},i})$$

so

$$[\![[A_{\alpha\beta}B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = [\![A_{\alpha\beta}]\!]^{PM}_{\mathbf{pa},i}([\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}) = (\mathbf{ma}([\![A_{\alpha\beta}]\!]^{M}_{\mathbf{a},i}))(\mathbf{ma}([\![B_\alpha]\!]^{M}_{\mathbf{a},i})).$$

But by Proposition 21(i),

$$(\mathbf{ma}(\llbracket A_{\alpha\beta}\rrbracket^{M}_{\mathbf{a},i})(\mathbf{ma}(\llbracket B_\alpha\rrbracket^{M}_{\mathbf{a},i})) = \mathbf{ma}(\llbracket A_{\alpha\beta}\rrbracket^{M}_{\mathbf{a},i}(\llbracket B_\alpha\rrbracket^{M}_{\mathbf{a},i}))$$

and by definition,

$$\mathbf{ma}(\llbracket A_{\alpha\beta}\rrbracket^{M}_{\mathbf{a},i}(\llbracket B_\alpha\rrbracket^{M}_{\mathbf{a},i})) = \mathbf{ma}(\llbracket [A_{\alpha\beta}B_\alpha]\rrbracket^{M}_{\mathbf{a},i}).$$

Therefore, $\llbracket [A_{\alpha\beta}B_\alpha]\rrbracket^{PM}_{\mathbf{pa},i} = \mathbf{ma}(\llbracket (A_{\alpha\beta}B_\alpha]\rrbracket^{M}_{\mathbf{a},i})$.

- Consider a term ${}^\wedge A_\alpha$. By definition, for every $j \in I$:

$$\llbracket {}^\wedge A_\alpha\rrbracket^{PM}_{\mathbf{pa},i}(j) = \llbracket A_\alpha\rrbracket^{PM}_{\mathbf{pa},j}$$

$$\llbracket {}^\wedge A_\alpha\rrbracket^{M}_{\mathbf{a},i}(j) = \llbracket A_\alpha\rrbracket^{M}_{\mathbf{a},j},$$

and by the induction hypothesis:

$$\llbracket A_\alpha\rrbracket^{PM}_{\mathbf{pa},j} = \mathbf{ma}(\llbracket A_\alpha\rrbracket^{M}_{\mathbf{a},j}).$$

So $\llbracket {}^\wedge A_\alpha\rrbracket^{PM}_{\mathbf{pa},i}(j) = \mathbf{ma}(\llbracket A_\alpha\rrbracket^{M}_{\mathbf{a},j}) = \mathbf{ma}(\llbracket {}^\wedge A_\alpha\rrbracket^{M}_{\mathbf{a},i}(j))$. But by Proposition 21(ii):

$$\mathbf{ma}(\llbracket {}^\wedge A_\alpha\rrbracket^{M}_{\mathbf{a},i}(j)) = (\mathbf{ma}(\llbracket {}^\wedge A_\alpha\rrbracket^{M}_{\mathbf{a},i}))(j).$$

Hence $\llbracket {}^\wedge A_\alpha\rrbracket^{PM}_{\mathbf{pa},i}(j) = (\mathbf{ma}(\llbracket {}^\wedge A_\alpha\rrbracket^{M}_{\mathbf{a},i}))(j)$, which implies $\llbracket {}^\wedge A_\alpha\rrbracket^{PM}_{\mathbf{pa},i} = \mathbf{ma}(\llbracket {}^\wedge A_\alpha\rrbracket^{M}_{\mathbf{a},i})$.

- Consider a term ${}^\vee A_{s\alpha}$. By definition, for every $i \in I$:

$$\llbracket {}^\vee A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = \llbracket A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i}(i)$$

$$\llbracket {}^\vee A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i} = \llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}(i),$$

and by the induction hypothesis, $\llbracket A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = \mathbf{ma}(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i})$. Hence $\llbracket {}^\vee A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = (\mathbf{ma}(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}))(i)$. But by Proposition 21(ii):

$$(\mathbf{ma}(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}))(i) = \mathbf{ma}(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}(i)).$$

Therefore, $\llbracket {}^\vee A_{s\alpha}\rrbracket^{PM}_{\mathbf{pa},i} = \mathbf{ma}(\llbracket A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i}(i)) = \mathbf{ma}(\llbracket {}^\vee A_{s\alpha}\rrbracket^{M}_{\mathbf{a},i})$.

- Consider a term $\lambda x_\alpha A_\beta$. It is sufficient to show that for every $z \in M_\alpha$:

$$\llbracket \lambda x_\alpha A_\beta\rrbracket^{PM}_{\mathbf{pa},i}(\mathbf{ma}(z)) = \mathbf{ma}(\llbracket \lambda x_\alpha A_\beta\rrbracket^{M}_{\mathbf{a},i}(z))$$

for by Proposition 21(i), $\mathbf{ma}(\llbracket \lambda x_\alpha A_\beta\rrbracket^{M}_{\mathbf{a},i}(z)) = (\mathbf{ma}(\llbracket \lambda x_\alpha A_\beta\rrbracket^{M}_{\mathbf{a},i}))(\mathbf{ma}(z))$, which shows effectively that $\llbracket \lambda x_\alpha A_\beta\rrbracket^{PM}_{\mathbf{pa},i} = \mathbf{ma}(\llbracket \lambda x_\alpha A_\beta\rrbracket^{M}_{\mathbf{a},i})$.

First, by definition, it is the case that for every $z \in M_\alpha$:

$$\llbracket \lambda x_\alpha A_\beta\rrbracket^{PM}_{\mathbf{pa},i}(\mathbf{ma}(z)) = \llbracket A_\beta\rrbracket^{PM}_{\mathbf{pa}',i}, \text{ where } \mathbf{pa}' = \mathbf{pa}(x_\alpha/\mathbf{ma}(z))$$

$$\llbracket \lambda x_\alpha A_\beta\rrbracket^{M}_{\mathbf{a},i}(z) = \llbracket A_\beta\rrbracket^{M}_{\mathbf{a}',i}, \text{ where } \mathbf{a}' = \mathbf{a}(x_\alpha/z).$$

Secondly, by the induction hypothesis, $\llbracket A_\beta\rrbracket^{PM}_{\mathbf{pa},i} = \mathbf{ma}(\llbracket A_\beta\rrbracket^{M}_{\mathbf{a},i})$ and obviously, $\mathbf{pa}(x_\alpha/\mathbf{ma}(z))(x_\alpha) = \mathbf{ma}(\mathbf{a}(x_\alpha/z)(x_\alpha))$ for every $z \in M_\alpha$. Therefore, for every $z \in M_\alpha$, $\llbracket \lambda x_\alpha A_\beta\rrbracket^{PM}_{\mathbf{pa},i}(\mathbf{ma}(z)) = \llbracket A_\beta\rrbracket^{PM}_{\mathbf{pa}',i} = \mathbf{ma}(\llbracket A_\beta\rrbracket^{M}_{\mathbf{a}',i}) = \mathbf{ma}(\llbracket \lambda x_\alpha A_\beta\rrbracket^{M}_{\mathbf{a},i}(z))$, where $\mathbf{a}' = \mathbf{a}(x_\alpha/z)$ and $\mathbf{pa}' = \mathbf{pa}(x_\alpha/\mathbf{ma}(z))$.

- Consider a term $[A_\alpha \equiv B_\alpha]$. By the induction hypothesis, $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} = \mathbf{ma}([\![A_\alpha]\!]^{M}_{\mathbf{a},i})$ and $[\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} = \mathbf{ma}([\![B_\alpha]\!]^{M}_{\mathbf{a},i})$. So $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i}, [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \in PT_\alpha$ and by Propositions 12 and 14, $\neg([\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} <> [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i})$ iff $[\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} \neq^* [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i}$. Hence we have:

$$[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = \begin{cases} 1 & \text{if } [\![A_\alpha]\!]^{PM}_{\mathbf{pa},i} <> [\![B_\alpha]\!]^{PM}_{\mathbf{pa},i} \\ 0 & \text{otherwise} \end{cases}$$

and so obviously:

$$[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = [\![[A_\alpha \equiv B_\alpha]]\!]^{M}_{\mathbf{a},i},$$

and a fortiori, $[\![[A_\alpha \equiv B_\alpha]]\!]^{PM}_{\mathbf{pa},i} = \mathbf{ma}([\![[A_\alpha \equiv B_\alpha]]\!]^{M}_{\mathbf{a},i})$.

Proposition 31 *Let PM be a standard partial model,* $\mathbf{pa} \in \mathrm{As}(PM)$, *$PM'$ a maximal extension of PM, and let* $\mathbf{pa}' \in \mathrm{As}(PM) = \mathrm{As}(PM')$ *be a maximal extension of* $\mathbf{pa}$. *There exists a standard classical model M and an assignment* $\mathbf{a} \in \mathrm{As}(M)$ *such that* $\mathrm{PR}(M) = PM'$ *and* $\mathrm{PR}(\mathbf{a}) = \mathbf{pa}'$. *A corollary is that for every partial model which is total there corresponds exactly one classical model.*

Proof: Let $PM' = \langle\{PM_\alpha\}_{\alpha\in T}, pm'\rangle$ be a maximal extension of the model $PM = \langle\{PM_\alpha\}_{\alpha\in T}, pm\rangle$. By definition, for every $\alpha \in T$ and every constant $c_\alpha, pm(c_\alpha) \leq pm'(c_\alpha) = \bigvee\langle x\rangle$ for some $x \in PT_{s\alpha}$. Let $M = \langle\{M_\alpha\}_{\alpha\in T}, m\rangle$ be a standard classical model such that for every $\alpha \in T$ and every constant c_α, $m(c_\alpha) = \Theta(\langle pm'(c_\alpha)\rangle)$. Obviously, M exists. Moreover, for every $\alpha \in T$ and every constant $c_\alpha, pm'(c_\alpha) = \bigvee(\langle pm'(c_\alpha)\rangle) = \mathbf{ma}(\Theta(\langle pm'(c_\alpha)\rangle)) = \mathbf{ma}(m(c_\alpha))$. Therefore $\mathrm{PR}(M) = PM'$. On the other hand, let $\mathbf{a} \in \mathrm{As}(M)$ be an assignment such that for every $\alpha \in T$ and every variable x_α, $\mathbf{a}(x_\alpha) = \Theta(\langle \mathbf{pa}'(x_\alpha)\rangle)$. Again, $\mathbf{ma}(\mathbf{a}(x_\alpha)) = \mathbf{pa}'(x_\alpha)$ for every $\alpha \in T$ and every variable x_α. Therefore $\mathrm{PR}(\mathbf{a}) = \mathbf{pa}'$.

Proposition 32 *Let* Γ *be a set of formulas and* A_t *be a formula of* **IL**. *Then* $\Gamma \Vdash A_t$ *iff* $\Gamma \vDash A_t$.

Proof: First suppose that $\Gamma \Vdash A_t$ but not $\Gamma \vDash A_t$. This means that there exists a standard classical model M, an assignment $\mathbf{a} \in \mathrm{As}(M)$ and $i \in I$ such that $\vDash_{M,\mathbf{a},i}\Gamma$ and $\nvDash_{M,\mathbf{a},i}A_t$. Let $PM = \mathrm{PR}(M)$ and $\mathbf{pa} = \mathrm{PR}(\mathbf{a})$. So by the corollary of Proposition 30, $\vDash_{PM,\mathbf{pa},i}\Gamma$ and $=\!\mid_{PM,\mathbf{pa},i}A_t$. But this contradicts the assumption that $\Gamma \Vdash A_t$. Therefore $\Gamma \vDash A_t$. On the other hand, suppose that $\Gamma \vDash A_t$ but not $\Gamma \Vdash A_t$. This means that there exists a standard partial model PM, an assignment $\mathbf{pa} \in \mathrm{As}(PM)$ and $i \in I$ such that $\vDash_{PM,\mathbf{pa},i}\Gamma$ and $=\!\mid_{PM,\mathbf{pa},i}A_t$. By Proposition 27 and the corollary of Proposition 28, there exists a maximal extension PM' of PM and a maximal extension $\mathbf{pa}'$ of $\mathbf{pa}$ such that $\vDash_{PM',\mathbf{pa}',i}\Gamma$ and $=\!\mid_{PM',\mathbf{pa}',i}A_t$. But by Proposition 31, there exists a standard classical model M and an assignment $\mathbf{a} \in \mathrm{As}(M)$ such that $PM' = \mathrm{PR}(M)$ and $\mathbf{pa}' = \mathrm{PR}(\mathbf{a})$. Therefore, by the corollary of Proposition 30, $\vDash_{M,\mathbf{a},i}\Gamma$ and $\nvDash_{M,\mathbf{a},i}A_t$. But this contradicts the assumption that $\Gamma \vDash A_t$. Therefore $\Gamma \Vdash A_t$.

Remark 33 Consider the deductive system **IL** described in [3], Chapter 1, Section 3. Since this system is sound in both the standard and classical sense (i.e., every theorem of **IL** is valid in standard classical semantics), Proposition 32 allows us to claim that it is also sound in both the standard and partial sense (one may easily verify that the rules of inference preserve validity in the partial sense).

Moreover, we know that restricted to a certain class Σ of formulas (the class of *persistent formulas*), the deductive system **IL** is complete for standard classical semantics. This of course allows us to conclude that relative to Σ, the deductive system **IL** is also complete for standard partial semantics.

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NOTE

1. In this definition we assume that $Z_{t(tt)}$ in $\wedge_{t(tt)}$ is the first variable of the indicated type not occurring free in any of A and B.

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