Notre Dame Journal of Formal Logic
Volume 33, Number 4, Fall 1992

Partial Functions in Type Theory

FRANÇOIS LEPAGE

Abstract This paper proposes a general recursive definition of the notion of partial functions in type theory. According to the given definition, the set of partial functions is a meet semi-lattice for the order "$\leq$" where "$f \leq g$" means "g is at least as defined as f". Two notions are introduced. These are, first, the notion of total partial functions (functions that behave like ordinary functions, the latter being called standard functions), and second, the notion of maximal functions (functions that make the most of the information of their arguments). It is then proved that the set of total maximal partial functions is isomorphic to the set of standard functions. The rest of the paper studies the properties of the space of partial functions. Applications to the semantics of intensional logic are suggested.

0 Introduction There are a number of good reasons to pay attention to *partiality* in semantics, that is, to the fact that expressions may not always be defined. One of these is that the hypothesis that all expressions are fully defined is equally unnatural both for natural language and formal languages (consider, for example, such expressions as "the present King of France is bald" or "$\sqrt{1/(x^2 - 1)}$"). Considerations of this sort have led some to begin working on partial logic (see Blamey [3], Alves and Guerzoni [1], Farmer [5], and van Benthem [2]).

There is another reason—an epistemic one—for taking partiality seriously. Classical semantics seems to have been created for a demiurge: in classical semantics there is no place for a gap between the semantic value of an expression and the semantic value an agent gives to the expression. From this point of view, classical semantics, i.e., semantics using only totally defined values, appears to be a limiting case of partial semantics: the limiting case where the undefined character vanishes. In other words, partiality may be seen as an approximation of totality.

This very general property of partial functions rests on a vague idea of continuity between the completely undefined and the completely defined. For exam-

Received November 14, 1991; revised June 30, 1992

ple, let us suppose that a complex expression A is built from the expressions $a_1, a_2, \ldots, a_n$, i.e., $A = f(a_1, a_2, \ldots, a_n)$. As the partial values of the a_i become more and more defined, so should the value of A. This is desirable from a computational point of view as well as from an epistemic point of view: as information concerning the parts increases, so does information concerning the whole.

In the last few years, many attempts have been made to define model structures for formal languages using partial relations or partial functions. Many different positions have been adopted (cf. Tichý [13], Kindt [7], and Farmer [5]), not all of them compatible with the idea of continuity or approximation. In particular, what should be the truth value of a statement containing undefined terms? Always undefined? Sometimes undefined? Sometimes true and sometimes false? Before trying to answer these questions, we need to clarify the concept of a partial semantic value. In doing so, the simple theory of types will be our framework (after Church [4]). In what follows, I shall characterize the notion of a *partial function in type theory* following the idea of continuity, not using partial relations (cf. Muskens [11]), and without loss of generality. The basic idea is to postulate the identity of a partial function in type theory with a certain kind of *monotonic* function; I shall then study the general properties of the space of these functions.

To begin with, let us first define the set of *types*.

Definition 0.1 The set of types T is the smallest set such that
(i) $e \in T$ (e is the type of individuals)
(ii) $t \in T$ (t is the type of truth values)
(iii) if $\alpha, \beta \in T$, then $\langle\alpha\beta\rangle \in T$ ($\langle\alpha\beta\rangle$ is the type of functions from entities of type α into entities of type β).[1]

Entities of each type can now be defined:

Definition 0.2 For every $\alpha \in T$, the set of entities of type α is the set D_α such that
(i) $D_e = E$ (where E is any nonempty set E is the set of individuals of the universe);
(ii) $D_i = \{0,1\}$ (the set of truth values);
(iii) $D_{\alpha\beta} = D_\beta^{D_\alpha}$ (the set of functions from D_α into D_β).

At first sight, the introduction of partial functions seems unproblematic. Why not simply define a partial function from D_α into D_β as a function which is defined for some arguments and not for others? Two problems arise when one tries to apply this "definition" to the hierarchy of functions. The first is that some expressions are not interpretable because they do not refer to any plausible object. Suppose that $f \in B^A$, $g \in C^B$, and that $f(a)$ is undefined. What about $g(f(a))$? We surely want to say that this expression is undefined, but it is not clear what that will commit us to. Strictly speaking, if $f(a)$ is undefined, $g(f(a))$ is not an undefined expression but a *meaningless* one because in order for $g(f(a))$ to be undefined, $f(a)$ must be an argument for which g is not defined. Yet if $f(a)$ is undefined, it is not an argument at all. The solution to this problem will be to give to the "undefined" a status *inside* the theory.

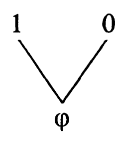
The second problem is about uniformity. At the basic level, i.e., for functions that take as arguments individuals or truth values and give as values individuals or truth values, it is possible simply to state that a partial function is a function that is undefined for some arguments, but this cannot be done at other levels. Let P_α be the set of partial functions of type α. According to the above "definition", we should have $P_{\alpha\beta} = P(D_\beta^{D_\alpha})$, but one of our intuitive and incompatible desiderata is that $P_{\alpha\beta} = P(P_\beta^{P_\alpha})$, i.e., a partial function of type $\alpha\beta$ should be a function that takes partial functions of type α as arguments. In other words, a function from D_α into D_β which is not always defined *is not* the same object as a function from P_α to P_β. This is essential in order to give content to the idea of continuity; partial functions should take partial functions as arguments and their values must be more defined as the arguments are more defined.

A much more recursive definition of a partial function is clearly needed. I will now give a rigorous definition of partial functions, after which I shall study the properties of the space of partial functions.

1 What is a partial function? The first step is to give a precise characterization of the concept of "undefinedness". In order to do this, let us borrow the concept of an undefined object introduced by Scott [12].

Consider, for example, the set $\{0,1\}^{\{0,1\}}$ of functions from truth values into truth values. Let φ be the *undefined object*. A partial function from $\{0,1\}$ into $\{0,1\}$ may be identified with a function from $\{0,1\}$ into $\{0,1,\varphi\}$, if we adopt the convention that $f(x) = \varphi$ means that "$f(x)$" is undefined. This trick allows us to replace a partial function with a certain kind of total function, for there is a natural isomorphism between the set of partial functions $P(A^B)$ and $(A \cup \{\varphi\})^B$. The question now is how to generalize to higher types.

The first obstacle to this generalization is the loss of symmetry between the domain and the codomain. Partial functions from $\{0,1\}$ into $\{0,1\}$ cannot simply be replaced with functions from $\{0,1,\varphi\}$ into $\{0,1,\varphi\}$, because there are some functions from $\{0,1,\varphi\}^{\{0,1,\varphi\}}$ which are, according to our convention, uninterpretable. For example, the existence of the function f such that $f(1) = \varphi$, $f(0) = \varphi$, and $f(\varphi) = 0$ is incompatible with our convention: this object cannot be seen as a partial function. This difficulty can be bypassed by ordering our sets with the relation $\leq$ such that $1 \leq 1$, $\varphi \leq 1$, $0 \leq 0$, $\varphi \leq 0$, and $\varphi \leq \varphi$ or, graphically:



Clearly, $x \leq y$ means x is either less defined than or equal to y. Now, if we consider the set $(\{0,1,\varphi\} \to \{0,1,\varphi\})$ of *monotonic* functions from $\{0,1,\varphi\}$ into $\{0,1,\varphi\}$ the parallelism produced by our convention is restored.

This trick can be generalized and the set of partial functions can now be defined for any type.

Definition 1.1 For any $\alpha \in T$, the set PM_α of partial functions of type α is
(i) $PM_e = E \cup \{\varphi\}$
(ii) $PM_t = \{0,1,\varphi\}$
(iii) $PM_{\alpha\beta} = (PM_\alpha \to PM_\beta)$

where $(PM_\alpha \to PM_\beta)$ is the set of monotonic functions from PM_α into PM_β, monotonicity being relative to the following order:

(i) for any $x \in PM_e$, $x \leq x$ and $\varphi \leq x$;
(ii) for any $x \in PM_t$, $x \leq x$ and $\varphi \leq x$;
(iii) for any $f, g \in PM_{\alpha\beta}$, $f \leq g$ if, and only if, for any $x \in PM_\alpha$, $f(x) \leq g(x)$.

Proposition 1.2[2] *For any α, PM_α is a complete meet semi-lattice, where the meet $\wedge$ and (when defined) the join $\vee$ are respectively*:

(i) *for $x, y \in PM_e$, $x \wedge y = x$ if $x = y$ and φ otherwise*;
(ii) *for $x, y \in PM_t$, $x \wedge y = x$ if $x = y$ and φ otherwise*;
(iii) *for $f, g \in PM_{\alpha\beta}$, $f \wedge g$ is that function h such that, for any $x \in PM_\alpha$, $h(x) = f(x) \wedge g(x)$*;

and

(iv) *for $x, y \in PM_e$, $x \vee y = x$ if $y = \varphi$ or $x = y$, $x \vee y = y$ if $x = \varphi$ and does not exist otherwise*;
(v) *for $x, y \in PM_t$, $x \vee y = x$ if $y = \varphi$ or $x = y$, $x \vee y = y$ if $x = \varphi$ and does not exist otherwise*;
(vi) *for $f, g \in PM_{\alpha\beta}$, $f \vee g$ is that function h such that, for any $x \in PM_\alpha$, $h(x) = f(x) \vee g(x)$ if $f(x) \vee g(x)$ exists.*

I will denote φ_α the lowest element of type α, i.e., $\varphi_\alpha = \wedge \{x \mid x \in PM_\alpha\}$.

For the rest of this paper, I will use the expression *standard functions* to designate ordinary classical total functions, and the expression *total function* will be used for another purpose. The relation "$\leq$" is defined for the set of partial objects.[3] Because the set of these objects is ordered and is a complete meet semilattice, one can hope to find the standard functions at the top. This is not exactly the case, because standard functions do not have the same domain as partial functions: they are not exactly objects of the same kind. In fact, the relation "$\leq$" holds between partial functions. If we want to compare directly standard and partial functions, something else must be defined.

Definition 1.3 For any $\alpha \in T$, let $\leq^* \subseteq PM_\alpha \times D_\alpha$ be the following relation:

(i) for any $x \in PM_e \cap D_e$, $x \leq^* x$ and $\varphi \leq^* x$;
(ii) for any $x \in PM_t \cap D_t$, $x \leq^* x$ and $\varphi \leq^* x$;
(iii) for any $f \in PM_{\alpha\beta}$, $g \in D_{\alpha\beta}$, $f \leq^* g$ iff for any $x \in D_\alpha$ and any $y \in PM_\alpha$ such that $y \leq^* x$, $f(y) \leq^* g(x)$.

This definition seems formally appropriate and sufficient to show that there is an isomorphism between the set $\{h \in PM_\alpha \mid \exists g \in D_\alpha, h = \vee\{f \mid f \leq^* g\}\}$ and D_α. If such an isomorphism exists, we can forget every standard function g and work with $h = \vee\{f \mid f \leq^* g\}$. The trouble is that there seems to be no simple proof for the existence of $\vee\{f \mid f \leq^* g\}\}$. Whereas there is a close connection between $\leq^*$ and $\leq$, I need much more refined analytical instruments to make the link explicit. The general strategy will be to characterize directly some specific classes of partial functions (classes of partial *total* functions) and then prove that there is an isomorphism between these classes and the class of standard functions. The most fundamental tool is introduced by the following definition.

Definition 1.4 For any $\alpha \in T$, let $\neq^* \subseteq PM_\alpha \times PM_\alpha$ be the following relation (I will read $\neq^*$ as "incompatibility" or "strong difference" and "$x \neq^* y$" as "x is incompatible with y" or "x strongly differs from y"):

(i) for $x, y \in PM_e$, $x \neq^* y$ iff $x \neq \varphi$ and $y \neq \varphi$ and $x \neq y$;
(ii) for $x, y \in PM_t$, $x \neq^* y$ iff $x \neq \varphi$ and $y \neq \varphi$ and $x \neq y$;
(iii) for $f, g \in PM_{\alpha\beta}$, $f \neq^* g$ iff there is an $x \in PM_\alpha$ such that $f(x) \neq^* g(y)$.

This relation expresses a very intuitive property. Different partial functions may differ in two ways. One might be defined for an argument while the other is undefined for the same argument. On the other hand, they might both be defined but take different values. This is incompatibility. But in a meet semi-lattice, two different objects may be such that their join exists or such that their join does not exist. The following proposition shows that the two properties are equivalent.

Proposition 1.5 *For any $\alpha \in T$, any $x, y \in PM_\alpha$, $x \neq^* y$ iff $x \vee y$ does not exist.*

Our problem can now be expressed in the following terms: is it possible to prove that if $f \leq^* h$ and $g \leq^* h$, then it is not the case that $f \neq^* g$? The problem is still the same. It is easy to show the relations between $\neq^*$, $\leq$, and $\vee$, but the link with $\leq^*$ is still to come. The only way to do that is to compare directly the "positive" contribution of a partial function with the contribution of a standard function. The following tool is quite simple and will be very useful.

Proposition 1.6 *For any $\alpha \in T$, and any $f \in PM_\alpha$ there is a (possibly empty) sequence $\langle \alpha_1, \ldots, \alpha_n \rangle$, with $\alpha_i \in T$ such that for any (possibly empty) sequence $\langle f_1, \ldots, f_n \rangle$, with $f_i \in PM_{\alpha_i}$*

$$f(f_1)\ldots(f_n) \in PM_e \text{ or } f(f_1)\ldots(f_n) \in PM_t.$$

This very simple and natural property is a surprisingly powerful instrument which can be used to materialize at the lowest levels the difference between two functions of any level. For any f, let us call a sequence $\langle f_1, \ldots, f_n \rangle$, such that $f_i \in PM_{\alpha_i}$, and $f(f_1)\ldots(f_n) \in PM_e$ or $f(f_1)\ldots(f_n) \in PM_t$ a *projector* for f and $f(f_1)\ldots(f_n)$ the corresponding *projection* of f.

Proposition 1.7 *For any $\alpha \in T$, and any $f, g \in PM_\alpha$, $f \neq g$ iff there is a (possibly empty) projector $\langle f_1, \ldots, f_n \rangle$ such that $f(f_1)\ldots(f_n) \neq g(f_1)\ldots(f_n)$.*

Proposition 1.7 provides us with the interesting extensional property that a partial function is entirely determined by its projections. This expected property rests on the fact that all our functions are extensional in the set-theoretical sense. The next proposition will also be useful.

Proposition 1.8 *For any $\alpha \in T$, and any $f, g \in PM_\alpha$, $f \neq^* g$ iff there is a (possibly empty) projector $\langle f_1, \ldots, f_n \rangle$ such that $f(f_1)\ldots(f_n) \neq^* g(f_1)\ldots(f_n)$.*

These two propositions give us the possibility of discriminating between any two different and any two incompatible functions. But the main reason this result is of interest is that I can now characterize a function in terms of its contribution at the lowest level. I will use this property to define a special hierarchy of partial functions—I will call them the *total* functions—that behave, when their arguments are themselves *total*, exactly as standard functions behave. But first, we might note that Propositions 1.6 and 1.7 hold if PM_α is replaced with D_α.

I will use the twin notions of a projector and a projection both for standard functions and for partial functions.

Definition 1.9 For any $\alpha \in T$, the set $PT_\alpha \subseteq PM_\alpha$ (of *partial total functions*) is the smallest set such that
(i) for $\alpha = e$, and $x \in PM_e$, $x \in PT_e$ iff $x \neq \varphi$;
(ii) for $\alpha = t$, and $x \in PM_t$, $x \in PT_t$ iff $x \neq \varphi$;
(iii) for $\alpha = \beta\gamma$, and $f \in PM_{\alpha\beta}$, $f \in PT_{\alpha\beta}$ iff for any $x \in PT_\alpha$, $f(x) \in PT_\beta$.

Also of use is the following property, which intuitively expresses the fact that these functions are total:

Proposition 1.10 *For any $f \in PT_\alpha$, $f \in PT_\alpha$ iff for any (possibly empty) projector $\langle f_1, \ldots, f_n \rangle$ such that $f_i \in PT_{\alpha_i}$ for some α_i, we have $f(f_1) \ldots (f_n) \neq \varphi$.*

Partial total functions behave like standard functions. But as I will now show, the correspondence is not one to one. In fact, in some cases, partial total functions behave like standard ones for total arguments but give different values for nontotal arguments. Before showing this, however, I must express that these functions are *equivalent*.

Definition 1.11 Let "<>" be the following relation on PM_α:
(i) for any $x \in PM_e$, $x <> x$;
(ii) for any $x \in PM_t$, $x <> x$;
(iii) for any $f, g \in PM_{\alpha\beta}$, $f <> g$ iff for any $x \in PT_\alpha$, $f(x) <> g(x)$.

The following proposition shows that if $f <> g$ then f and g take the same value for total projectors.

Proposition 1.12 *If $f, g \in PM_{\alpha\beta}$, then $f <> g$ iff for any (possibly empty) projector $\langle f_1, \ldots, f_n \rangle$ such that $f_i \in PT_{\alpha_i}$ for some α_i:*

$$f(f_1) \ldots (f_n) = g(f_1) \ldots (f_n).$$

One of the obvious consequences of Proposition 1.12 is that "<>" is an equivalence relation, namely the equivalence relation of having exactly the same projections for the same *total* projectors. The equivalence class of f will be denoted "$\langle f \rangle$". The following properties are quite obvious:

Proposition 1.13 *For any projector $\langle f_1, \ldots, f_n \rangle$ of $f, g \in PT_\alpha$,*

$$(f \wedge g)(f_1) \ldots (f_n) = f(f_1) \ldots (f_n) \wedge g(f_1) \ldots (f_n).$$

Proposition 1.14 *For any $\alpha \in T$ and $f \in PT_\alpha$, $(\wedge \langle f \rangle) \in PT_\alpha$, and $(\wedge \langle f \rangle) <> f$.*

Proof: $\wedge \langle f \rangle$ exists in PM_α because the meet of any set exists; on the other hand, it is clear from Definition 1.11 that for any projector $\langle f_1, \ldots, f_n \rangle$ such that $f_i \in PT_{\alpha_i}$ for some α_i, $f(f_1) \ldots (f_n) = g(f_1) \ldots (f_n)$ for any $g \in \langle f \rangle$. So, by Proposition 1.13, $\wedge \langle f \rangle (f_1) \ldots (f_n) = f(f_1) \ldots (f_n)$.

We can now explicitly describe a partial function that behaves exactly like a standard one. I will further show that this function is the meet of its equivalence class: it behaves like a standard function for total arguments, but is totally undefined for other arguments.

Definition 1.15 For any α, let $\mathfrak{J}: D_\alpha \to PM_\alpha$ be the following function:

(i) for any $x \in D_e$, $\mathfrak{J}(x) = x$;

(ii) for any $x \in D_t$, $\mathfrak{J}(x) = x$;

(iii) for any $f \in D_{\alpha\beta}$, $\mathfrak{J}(f)$ is that function such that for any $x \in D_\alpha$ and for any $y \in \langle \mathfrak{J}(x) \rangle$,

$$\mathfrak{J}(f)(y) = \mathfrak{J}(f(x))$$

and for any other y

$$\mathfrak{J}(f)(y) = \varphi_\beta.$$

There is clearly at least one such function. The following proposition will show that there is only one because if there were two, they would have both the same projections.

Proposition 1.16 *$\mathfrak{J}$ is that function such that for any (possibly empty) projector $\langle f_1, \ldots, f_n \rangle$ of $f, f_i \in D_{\alpha_i}$ and for any $\langle g_1, \ldots, g_n \rangle$ such that $g_i \in \langle \mathfrak{J}(f_i) \rangle$,*

$$\mathfrak{J}(f)(g_1) \ldots (g_n) = f(f_1) \ldots (f_n)$$

and such that, for any $\langle g_1, \ldots, g_n \rangle$ such that $g_i \in PM_{\alpha_i}$ and for some $g_i, g_i \notin \langle \mathfrak{J}(f_i) \rangle$,

$$\mathfrak{J}(f)(g_1) \ldots (g_n) = \varphi.$$

For any standard function f, $\mathfrak{J}(f)$ is a total function but is completely undefined if its argument is not itself total. This indicates that each $\mathfrak{J}(f)$ is the lowest function of set of total equivalent functions.

Proposition 1.17 *Let $g \in PM_\alpha$. Then $g = \wedge\langle g \rangle \in PT_\alpha$ iff there is an $f \in D_\alpha$, such that $\mathfrak{J}(f) = g$.*

Proof:

(i) For any $x \in D_e$, $\mathfrak{J}(x) = x \in PT_e$, and $\wedge\langle x \rangle = x$.

(ii) For any $x \in D_t$, $\mathfrak{J}(x) = x \in PT_t$, and $\wedge\langle x \rangle = x$.

(iii) $\Rightarrow$. Let $g \in PM_{\beta\gamma}$. If $g = \wedge\langle g \rangle \in PT_{\beta\gamma}$, then for any projector $\langle g_1, \ldots, g_n \rangle$ such that $g_i \in PT_{\alpha_i}$, for some $\alpha_i, g(g_1) \ldots (g_n) \neq \varphi$. This implies that $g(\wedge\langle g_1 \rangle) \ldots (\wedge\langle g_n \rangle) \neq \varphi$. By the induction hypothesis, for any such $\langle g_1, \ldots, g_n \rangle$ there is a projector $\langle f_1, \ldots, f_n \rangle$, such that $f_i \in D_{\alpha_i}$ for some α_i, $\mathfrak{J}(f_i) = \wedge\langle g_i \rangle$. The function $f \in D_{\beta\gamma}$, such that

$$f(f_1) \ldots (f_n) = g(\wedge\langle g_1 \rangle) \ldots (\wedge\langle g_n \rangle)$$

is such that $\mathfrak{J}(f) \in \langle g \rangle$, $\mathfrak{J}(f) \leq h$ for any $h \in \langle g \rangle$ (because, by Proposition 1.16, $\mathfrak{J}(f)$ behaves like h on total arguments and is undefined for any other argument) and so $\mathfrak{J}(f) = \wedge\langle g \rangle = g$.

$\Leftarrow$. Let $f \in D_\alpha$ be such that $\mathfrak{J}(f) = g$ for some g. By the induction hypothesis and Proposition 1.16, for any projector $\langle f_1, \ldots, f_n \rangle$, such that $f_i \in D_{\alpha_i}$,

$$f(f_1) \ldots (f_n) = \mathfrak{J}(f)(\wedge\langle g_1 \rangle) \ldots (\wedge\langle g_n \rangle)$$

for some $g_i \in PT_{\alpha_i}$ and

$$\mathfrak{J}(f)(h_1) \ldots (h_n) = \varphi$$

if $h_i \notin PT_{\alpha_i}$. Thus $\mathfrak{J}(f) \in PT_\alpha$ and $\mathfrak{J}(f) = \wedge\langle \mathfrak{J}(f) \rangle$.

These results show that for any standard function f the function $\Im(f)$ is not only a partial total function, i.e., a function that behaves exactly like f when its argument is itself an image of a standard argument by $\Im$, but it is the lowest partial total function that behaves like this: when the argument of $\Im(f)$ is not total, the value is totally undefined. We will have more on that later.

The following very intuitive property will be useful.

Proposition 1.18 *For any $f \in PT_\alpha$, there is no $g \notin PT_\alpha$, such that $f \leq g$.*

The following proposition provides the means to compare partial and standard functions. According to this proposition: (1) if all the partial functions of a given set agree with regard to total arguments, their join exists and (2) any partial function is dominated by a total one. The first property will be used to make a correspondence between a class of total equivalent partial functions and a standard function and the second to compare any partial function to a standard one through this correspondence.

Proposition 1.19 *For any $f \in PM_\alpha$, $\vee\langle f\rangle$ exists, and there is a $g \in PT_\alpha$ such that $f \leq g$.*

Proof:

(i) For $\alpha = e$, it is straightforward.

(ii) For $\alpha = t$, it is straightforward.

(iii) For $\alpha = \beta\gamma$, let us consider $f, g \in PM_{\beta\gamma}$, such that $f <> g$. Let us suppose that $f \vee g$ does not exist. By Proposition 1.5, $f \neq^* g$. Therefore, there is an x such that

$$f(x) \neq^* g(x).$$

By the induction hypothesis, there is a $y \in PT_\alpha$, such that $x \leq y$. This implies, by monotonicity, that $f(y) \neq^* g(y)$, which contradicts that $f <> g$. So $f \vee g$ exists.

Let us now show that for any $f \in PM_{\beta\gamma}$, there is a $g \in PT_{\beta\gamma}$ such that $f \leq g$. If $f \in PT_{\beta\gamma}$, let us take $f = g$. If not, let $f(f_1)\ldots(f_n) = \varphi$ with $f_i \in PT_{\alpha_i}$. By the induction hypothesis, each $\vee\langle f_i\rangle$ exists. There are two cases:

Case 1. $f(\vee\langle f_1\rangle)\ldots(\vee\langle f_n\rangle) = \varphi$. In this case, let $g(g_1)\ldots(g_n) = x$, with $x \neq \varphi$ is any value of the corresponding type if $g_i \in \langle f_i\rangle$ and $g(g_1)\ldots(g_n) = f(g_1)\ldots(g_n)$ for all other projectors. *Claim:* g is monotonic. For, let $g_1 \leq g_2 \in PM_\alpha$. One of only three possibilities is the case, that is,

(a) $g_1 \in PM_\alpha$, $g_2 \in PM_\alpha$;
(b) $g_1 \in PM_\alpha$, $g_2 \in PT_\alpha$, with $g_2 \in \langle f_2\rangle$;
(c) $g_1 \in PT_\alpha$, $g_2 \in PT_\alpha$, with $g_1 \in \langle f_1\rangle$ and $g_2 \in \langle f_2\rangle$;

(the fourth combination is impossible by Proposition 1.18).

If (a), then $g(g_1) = f(g_1) \leq f(g_2) = g(g_2)$.

If (b), then for any $\langle h_2,\ldots,h_n\rangle$, if one of the $h_i \notin \langle f_i\rangle$, then

$$\begin{aligned} g(g_1)(h_2)\ldots(h_n) &= f(g_1)(h_2)\ldots(h_n) \text{ by definition of } g \\ &\leq f(g_2)(h_2)\ldots(g_n)\ f \text{ is monotonic} \\ &= g(g_2)(h_2)\ldots(g_n) \text{ by definition of } g \end{aligned}$$

if all of the $h_i \in \langle f_i \rangle$, then

$$\begin{aligned} g(g_1)(h_2)\dots(h_n) &= f(g_1)(h_2)\dots(h_n) \text{ by definition of } g \\ &\leq f(g_2)(h_2)\dots(h_n)\ f \text{ is monotonic} \\ &\leq g(g_2)(h_2)\dots(h_n) \text{ by definition of } g. \end{aligned}$$

If (c), then for any $\langle h_2, \dots h_n \rangle$; if one of the $h_i \notin \langle f_i \rangle$, then

$$\begin{aligned} g(g_1)(h_2)\dots(h_n) &= f(g_1)(h_2)\dots(h_n) \text{ by definition of } g \\ &\leq f(g_2)(h_2)\dots(h_n)\ f \text{ is monotonic} \\ &= g(g_2)(h_2)\dots(h_n) \text{ by definition of } g; \end{aligned}$$

if all of the $h_i \in \langle f_i \rangle$, then

$$g(g_1)(h_2)\dots(h_n) = g(g_2)(h_2)\dots(h_n) \text{ by definition of } g.$$

Case 2. $f(\vee\langle f_1 \rangle)\dots(\vee\langle f_n \rangle) = x \neq \varphi$. In this case, let $g(g_1)\dots(g_n) = x$, if $g_i \in \langle f_i \rangle$ and $g(g_1)\dots(g_n) = f(g_1)\dots(g_n)$ for all other projectors. *Claim:* g is also monotonic. For, let $g_1 \leq g_2 \in PM_\alpha$. One of the three following possibilities is the case:

(a) $g_1 \in PM_\alpha$, $g_2 \in PM_\alpha$
(b) $g_1 \in PM_\alpha$, $g_2 \in PT_\alpha$ (with $g_2 \in \langle f_2 \rangle$)
(c) $g_1 \in PT_\alpha$, $g_2 \in PT_\alpha$ (with $g_1 \in \langle f_1 \rangle$ and $g_2 \in \langle f_2 \rangle$).

If (a), then $g(g_1) = f(g_1) \leq f(g_2) = g(g_2)$.
If (b), then for any $\langle h_2, \dots, h_n \rangle$, if one of the $h_i \notin \langle f_i \rangle$, then

$$\begin{aligned} g(g_1)(h_2)\dots(h_n) &= f(g_1)(h_2)\dots(h_n) \text{ by definition of } g \\ &\leq g(g_2)(h_2)\dots(g_n)\ f \text{ is monotonic} \\ &= g(g_2)(h_2)\dots(g_n) \text{ by definition of } g; \end{aligned}$$

if all of the $h_i \in \langle f_i \rangle$, then

$$\begin{aligned} g(g_1)(h_2)\dots(h_n) &= f(g_1)(h_2)\dots(h_n) \text{ by definition of } g \\ &\leq f(g_2)(h_2)\dots(h_n)\ f \text{ is monotonic} \\ &\leq g(g_2)(h_2)\dots(h_n) \text{ by definition of } g. \end{aligned}$$

If (c), then for any $\langle h_2, \dots h_n \rangle$, if one of the $h_i \notin \langle f_i \rangle$, then

$$\begin{aligned} g(g_1)(h_2)\dots(h_n) &= f(g_1)(h_2)\dots(h_n) \text{ by definition of } g \\ &\leq f(g_2)(h_2)\dots(h_n)\ f \text{ is monotonic} \\ &= g(g_2)(h_2)\dots(h_n) \text{ by definition of } g; \end{aligned}$$

if all of the $h_i \in \langle f_i \rangle$, then

$$g(g_1)(h_2)\dots(h_n) = g(g_2)(h_2)\dots(h_n) \text{ by definition of } g.$$

Therefore for any projector $\langle g_1, \dots, g_n \rangle$ such that $g_i \in PT_{\alpha_i}$ and such that $f(g_1), \dots (g_n) = \varphi$, there is a g with $f \leq g$ such that $g(g_1), \dots (g_n) \neq \varphi$. If g is not total, the process can be repeated and will give g^1 such that $g \leq g^1$, $g^1 \leq g^2$, and so on. Clearly, $\vee\{g^n\}$ exists, is total, and $f \leq \vee\{g^n\}$.[4]

Proposition 1.20 *For any $f, g \in PT_\alpha$, $f \vee g$ exists iff $f <> g$.*

Proposition 1.21 *For any $f \in PM_\alpha$, $\langle f \rangle$ is a complete lattice.*

We can now explicitly describe a function that maps classes of equivalence of total function on the corresponding standard function.

Proposition 1.22 *There is an epimorphism $\Psi : PT_\alpha \to D_\alpha$ such that for any $f \in PT_\alpha$,*

$$\langle f \rangle = \Psi^{-1}(\Psi(f))$$

Proof: Let $\Psi : PT_\alpha \to D_\alpha$ be such that

(i) For $\alpha = e$, $\Psi(x) = x$. Clearly, Ψ is an isomorphism and thus an epimorphism.

(ii) For $\alpha = t$, $\Psi(x) = x$, same argument.

(iii) For $\alpha = \beta\gamma$, let $f \in PT_\alpha$. $\Psi(f) \in D_\alpha$ be the following function: For any projector $\langle g_1, \ldots, g_n \rangle$, such that $g_i \in D_{\alpha_i}$, there is, by the induction hypothesis, a $\langle f_1, \ldots, f_n \rangle$, such that $f_i \in PT_{\alpha_i}$ and $\Psi(f_i) = g_i$. For any such $\langle g_1, \ldots, g_n \rangle$, let $\Psi(f) \in D_\alpha$ be the following function:

$$\Psi(f)(g_1)\ldots(g_n) = \Psi(f(f_1)\ldots(f_n)).$$

Clearly,

$$\Psi(f(f_1)\ldots(f_n)) = \Psi(f)(g_1)\ldots(g_n) = \Psi(f)(\Psi(f_1))\ldots(\Psi(f_n))$$

and, thus, Ψ is a morphism.

Furthermore, if $f <> h$, then $\Psi(f) = \Psi(h)$. In order to see this, let us suppose that $\Psi(f) \neq \Psi(h)$ and that there is therefore, by Proposition 1.7, a projector $\langle g_1, \ldots, g_n \rangle$, such that $g_i \in D_{\alpha_i}$ and such that

$$\Psi(f)(g_1)\ldots(g_n) \neq \Psi(h)(g_1)\ldots(g_n).$$

But since Ψ is a morphism, we know by the induction hypothesis that for any such $\langle g_1, \ldots, g_n \rangle$ there is a $\langle f_1, \ldots, f_n \rangle$, such that $f_i \in PT_{\alpha_i}$ and $\Psi(f_i) = g_i$. Therefore,

$$\Psi(f)(\Psi(f_1))\ldots(\Psi(f_n)) \neq \Psi(h)(\Psi(f_1))\ldots(\Psi(f_n))$$

and

$$\Psi(f(f_1)\ldots(f_n)) \neq \Psi(h(f_1)\ldots(f_n)).$$

Ψ is a function, therefore,

$$f(f_1)\ldots(f_n) \neq h(f_1)\ldots(f_n)$$

and by the definition of $<>$, it is not the case that $f <> h$.

All that remains is to prove that Ψ is surjective. Let $g \in D_\alpha$, and by Proposition 1.17, let f be such that $\Im(g) = f = \wedge\langle f \rangle$. Therefore, $\Psi(\Im(g)) = \Psi(f)$. I will now show that $\Psi(\Im(g)) = g$, that is, $\Im$ is an inverse of Ψ.

Let $\langle g_1, \ldots, g_n \rangle$ be a projector with $g_i \in D_{\alpha_i}$. By the induction hypothesis,

$$\Psi(\Im(g))(g_1)\ldots(g_n) = \Psi(\Im(g))(\Psi(\Im(g_1)))\ldots(\Psi(\Im(g_n)))$$

with $\Im(g_i) = f_i = \wedge\langle f_i \rangle$, for $f_i \in PT_{\alpha_i}$ for some α_i. Thus, Ψ being a morphism,

$$\Psi(\Im(g))(g_1)\ldots(g_n) = \Psi(\Im(g)(\Im(g_1))\ldots(\Im(g_n))).$$

Now, by Proposition 1.16, we have

$$\Im(g)(\Im(g_1))\ldots(\Im(g_n)) = g(g_1)\ldots(g_n)$$

and then

$$\Psi(\Im(g))(g_1)\ldots(g_n) = \Psi(g(g_1)\ldots(g_n)).$$

But for PT_e and PT_t, Ψ is identity. So,

$$\Psi(\Im(g))(g_1)\ldots(g_n) = g(g_1)\ldots(g_n)$$

and

$$\Psi(\Im(g)) = g.$$

$\Im$ *is thus a right inverse of* Ψ which proves that Ψ is surjective.

Definition 1.23 Let $f \in PT_\alpha$. If $f = \vee\langle f \rangle$, we will say that f is *maximal.* If $f = \wedge\langle f \rangle$ we will say that f is *minimal.*

For any class of equivalence of total function, there is exactly one member that is maximal and one member that is minimal. The next result is then not surprising.

Proposition 1.24 *The set of the total maximal (resp. minimal) functions is isomorphic to the set of standard functions, i.e., if $TM_\alpha \subseteq PT_\alpha$ is such that for any $f \in TM_\alpha$, $f = \vee\langle f \rangle$ (resp. if $TB_\alpha \subseteq PT_\alpha$ is such that for any $f \in TB_\alpha$, $f = \wedge\langle f \rangle$), there is an isomorphism*

$$\Phi : TM_\alpha \to D_\alpha \quad (\text{resp. } \Theta : TB_\alpha \to D_\alpha)$$

such that for any projector $\langle f_1, \ldots, f_n \rangle$, such that $f_i \in TM_{\alpha_i}$ (resp. $f_i \in TB_{\alpha_i}$)

$$\Phi(f(f_1)\ldots(f_n)) = \Phi(f)(\Phi(f_1))\ldots(\Phi(f_n)) = f(f_1)\ldots(f_n)$$

$$(\text{resp. } \Theta(f(f_1)\ldots(f_n)) = \Theta(f)(\Theta(f_1))\ldots(\Theta(f_n)) = f(f_1)\ldots(f_n)).$$

Proof: We simply take as Φ (resp. Θ) the restriction of Ψ to TM_α (resp. TB_α). One easily proves that Φ (resp. Θ) is an injection and a surjection.

With these isomorphisms, we have a rigorous characterization of partial functions that behave exactly like standard functions and, via these isomorphisms, we can directly compare partial and standard functions. In fact, I can now prove the following proposition:

Proposition 1.25 *For any $g \in D_\alpha$, $\Phi^{-1}(g) = \vee\{f \in PM_\alpha \mid f \leq^* g\}$.*

Proof: Let us first show that $f \leq^* g$ iff $f \leq \Phi^{-1}(g)$.

$\Rightarrow$. If $f \leq^* g$, then for any projectors $\langle f_1, \ldots, f_n \rangle$, $\langle g_1, \ldots, g_n \rangle$ such that

$$f_i \leq^* g_i, f(f_1)\ldots(f_n) \leq^* g(g_1)\ldots(g_n)$$

(i) For $\alpha = e$, it is straightforward.

(ii) For $\alpha = t$, it is straightforward.

(iii) For $\alpha = \beta\gamma$, by Definition 1.3, $f \leq^* g$ and $f_1 \leq^* g_1$ imply that $f(f_1) \leq^* g(g_1)$. By applying the same argumentation n times, we have $f(f_1) \ldots (f_n) \leq^* g(g_1) \ldots (g_n)$.

But $\leq^*$ and $\leq$ are the same on e and t, so for $f_i \leq^* g_i$,

$$f(f_1) \ldots (f_n) \leq^* g(g_1) \ldots (g_n)$$

iff for $f_i \leq^* g_i$,

$$f(f_1) \ldots (f_n) \leq g(g_1) \ldots (g_n).$$

By the induction hypothesis, we have for $f_i \leq^* g_i$.

$$f(f_1) \ldots (f_n) \leq^* g(g_1) \ldots (g_n)$$

iff for $f_i \leq \Phi^{-1}(g_i)$,

$$f(f_1) \ldots (f_n) \leq g(g_1) \ldots (g_n).$$

By Proposition 1.24, we have for $f_i \leq^* g_i$,

$$f(f_1) \ldots (f_n) \leq^* g(g_1) \ldots (g_n)$$

iff for $f_i \leq \Phi^{-1}(g_i)$,

$$f(f_1) \ldots (f_n) \leq \Phi^{-1}(g)(\Phi^{-1}(g_1)) \ldots (\Phi^{-1}(g_n)).$$

$\Phi^{-1}(g)$ being total maximal and f being monotonic, this implies that

$$f \leq \Phi^{-1}(g).$$

$\Leftarrow$. Let us suppose that $f \leq \Phi^{-1}(g)$.

(i) For $\alpha = e$, it is straightforward.

(ii) For $\alpha = t$, it is straightforward.

(iii) For $\alpha = \beta\gamma$.

In that case, for any projectors $\langle f_1, \ldots, f_n \rangle$,

$$f(f_1) \ldots (f_n) \leq \Phi^{-1}(g)(f_1) \ldots (f_n).$$

By monotonicity, for any g_i such that $f_i \leq \Phi^{-1}(g_i)$, we have $f(f_1) \ldots (f_n) \leq \Phi^{-1}(g)(\Phi^{-1}(g_1)) \ldots (\Phi^{-1}(g_n))$. Using the induction hypothesis for any projectors $\langle f_1, \ldots, f_n \rangle$ such that $f_i \leq^* g_i$,

$$f(f_1) \ldots (f_n) \leq \Phi^{-1}(g)(\Phi^{-1}(g_1)) \ldots (\Phi^{-1}(g_n)).$$

But, by Proposition 1.24,

$$\Phi^{-1}(g)(\Phi^{-1}(g_1)) \ldots (\Phi^{-1}(g_n)) = g(g_1) \ldots (g_n).$$

So, by Definition 1.3, $f \leq^* g$. From this we can conclude that

$$\vee\{f \in PM_\alpha \mid f \leq^* g\} = \vee\{f \in PM_\alpha \mid f \leq \Phi^{-1}(g)\} = \Phi^{-1}(g).$$

All this brings a precise answer to the question: "What is a partial function?" but raises many other questions. One of the surprises was that Ψ is *not* an isomorphism. What is the difference between maximal and minimal functions and all the in-between functions belonging to the same equivalence class? What do the lattices of equivalent functions look like? Our next task is to answer some of these questions.

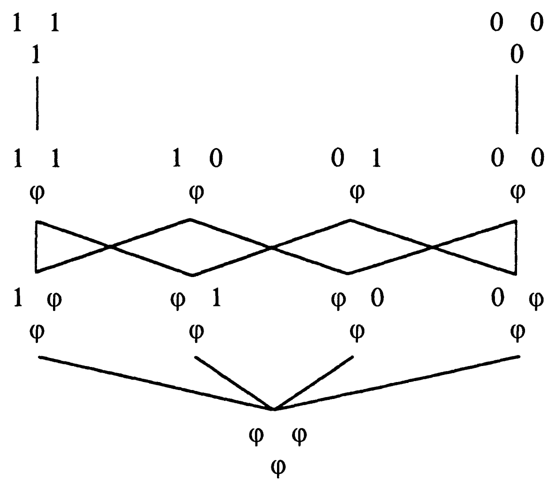
2 ***Some properties of the meet semi-lattice of partial functions*** Let us begin with some examples. I will restrict my attention to the hierarchy of Boolean functions, called propositional types (cf. Henkin [6]). Since all these spaces are finite, we can have direct representations of the function spaces. For example, we saw that PM_t can be represented by the following schema:

$$\begin{array}{cc} 1 & 0 \\ \multicolumn{2}{c}{\varphi} \end{array}$$

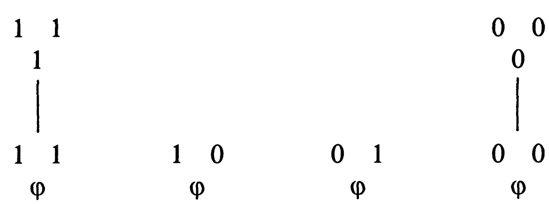
Furthermore, any element of PM_{tt} can be represented using the following convention[5]:

$$\begin{array}{cc} f(1) & f(0) \\ \multicolumn{2}{c}{f(\varphi)} \end{array}$$

Thus PM_{tt} can itself be represented as



The total functions are



Now, if I represent the element of $g \in D_{tt}$ with $[g(1) \;\; g(0)]$, we have

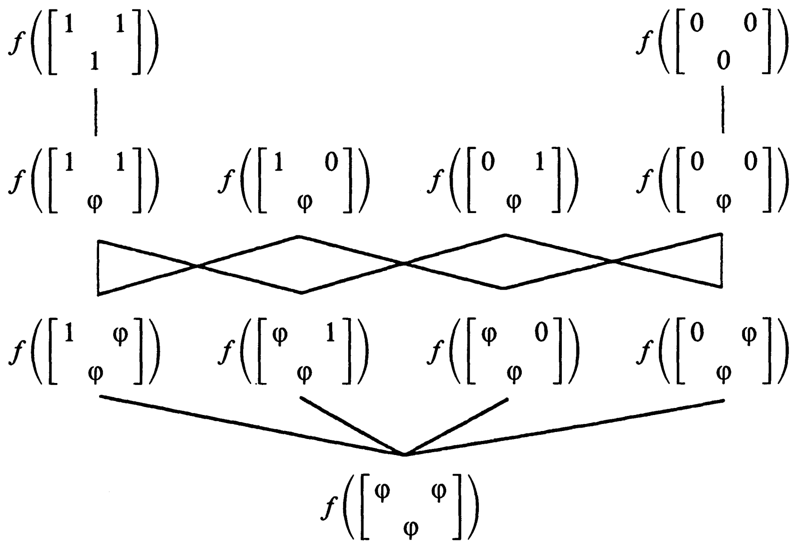
$$\Psi^{-1}([1 \;\; 1]) = \begin{bmatrix} 1 & & 1 \\ & 1 & \\ & & \\ 1 & & 1 \\ & \varphi & \end{bmatrix},$$

$$\Psi^{-1}([1 \;\; 0]) = \begin{bmatrix} 1 & & 0 \\ & \varphi & \end{bmatrix},$$

$$\Psi^{-1}([0 \;\; 1]) = \begin{bmatrix} 0 & & 1 \\ & \varphi & \end{bmatrix}, \quad \text{and}$$

$$\Psi^{-1}([0 \;\; 0]) = \begin{bmatrix} 0 & & 0 \\ & 0 & \\ & & \\ 0 & & 0 \\ & \varphi & \end{bmatrix}.$$

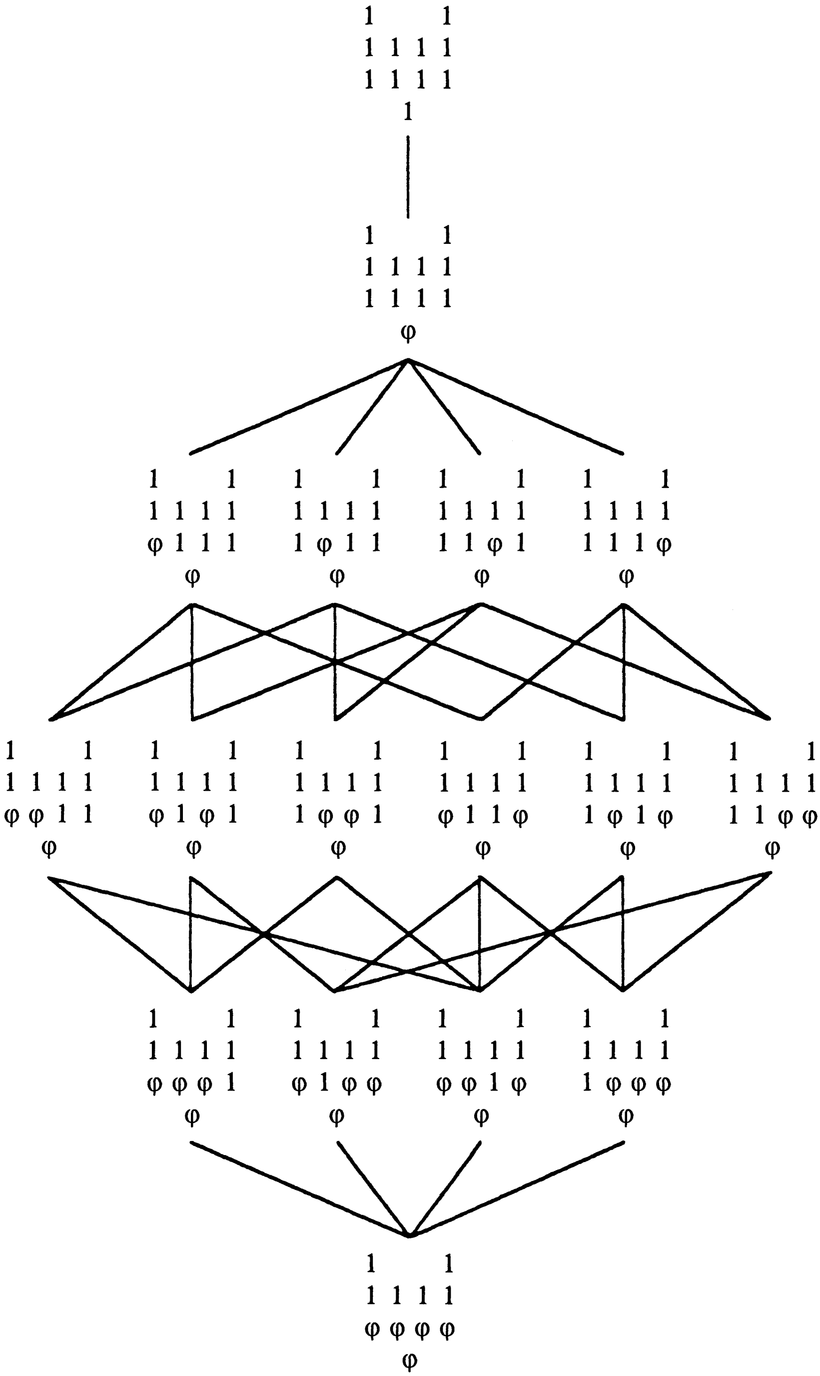
We now have examples of total maximal and minimal functions. These examples suggest the following interpretation of the distinction between minimal and maximal functions: a *minimal* total function is a function which is undefined whenever its argument is not totally defined, whereas a *maximal* function is a function which is defined whenever possible, without breaking monotonicity, even if its argument is *undefined*. How could this be? Let us consider $\begin{smallmatrix} 1 & & 1 \\ & 1 & \end{smallmatrix}$. This function is such that $f(\varphi) = 1$ but nevertheless monotonic because $f(\varphi)$ is not dominated by incompatible values. But is this naive interpretation exportable toward the higher levels of the hierarchy? Surely. The following are some examples taken from $PM\langle_{tt}\rangle_t$. Any element of $PM\langle_{tt}\rangle_t$ (which contains 397 functions[6]) can be represented by this schema:



Correspondingly, any $g \in D\langle_{tt}\rangle_t$ can be represented with

$$[g(1\quad 1)g(1\quad 0)g(0\quad 1)g(0\quad 0)].$$

Let us take a look at $\Psi^{-1}([1\ 1\ 1\ 1])$:

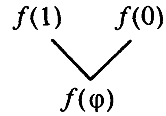


Another interesting set is D_{ttt}, the set of binary Boolean connectors. Any g can be represented by $[g(1)\ \ g(0)]$, where $g(1), g(0) \in D_{tt}$. Therefore, g can be represented by

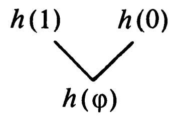
$$[g(1)(1)\ g(1)(0) \qquad g(0)(1)\ g(0)(0)],$$

which is the truth table of g. For example, we will use [1111] to represent the constant function f such that for any truth value x, $f(x)$ is the constant function such that for any truth value y, $f(x)(y) = 1$.

Accordingly, any $f \in D_{ttt}$ can be represented by



In turn, $f(x) = h$ can be represented by



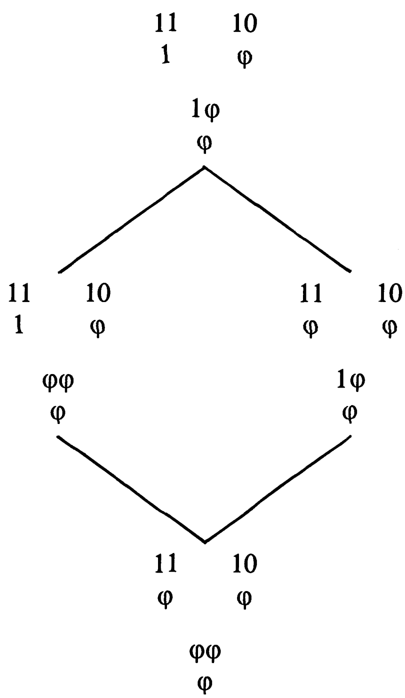
Let us now take a look at disjunction. Its diagram is

$$([11\quad 10])$$

and

$$\Psi^{1-}([11\quad 10])$$

will be



The top value is Kleene's three-valued disjunction. On the bottom the function is a disjunction when both arguments are defined and is undefined otherwise. The middle ones are a bit odd: the symmetry between arguments has been lost. For the left one, if the first argument is defined, the behavior of this connector is that of Kleene's and if the first argument is not defined, the second one is not even considered, i.e., the resulting function is always undefined. The right one has exactly the converse behavior. These are the only four possible monotonic partial disjunctions.

The equivalence has a special property. Let us consider

$$\Psi^{-1}([10 \quad 01]).$$

This set has only one member, namely

$$\begin{array}{ccc} 10 & & 01 \\ \varphi & & \varphi \\ & \varphi\varphi & \\ & \varphi & \end{array}$$

which is simultaneously minimal and maximal. The philosophical moral is that there are no nontrivial ways to approach propositional identity.

These "empirical" examples strongly suggest the following interpretation. A partial function is a function which is more or less defined. But a partial function is also a function which uses more or less the information of its arguments. This suggests that if two functions differ, it is not necessarily the case that they differ for total arguments. For example:

$$f = \begin{array}{ccc} 11 & & 10 \\ \varphi & & \varphi \\ & 1\varphi & \\ & \varphi & \end{array} \quad \text{and} \quad g = \begin{array}{ccc} 11 & & 10 \\ \varphi & & \varphi \\ & \varphi\varphi & \\ & \varphi & \end{array}$$

are such that $f(1) = g(1) = {}^{1}{}_{\varphi}{}^{1}$, $f(0) = g(0) = {}^{1}{}_{\varphi}{}^{0}$ and $f(\varphi) \neq g(\varphi)$. But if two functions are incompatible, we know by Proposition 1.8 that this incompatibility can always be shown for total arguments. Let us generalize.

Definition 2.1 A set $A \subseteq PM_{\alpha\beta}$ is *separable* iff for any $f \in A$, there is a $g \in A$ such that $f \neq^* g$.

Definition 2.2 Let $A \subseteq PM_{\alpha\beta}$ be a separable set. A set $B \subseteq PM_{\alpha}$ will be called a *riddle* for A iff for any $f, g \in A$, there is an $x \in B$ such that $f(x) \neq^* g(x)$.

Definition 2.3 A riddle is *universal* iff it is a riddle for any separable set.

Proposition 2.4 *For any α, PT_{α} is a universal riddle.*

Proposition 2.5 *Let $A \subseteq PT_{\alpha}$ be such that for any $f \in PT_{\alpha}$, there is a $g \in A$ such that $f <> g$. Then A is a universal riddle.*

Proposition 2.6 *For any α, the set TM_{α} of all the total maximal functions is a universal riddle.*

Proposition 2.7 *For any α, the set TB_{α} of all the total minimal functions is a universal riddle.*

Proposition 2.5 gives us a sufficient condition for a set to be a universal riddle. This condition may seem to be too strong. This is not the case.

Proposition 2.8 *Let $A \subseteq PM_\alpha$ be a universal riddle. Then for any $f \in PT_\alpha$, there is a $g \in A$ such that $f <> g$.*

Proof: Let us suppose that A is a universal riddle and that there is an $f \in PT_\alpha$ such that for any $g \in A$, $f \notin \langle g \rangle$. Let $h_1, h_2 \in PT_{\alpha\beta}$ be the following functions: for any $x \notin \langle g \rangle$, $h_1(x) = h_2(x) = \varphi_\beta$, and for any $x \in \langle g \rangle$, $h_1(x) \in PT_\alpha$ and $h_2(x) \in PT_\alpha$ are such that $h_1(x) \neq^* h_2(x)$. Such functions clearly exist and A is not a riddle for $\{h_1, h_2\}$.

Proposition 2.9 *If $f \neq^* g$, then for any $h \in \langle f \rangle$, $h \neq^* g$.*

Proof: $f \neq^* g$ implies that there is a projector $\langle f_1, \ldots, f_n \rangle$ such that $f_i \in PT_{\alpha_i}$ and $f(f_1) \ldots (f_n) \neq^* g(f_1) \ldots (f_n)$. If there is an $h \in \langle f \rangle$ such that $h \vee g$ exists, then

$$\begin{aligned} &(h \vee g)(f_1) \ldots (f_n) = \\ &(h(f_1) \vee g(f_1))(f_2) \ldots (f_n) = \\ &\vdots \\ &= h(f_1) \ldots (f_n) \vee g(f_1) \ldots (f_n). \end{aligned}$$

But, by Definition 1.10 $f(f_1) \ldots (f_n) = h(f_1) \ldots (f_n)$. Therefore, $f(f_1) \ldots (f_n) \vee g(f_1) \ldots (f_n)$ exists, which is impossible by Proposition 1.5.

These propositions express properties of total functions as arguments. Now, here is a property of maximal total functions as functions.

One could expect that if one of two equivalent nontotal functions is dominated by a total third one, the second must be as well. But this is not the case. Here is a counterexample:

11 1φ
φ φ

φφ
φ

is equivalent to

11 1φ
φ φ

1φ
φ

and the first, but not the second, is dominated by

11 11
φ φ

φφ
φ

Now, is it possible to prove that any nontotal function is dominated by two incompatible ones? Again, this is not generally the case. To see this, let us consider a function that behaves like a total function on TM_α but not on TB_α. For example, the already noted $f \in PM\langle_{tt}\rangle_t$

$$\begin{array}{cccc} 1 & & & 1 \\ \varphi & 0 & 0 & 1 \\ \varphi & \varphi & \varphi & \varphi \\ & \varphi & & \end{array}$$

is not total. It belongs to a lattice

$$\begin{array}{cccc} 1 & & & 1 \\ 1 & 0 & 0 & 1 \\ \varphi & \varphi & \varphi & \varphi \\ & \varphi & & \end{array}$$

$$\begin{array}{cccc} 1 & & & 1 \\ \varphi & 0 & 0 & 1 \\ \varphi & \varphi & \varphi & \varphi \\ & \varphi & & \end{array} \qquad\qquad \begin{array}{cccc} 1 & & & 1 \\ 1 & 0 & 0 & \varphi \\ \varphi & \varphi & \varphi & \varphi \\ & \varphi & & \end{array}$$

$$\begin{array}{cccc} 1 & & & 1 \\ \varphi & 0 & 0 & \varphi \\ \varphi & \varphi & \varphi & \varphi \\ & \varphi & & \end{array}$$

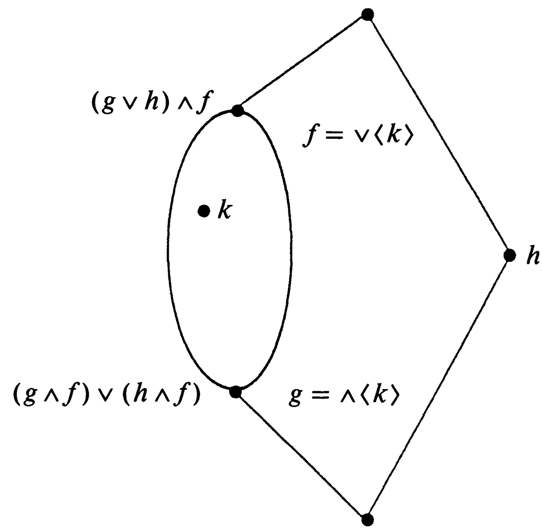
which is dominated by a total maximal function.

In general, the lattice $\Psi^{-1}(f)$ is a sublattice of a larger one. Now one might suppose it were possible to prove that any maximal nontotal function is dominated by two incompatible ones. Yet this is not so. A function can be maximal but undefined for some total argument. We need a much stronger notion of maximality. But the problem is that to be maximal is to be the join of a class of functions, all of which take the same value on the same *total* functions but not necessarily on *equivalent* functions. For example,

$$\begin{array}{cccc} 1 & & & 1 \\ \varphi & 1 & 1 & 1 \\ \varphi & \varphi & 1 & 1 \\ & \varphi & & \end{array}$$

is *maximal* according to our definition (it is not dominated by a function which takes the same values on total arguments). It is also nontotal (it is undefined for ${}^{1}{}_{\varphi}{}^{1}$), but not dominated by incompatible functions.

From an algebraic point of view, the consequence of the fact that some partial functions are equivalent is that the meet semi-lattice is not distributive and not even modular; that is, it is not the case, in general, that $(g \vee h) \wedge f = (g \wedge f) \vee (h \wedge f)$ even if both sides are defined. Let k be any function which is such that $\wedge\langle k\rangle \neq \vee\langle k\rangle$. In that case, we will have:



Another consequence of this is that the subset of maximal (resp. minimal) functions is not a sub meet semi-lattice. The join of two maximal functions is generally not a maximal function. The question is now whether we can define a subset of PM_α such that (a) this subset is a sub meet semi-lattice and (b) its elements have the same behavior for equivalent functions. The answer is that we can.

Let us consider the following definition.

Definition 2.10 A function $f \in PM_\alpha$ is *uniform* iff for any projectors $\langle f_1, \ldots, f_n\rangle$ and $\langle g_1, \ldots, g_n\rangle$ of f such that $f_i <> g_i$, $f(f_1) \ldots (f_n) = f(g_1) \ldots (g_n)$. I will call this set U_α.

Proposition 2.11 *A function $f \in PM_{\alpha\beta}$ is uniform iff for any $x, y \in PM_\alpha$, with $x <> y$, $f(x) = f(y)$ and $f(x)$ is uniform.*

Proof:

$\Rightarrow$. *Let $f \in PM_{\alpha\beta}$* be uniform and $x, y \in PM_\alpha$, be such that $x <> y$. Let us further suppose that $f(x) \neq f(y)$. There is thus a projector $\langle f_1, \ldots, f_n\rangle$ such that $f(x)(f_2) \ldots (f_n) \neq f(y)(f_2) \ldots (f_n)$. But $x <> y$ and $f_i <> f_i$ for any i, which means that f is not uniform, which is absurd.

Let us now suppose that $f(x)$ is not uniform. In that case there are two projectors $\langle f_2, \ldots, f_n\rangle$ and $\langle g_2, \ldots, g_n\rangle$ such that $f_i <> g_i$, $f(x)(f_2) \ldots (f_n) \neq f(x)(g_2) \ldots (g_n)$. But $x <> x$, thus f is not uniform, which is absurd.

$\Leftarrow$. Let f be such that for any x and y, if $x <> y$ then $f(x) = f(y)$ and $f(x)$ is not uniform. So there are two projectors $\langle f_1, \ldots, f_n\rangle$ and $\langle g_1, \ldots, g_n\rangle$ such that $f_i <> g_i$, and $f(f_1) \ldots (f_n) \neq f(g_1) \ldots (g_n)$. But $f_1 <> g_1$ implies that $f(f_1) = f(g_1)$. This implies that $f(f_1) \ldots (f_n) \neq f(f_1) \ldots (g_n)$ which contradicts the hypothesis that $f(f_1)$ is uniform.

I will need the following property:

Proposition 2.12 *For any $x, y \in PM_{\alpha\beta}$, if $x \leq y$, then $\vee\langle x\rangle \leq \vee\langle y\rangle$.*

The following proposition shows that uniform functions are not "scarce": every function is either uniform or dominated by a uniform function.

Proposition 2.13 *Let $f \in PM_\alpha$. There is $g \in PM_\alpha$, g uniform, such that $f \leq g$.*

Proof:
(i) For $\alpha = e$, any $x \in PM_e$ is uniform.
(ii) For $\alpha = t$, any $x \in PM_t$ is uniform.
(iii) For $\alpha = \beta\gamma$, for $x \in PM_\beta$, let g be such that $g(x) = f(\vee\langle x\rangle)$. I have to show that such a g exists, i.e., that g is monotonic. Let x, y be such that $x \leq y$. In that case, by Proposition 2.13, $\vee\langle x\rangle \leq \vee\langle y\rangle$ and f being monotonic, $f(\vee\langle x\rangle) \leq f(\vee\langle y\rangle)$ and finally $g(x) \leq g(y)$.

Proposition 2.14 *All maximal total functions are uniform.*

Proposition 2.15 *For any α, φ_α is uniform.*

Proposition 2.16 *The set $U_{\alpha\beta}$ is a sub meet semi-lattice of PM_α.*

Proof:
(i) For $\alpha = e$, $U_\alpha = PM_\alpha$.
(ii) For $\alpha = t$, $U_\alpha = PM_\alpha$.
(iii) For $\alpha = \beta\gamma$, let $f, g \in U_{\beta\gamma}$ and let us suppose that $f \vee g$ exists. Let $y \in U_\beta$, and $x \in \langle y\rangle$.

$$(f \vee g)(x) = f(x) \vee g(x) = f(y) \vee g(y) = (f \vee g)(y).$$

Thus, $f \vee g$ is uniform. The argument is the same for the meet.

The following proposition is very strong: it means that maximal uniform nontotal partial functions draw the maximum of information from their arguments without breaking monotonicity.

Proposition 2.18 *Let $f \in PM_\alpha$ be a maximal uniform nontotal partial function. If E contains at least two elements, then there are $g, h \in PM_\alpha$, $g \neq^* h$, such that $f \leq g$ and $f \leq h$.*

Proof:
(i) For $\alpha = e$, the only such function is φ, which is dominated by all the elements of E.
(ii) For $\alpha = t$, the only such function is φ, which is dominated by 0 and 1.
(iii) For $\alpha = \beta\gamma$, f being nontotal, there is an projector $\langle f_1, \ldots, f_n\rangle$, with $f_i \in PT_{\alpha_i}$ such that $f(f_1)\ldots(f_n) = \varphi$. f being uniform, for any projector $\langle g_1, \ldots, g_n\rangle$ of f such that $f_i <> g_i$, $f(g_1)\ldots(g_n) = \varphi$. Let g, h be such that $g(g_1)\ldots(g_n) = x$ and $h(g_1)\ldots(g_n) = y$, x and y being such that $x \neq^* y$. One can easily verify that g and h exist and are monotonic.

3 Conclusion It was claimed in the introduction that our construction is very general. But what about partial relations? It is well known that for standard objects any relation can be identified with one and only one function. This is based on Schönfinkel's theorem

$$\{0,1\}^{A_1 \times \ldots \times A_n} \cong (\ldots(\{0,1\}^{A_1})\ldots)^{A_n}$$

through the natural identification of a relation with its characteristic function

$$\mathcal{P}(A_1 \times \ldots \times A_n) \cong \{0,1\}^{A_1 \times \ldots \times A_n}.$$

It has been argued (in [13]) that this isomorphism is broken when dealing with partial objects. The argument is the following:

Let $f: \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$ be such that $f(x,y) = y$ if $x = 0$ and undefined otherwise. f is clearly the characteristic function of a relation. Now, consider the following two functions:

$f_1: \{0,1\} \rightarrow \{0,1\}^{\{0,1\}}$ such that $f_1(0)$ is the identity function and $f_1(1)$ is undefined and

$f_2: \{0,1\} \rightarrow \{0,1\}^{\{0,1\}}$ such that $f_2(0)$ is the identity function and $f_2(1)$ is such that $f_2(1)(0)$ is undefined and $f_2(1)(1)$ is also undefined.

Strictly speaking, f_1 and f_2 are not the same object: $f_1(1)$ is defined whereas $f_2(1)$ is not.

This is a bad argument which rests on the inaccuracy of the notion of "undefinition". Now we can easily see that if we think of the undefined function from $A \rightarrow B$, as the function such that for any $x \in A$, $f(x)$ is the undefined object, the isomorphism is restored!

Let us conclude with an overview of possible applications to the semantics of intensional logic. It is well known from Henkin [6] and Montague [10] that we need only three syncategorematic operators in type theory and in intensional logic: one abstractor and its converse, and identity. Suppose that there is a model M, in which each constant c has received as its value a standard function $v(c)$ and that to each variable x is assigned a value $v(x)$ of the corresponding type. A partial model compatible with M will be a model M' such that c has received a partial value $v'(c) \leq^* v(c)$ and x a partial value $v'(x) \leq^* v(x)$. We can now have sequences of partial models in which each constant takes values that are nearer and nearer to the given total value in M. It can easily be shown (see Lepage [9]) that in any such partial model M', every complex expression constructed by functional application or functional abstraction receives a value in M' which is a partial function y' such that $y' \leq^* y$, where y is the value of the expression in M. In other words, functional application and functional abstraction are monotonic for $\leq$ and for $\leq^*$.

The problem is with identity. The classical standard value of identity is such that:

$$v(a \equiv b) = v(\equiv)(v(a)(v(b))) = 1 \text{ iff } v(a) = v(b)$$

$$v(a \equiv b) = v(\equiv)(v(a)(v(b))) = 0 \text{ iff } v(a) \neq v(b).$$

Going partial, there is no natural generalization of this "definition". We clearly cannot adopt a definition like $v'(a \equiv b) = v'(\equiv)(v'(a)(v'(b))) = 1$ iff $v'(a) = v'(b)$, because such a definition is incompatible with our requirement of monotonicity (we need only consider the case where $v'(a) = v'(b) = \varphi_\alpha$).

Now, if we consider the following definition:

$$v'(a \equiv b) = v'(\equiv)(v'(a)(v(b))) = 1 \text{ iff } v'(a) = v'(b) \text{ and } v'(a) \text{ is total}$$

$$v'(a \equiv b) = v'(\equiv)(v'(a)(v'(b))) = 0 \text{ iff } v'(a) \neq^* v'(b)$$

$$v'(a \equiv b) = v'(\equiv)(v'(a)(v'(b))) = \varphi \text{ otherwise,}$$

it can be shown (see Lapierre [8]) that, using this definition, the class of expressions of intensional logic that are *never false* under an arbitrary partial interpretation is exactly the class of valid expressions, i.e., the class of expressions which are true under any standard interpretation.

Furthermore, according to this definition, there is a class of expressions which are always true under arbitrary partial interpretations. For example, $(\lambda xx \equiv \lambda xx)$ is always true (λxx is always a total function). The question now is to provide a complete axiomatic system for this class. This will be the object of another paper.

Acknowledgment This research was supported by FCAR and CRSHC grants. Thanks to the referee for valuable comments.

NOTES

1. Coin brackets will be omitted when possible, using the convention of association on the right: for example, $\langle\alpha\beta\rangle$ will be written $\alpha\beta$; $\langle\alpha\langle\beta\gamma\rangle\rangle$ will be written $\alpha\beta\gamma$; and $\langle\langle\alpha\beta\rangle\gamma\rangle$ will be written $\langle\alpha\beta\rangle\gamma$. For the sake of simplicity, we do not consider the type of possible worlds. It should be clear that all the propositions will still hold if we introduce such a type.
2. When a proof uses only elementary results of set theory, it will be skipped.
3. Strictly speaking, there are as many relations as there are types. Each relation should be indexed: for example, for PM_α the relation is "$\leq_\alpha$". Since no confusion should arise, we will omit the index.
4. If E is denumerable, the range of n is not ω but a transfinite ordinal which depends on the type of f. If E is finite, this proposition provides an explicit construction of the total function.
5. These very useful diagrams were introduced by Serge Lapierre.
6. The entire meet semi-lattice $PM_{\langle tt\rangle t}$ has been drawn by Saint-Louis using Prolog. It is too big to be shown here.

REFERENCES

[1] Alves, E. H. and J. A. D. Guerzoni, "Extending Montague's system: a three valued intensional logic," *Studia Logica*, vol. 49 (1990), pp. 127–132.

[2] van Benthem, J., "Partiality and nonmonotonicity in classical logic," *Logique et Analyse*, vol. 29 (1986), pp. 225–247.

[3] Blamey, S., "Partial logic," pp. 1–70 in *Handbook of Philosophical Logic*, edited by D. Gabbay and F. Guenthner, Reidel, Dordrecht, 1986.

[4] Church, A., "A formulation of the simple theory of types," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 56–68.

[5] Farmer, W. M., "A partial functions version of Church's simple theory of types," *The Journal of Symbolic Logic*, vol. 55 (1990), pp. 1269–1271.

[6] Henkin, L., "A theory of propositional types," *Fundamenta Mathematica*, vol. 52 (1983), pp. 323–344.

[7] Kindt, W., "Two approaches to vagueness: theory interaction and topology," pp. 361–392 in *Approaching Vagueness*, edited by T. T. Ballmer and M. Pinkal, Elsevier Science Publisher, Amsterdam, 1983.

[8] Lapierre, S., "A partial semantics for intensional logic," *Notre Dame Journal of Formal Logic*, vol. 33 (1992), pp. 517–541.

[9] Lepage, F., "Intensional Logic and the pragmatics of belief," pp. 143–151 in *Entities and Individuation*, edited by D. Stewart, Edwin Mellen Press, New York, 1989.

[10] Montague, R., "Universal grammar," *Theoria*, vol. 36 (1970), pp. 373–398.

[11] Muskens, R., "Going partial in Montague grammar," *ITLI Prepublication Series*, LP–88–04, University of Amsterdam, 1988.

[12] Scott, D., "Models for various type-free calculi," pp. 157–187 in *Logic, Methodology and Philosophy of Science IV*, edited by P. Suppes, North-Holland, Amsterdam, 1973.

[13] Tichý, P., "Foundations of partial type theory," *Reports on Mathematical Logic*, vol. 14 (1982), pp. 59–72.

Département de Philosophie
Université de Montréal
C.P. 6128, Succ. 'A'
Montréal, Québec
Canada H3C 3J7