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On Potential Embedding and Versions of Martin's Axiom

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Abstract We give a characterization of versions of Martin's axiom and some other related axioms by means of potential embedding of structures.

1 Introduction Let A and B be structures. For a condition $\mathcal{E}$ on p.o.-sets (e.g., ccc, proper, $<\kappa$-closed, etc.) let us say that A is $\mathcal{E}$-potentially embeddable into B if there exists a p.o.-set P with the property $\mathcal{E}$ such that $\Vdash_P$ "A is embeddable into B". Similarly we shall say that A and B are $\mathcal{E}$-potentially isomorphic if there exists a p.o.-set P with the property $\mathcal{E}$ such that $\Vdash_P$ "$A \cong B$".

The notion of $(<\kappa,\infty)$-distributive-potentially isomorphism and $<\kappa$-closed-potentially isomorphism have been studied in Nadel and Stavi [7]. In Fuchino, Koppelberg, and Takahashi [4] a characterization of $(<\kappa,\infty)$-distributive-potentially isomorphism to a free Boolean algebra is given under certain set theoretic assumptions on κ.

In this note we shall consider the question if $\mathcal{E}$-potential embedding ($\mathcal{E}$-potential isomorphism) implies the real embedding (isomorphism).

The following examples suggest that this question is by no means trivial for some instances of A and B even when we consider the ccc as the condition $\mathcal{E}$. Example 1.1c is due to S. Kamo.

Example 1.1

(a) Let A be the subalgebra of the Boolean algebra $\wp(\omega_1)$ consisting of finite and co-finite subsets of ω_1. Assume that there exists a ccc Boolean algebra B which is not productively ccc. Let C be a ccc Boolean algebra such that $B \oplus C$ does not satisfy the ccc. By the ccc of B, A is not embeddable in B. But, since $\Vdash_{C^+}$ "B does not satisfy the ccc", we obtain the result that $\Vdash_{C^+}$ "A is embeddable into B".

This situation can also be coded in structures in a language with only a binary relation symbol: Let B and C be as above. Let D and E be the structures defined by

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$$D = (\omega_1, \{(\alpha,\beta) : \alpha,\beta \in \omega_1,\ \alpha \neq \beta\}),$$
$$E = (B^+, \{(a,b) : a,b \in B^+,\ a\cdot b = 0\}).$$

Then no uncountable substructure of D is embeddable into E. But $\Vdash_{C^+}$ "D is embeddable into E".

(b) (CH) Let A and B be mutually nonembeddable $\aleph_1$-dense suborderings of $\mathbb{R}$. Let P be the standard p.o.-set forcing $MA + 2^{\aleph_0} = \aleph_2$. Then, by Baumgartner [1], $\Vdash_P A \cong B$ holds.

(c) Let $A = (\mathbb{R},<)$ and $B = (\mathbb{R}\setminus\{0\},<)$. Clearly $A \ncong B$. However if P is a p.o.-set which adds any new real then $\Vdash_P$ "$A \cong B$" holds. This can be seen as follows. Working in the generic extension $V[G]$ of the ground model V, where G is a P-generic filter over V, let $x \in \mathbb{R}^{V[G]}\setminus\mathbb{R}^V$. Let $\{a_n\}_{n\in\omega}, \{b_n\}_{n\in\omega}, \{c_n\}_{n\in\omega}, \{d_n\}_{n\in\omega}$ be sequences of elements of $Q \cup \{-\infty,+\infty\}$ such that $a_0 = c_0 = -\infty$, $b_0 = d_0 = +\infty$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$ and $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = 0$ where $\{a_n\}_{n\in\omega}, \{c_n\}_{n\in\omega}$ are strictly increasing and $\{b_n\}_{n\in\omega}, \{d_n\}_{n\in\omega}$ are strictly decreasing sequences. Then there exist $(f_n)_{n\in\omega}, (g_n)_{n\in\omega} \in V[G]$ such that $f_n, g_n \in V$, $V \models$ "$f_n : (a_n, a_{n+1}] \overset{\cong}{\to} (c_n, c_{n+1}]$" and $V \models$ "$g_n : [b_{n+1}, b_n) \overset{\cong}{\to} [d_{n+1}, d_n)$" for $n \in \omega$. Let $f = \bigcup_{n\in\omega} f_n \cup \bigcup_{n\in\omega} g_n$. Then $f \in V[G]$ and f is an isomorphism of A to B.

If B has some "good" properties we can deduce the real embedding (isomorphism) to B from the potential embedding (potential isomorphism). The following trivial lemma is such an example:

Lemma 1.2 *Let B be κ^+-universal for $\kappa = |A|$. If there exists a p.o.-set P such that $\Vdash_P$ "A is embeddable into B" then A is embeddable into B.*

In Section 2 we show that the ccc-potential isomorphism to a free algebra implies the real isomorphism (Theorem 2.1).

For a condition $\mathcal{E}$ on p.o.-sets and a cardinal κ let $\mathrm{PE}_\kappa^{\mathcal{E}}$ ($\mathrm{PI}_\kappa^{\mathcal{E}}$) denote the following axiom:

> For all structures A, B such that $|A| = \kappa$, if A is $\mathcal{E}$-potentially embeddable into B (if A is $\mathcal{E}$-potentially isomorphic to B and $|A| = |B|$) then A is embeddable into (isomorphic to) B.

Similarly we shall also consider a class of axioms on potential partial embedding. Let $\mathrm{PPE}_\kappa^{\mathcal{E}}$ be the axiom saying:

> For all structures A, B in some relational language L, if $\Vdash_P$ "there exists a substructure C of A of size κ which is embeddable into B" for some p.o.-set P with the property $\mathcal{E}$ then there exists a substructure C of A of size κ which is embeddable into B.

Here we call a language L relational when L contains no function symbols. Note that the definition of $\mathrm{PE}_\kappa^{\mathcal{E}}$ and $\mathrm{PI}_\kappa^{\mathcal{E}}$ would not be changed if the structures considered would have been restricted to be in relational languages.

In Section 3 we show that the axioms $\mathrm{PE}_\kappa^{\mathcal{E}}$, $\mathrm{PI}_\kappa^{\mathcal{E}}$, and $\mathrm{PPE}_\kappa^{\mathcal{E}}$ are closely related to the corresponding version of Martin's axiom. We show that Martin's axiom on the property $\mathcal{E}$ for κ dense sets is equivalent to $\mathrm{PE}_\kappa^{\mathcal{E}}$ (Theorem 3.3). In particular, $\mathrm{PE}_\kappa^{\mathrm{ccc}}$ is equivalent to Martin's axiom for κ dense sets.

In Section 4 we shall give still another characterization of versions of Martin's axiom which is related to the characterization of Martin's axiom given in Todorčević and Veličković [9].

Using the axioms on potential partial embedding introduced above and their weakenings to be defined in Section 5, we also give characterizations of some other known axioms related to Martin's axiom (Theorem 3.6 and Theorem 5.4).

2 *Free algebras* Let V be any variety. For an algebra A and a subset X of A let $[X]_A$ denote the subalgebra of A generated by X. An algebra A in V is called *V-free* (or simply *free* for short) if there is a subset $X \subseteq A$ such that $A = [X]_A$ and for any $B \in V$ and any mapping $f: X \to B$ there exists a homomorphism $\hat{f}: A \to B$ extending f. Such X is said to be a *free basis* of A.

Clearly X is a free basis of A if and only if $[X]_A = A$ and, for every finite $Y \subseteq X$, Y is a free basis of $[Y]_A$.

Theorem 2.1 *Let A be an algebra in a variety V. If there exists a* ccc *p.o.-set P such that $\Vdash_P$ "A is free" then A is really free.*

Proof: Let P be as above and $\dot{X}$ be a P-name such that $\Vdash_P$ "$\dot{X}$ is a free basis of A".

Claim 2.2 *Let $a \in A$ and B be a subalgebra of A such that $\Vdash_P$ "$B = [\dot{X} \cap B]_A$". Then there is a subalgebra B' of A such that B' is countably generated over B, $a \in B'$ and $\Vdash_P$ "$B' = [\dot{X} \cap B']_A$". If B is free then B' is also free and any free basis of B can be extended to a free basis of B'.*

Proof: For each $b \in A$ let Y_b be a countable subset of A such that $\Vdash_P$ "$[\dot{X} \cap Y_b]_A \ni b$". This is possible since P satisfies the ccc. Let Y be a countable subset of B such that $a \in [Y]_A$ and $Y_b \subseteq Y$ for every $b \in Y$. Then $B' = [B \cup Y]_A$ is as desired.

Now suppose that U is a free basis of B. Let $Y = \{y_n : n \in \omega\}$. Let $p_n \in P$, $k_n \in \omega$ and $u_{n,i} \in B' \setminus B$ for $i \le k_n$ and $n \in \omega$ be such that

$$p_n \ge p_{n+1} \text{ for every } n \in \omega,$$

$$p_n \Vdash_P \text{“} u_{n,0}, \ldots, u_{n,k_n} \in \dot{X} \text{” for every } n \in \omega,$$

$$p_n \Vdash_P \text{“} y_n \in [B \cup \{u_{n,0}, \ldots, u_{n,k_n}\}]_A \text{” for all } n \in \omega.$$

It follows that

$$B' = [B \cup \{u_{n,i} : i \le k_n,\ n \in \omega\}]_A.$$

Clearly $U \cup \{u_{n,i} : i \le k_n, n \in \omega\}$ is then a free basis of B'. This proves Claim 2.2.

Let $\kappa = |A|$. By Claim 2.2 we can construct sequences $(A_\alpha)_{\alpha<\beta}$ and $(X_\alpha)_{\alpha<\beta}$ for some $\beta \le \kappa$ inductively so that:

$(A_\alpha)_{\alpha<\beta}$ is a continuously increasing sequence of subalgebras of A;
$(X_\alpha)_{\alpha<\beta}$ is a continuously increasing sequence of subsets of A and X_α is a free basis of A_α for all $\alpha < \beta$;
$\Vdash_P$ "$A_\alpha = [\dot{X} \cap A_\alpha]_A$" for all $\alpha < \beta$;
$\bigcup_{\alpha<\beta} A_\alpha = A$.

Then $\bigcup_{\alpha<\beta} X_\alpha$ is a free basis of A. This completes the proof of Theorem 2.1.

For p.o.-set which collapses any cardinal, the theorem corresponding to Theorem 2.1 does not hold. In particular, Theorem 2.1 for proper p.o.-sets does not hold. This can be seen by the following example in the variety of Boolean algebras. Let P be a p.o.-set which collapses κ^+. Let $B = \mathrm{Fr}\,\kappa^+ \times \mathrm{Fr}\,\kappa$. B is not free since it is not homogeneous. But $\Vdash_P$ "B is free".

A Boolean algebra A is said to be projective if A is a retract of a free Boolean algebra, i.e., if there are a free Boolean algebra F and homomorphisms $e: A \to F$, $f: F \to A$ such that $f \circ e = id_A$. A theorem by Ščepin says that a Boolean algebra A is projective if and only if $A \oplus \mathrm{Fr}\,\kappa$ is free for $\kappa = |A|$ (see Koppelberg [6]). Hence we obtain the following corollary.

Corollary 2.3 *Let A be a Boolean algebra. If there is a* ccc *p.o.-set P such that $\Vdash_P$ "A is projective". Then A is really projective.*

Since not every subalgebra of a free Boolean algebra is projective, the following problem still remains open:

Problem 2.4 Does ccc-potential embedding into a free Boolean algebra imply the real embedding?

Note that in varieties such as groups, Abelian groups, etc., where any subalgebra of a free algebra is also free, the problem above does not arise.

In contrast to Problem 2.4 we have a complete answer to a similar problem on potential embeddability in the power-set algebra over ω:

Proposition 2.5 (Fuchino [3]) *Let B be a Boolean algebra. Then the following are equivalent*:

(1) *There exists a* ccc *p.o.-set P such that*

$$\Vdash_P \text{"}B \text{ is embeddable in } \wp(\omega)\text{"};$$

(2) *There exists a* ccc *p.o.-set P such that*

$$\Vdash_P \text{"}B \text{ has a finitely additive strictly positive measure"};$$

(3) $\overbrace{B \oplus \cdots \oplus B}^{n \text{ times}}$ *has the* ccc *for every $n \in \omega$.*

Note that condition (1) of Proposition 2.5 does not mean the ccc-potental embeddability of B into the power-set algebra $\wp(\omega)$ in our sense since the power-set algebra $\wp(\omega)$ in the generic extension is, in general, not equal to the power-set algebra $\wp(\omega)$ in the ground model.

3 Potential embedding and Martin's axiom For any condition $\mathcal{E}$ on p.o.-sets and cardinal κ, let $\mathrm{MA}_\kappa^{\mathcal{E}}$ be the following assertion:

> For any p.o.-set satisfying the condition $\mathcal{E}$ and for any family $\mathfrak{D} = \{D_\alpha : \alpha < \kappa\}$ of dense subsets of P, there exists a $\mathfrak{D}$-generic filter over P.

Using this notation, the proper forcing axiom (PFA) and Martin's maximum (MM) can be denoted by $\mathrm{MA}_{\aleph_1}^{\text{proper}}$ and $\mathrm{MA}_{\aleph_1}^{\text{stat.preserving}}$ respectively. For these axioms see, for example, Jech [5]. Martin's axiom for κ dense sets (i.e., $\mathrm{MA}_\kappa^{\text{ccc}}$) is also denoted, as usual, by MA_κ.

Lemma 3.1 *For any condition* $\mathcal{E}$ *on p.o.-sets and an infinite cardinal* κ, $\mathrm{MA}_\kappa^{\mathcal{E}}$ *implies* $\mathrm{PE}_\kappa^{\mathcal{E}}$, $\mathrm{PI}_\kappa^{\mathcal{E}}$, *and* $\mathrm{PPE}_\kappa^{\mathcal{E}}$.

Proof: Assume that $\mathrm{MA}_\kappa^{\mathcal{E}}$ holds. We shall show that $\mathrm{PI}_\kappa^{\mathcal{E}}$ and $\mathrm{PPE}_\kappa^{\mathcal{E}}$ hold. The proof of $\mathrm{PE}_\kappa^{\mathcal{E}}$ can be done similarly.

Let A and B be structures of size κ. Suppose that there exists a p.o.-set P satisfying the condition $\mathcal{E}$ and a P-name $\dot{f}$ such that $\Vdash_P$ "$\dot{f}$ is an isomorphism from A to B". For each $a \in A$ and $b \in B$ let

$$D_a = \{p \in P: \text{there exists some } d \in B \text{ such that } p \Vdash_P \text{“}\dot{f}(a) = d\text{”}\},$$
$$D'_b = \{p \in P: \text{there exists some } c \in A \text{ such that } p \Vdash_P \text{“}\dot{f}(c) = b\text{”}\}.$$

Clearly D_a and D'_b are dense subsets of P. Let $\mathfrak{D} = \{D_a : a \in A\} \cup \{D'_b : b \in B\}$ and let G be a $\mathfrak{D}$-generic filter over P. Then the mapping $f: A \to B$ defined by

$$f(a) = b \qquad \text{for some } b \in B \text{ such that there exists } p \in G \text{ with } p \Vdash_P \text{“}\dot{f}(a) = b\text{”}$$

is an isomorphism from A to B. This proves that $\mathrm{PI}_\kappa^{\mathcal{E}}$ holds.

For $\mathrm{PPE}_\kappa^{\mathcal{E}}$ let A and B be structures in a relational language L. Suppose that there exists a p.o.-set P satisfying the condition $\mathcal{E}$ and P-names $\dot{C}, \dot{h}$ such that $\Vdash_P$ "$\dot{C}$ is a substructure of A of size κ and $\dot{h}$ is an embedding of $\dot{C}$ into B". Let $\dot{g}$ be a P-name of injective enumeration of $\dot{C}$ of length κ, i.e., $\Vdash_P$ "$\dot{g}: \kappa \to \dot{C}$ is 1-1 onto". For each $\alpha < \kappa$ let

$$D_\alpha = \{p \in P: p \text{ decides } \dot{g}(\alpha) \text{ and } \dot{h} \circ \dot{g}(\alpha)\}.$$

Let $\mathfrak{D} = \{D_\alpha : \alpha \in \kappa\}$ and let G be a $\mathfrak{D}$ generic filter over P. Let

$$C = \{a \in A : \text{There exists } p \in G \text{ and } \alpha < \kappa \text{ such that } p \Vdash_P \text{“}\dot{g}(\alpha) = a\text{”}\}.$$

Then C is a substructure of A of size κ. Let $h: C \to B$ be defined by

$$h(a) = b \qquad \text{for } b \in B \text{ such that there is } p \in G \text{ and } \alpha < \kappa \text{ with } p \Vdash_P \text{“}\dot{g}(\alpha) = a \wedge \dot{h} \circ \dot{g}(\alpha) = b\text{”}.$$

It is easy to see that h is an embedding of C into B.

Lemma 3.2 $\mathrm{PI}_{2^{\aleph_0}}^{\mathcal{E}}$ *does not hold for any condition* $\mathcal{E}$ *such that there exists a p.o.-set* P *satisfying* $\mathcal{E}$ *which adds a new real. In particular* $\mathrm{PI}_{\aleph_1}^{\mathrm{ccc}}$ *implies* $\neg CH$.

Proof: Immediate from Example 1.1c. The second assertion can also be seen directly in Example 1.1b.

Similarly, using Example 1.1a, we could show that $\mathrm{PE}_{\aleph_1}^{\mathrm{ccc}}$ or $\mathrm{PPE}_{\aleph_1}^{\mathrm{ccc}}$ implies $\neg CH$. However we can actually prove much more general results. For a condition $\mathcal{E}$, we shall say that $\mathcal{E}$ is a regular condition on p.o.-sets if, for any p.o.-set P and any dense subordering Q of P, P satisfies $\mathcal{E}$ whenever Q satisfies $\mathcal{E}$. Note that the conditions $\mathcal{E}$ on p.o.-sets used to define the usual versions of Martin's axiom of the form $\mathrm{MA}_\kappa^{\mathcal{E}}$ (σ-centered, ccc, proper, stationary preserving, etc.) are regular.

Theorem 3.3 *For a regular condition* $\mathcal{E}$ *on p.o.-sets and an infinite cardinal* κ, $\mathrm{MA}_\kappa^{\mathcal{E}}$ *is equivalent to* $\mathrm{PE}_\kappa^{\mathcal{E}}$.

Proof: By Lemma 3.1 it is enough to show that $\mathrm{PE}_\kappa^{\mathcal{E}}$ implies $\mathrm{MA}_\kappa^{\mathcal{E}}$. For $\kappa = \aleph_0$ this is clear since $\mathrm{MA}_{\aleph_0}^{\mathcal{E}}$ and $\mathrm{PE}_{\aleph_0}^{\mathcal{E}}$ already hold in ZFC. Let us assume that $\mathrm{MA}_\kappa^{\mathcal{E}}$

does not hold for an uncountable κ. We shall show that $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ does not hold. Let P be a p.o.-set satisfying the condition $\mathcal{E}$ with a family $\mathfrak{D} = \{D_\alpha : \alpha < \kappa\}$ of dense subsets of P such that there exists no $\mathfrak{D}$-generic filter over P.

Claim 3.4 *There exists a p.o.-set P' satisfying the condition $\mathcal{E}$ and a family $\mathfrak{D}' = \{D'_\alpha : \alpha < \kappa\}$ of dense subsets of P' such that*
(a) *there exists no $\mathfrak{D}'$-generic filter over P',*
(b) *for every $\alpha < \kappa$ and $p \in D'_\alpha$ there exists $q \in P'$ such that $p \leq q$ and*

$$q \in D'_\alpha \Big\backslash \bigcup_{\beta \in \kappa \setminus \{\alpha\}} D'_\beta.$$

Proof: For every $p \in P$ let $T_p = \{q_{p,\alpha} : \alpha < \kappa\}$ where we assume that $q_{p,\alpha} \notin P$ and $q_{p,\alpha} \neq q_{p',\alpha'}$ for $p, p' \in P$ and $\alpha, \alpha' \in \kappa$ such that $(p,\alpha) \neq (p',\alpha')$. Let

$$P' = P \,\dot{\cup}\, \dot{\bigcup_{p \in P}} T_p$$

and

$$\begin{aligned} \leq^{P'} = \leq^{P} &\cup \{(p', q_{p,\alpha}) : p, p' \in P,\ \alpha < \kappa,\ p' \leq p\} \\ &\cup \{(q_{p,\alpha}, q_{p,\beta}) : p \in P,\ \alpha \leq \beta < \kappa\}. \end{aligned}$$

Since P is dense in $P' = (P', \leq^{P'})$, P' still satisfies the condition $\mathcal{E}$. For $\alpha < \kappa$ let

$$D'_\alpha = D_\alpha \cup \{q_{p,\alpha} : p \in D_\alpha\}.$$

Then P' and $\mathfrak{D}' = \{D'_\alpha : \alpha < \kappa\}$ are as desired.

By the claim above we may assume without loss of generality that for any $\alpha < \kappa$ and $p \in D_\alpha$ there exists $q \geq p$ such that

$$q \in D_\alpha \Big\backslash \bigcup_{\beta \in \kappa \setminus \{\alpha\}} D_\beta.$$

Now let $A = (\kappa, \{\alpha\}, \kappa^n)_{\alpha < \kappa, n < \omega}$ and $B = (P, D_\alpha, C_n)_{\alpha < \kappa, n < \omega}$ where

$$\begin{aligned} C_n = \{(p_1, \ldots, p_n) : p_1, \ldots, p_n \in P, &\text{ there exists } q \in P \\ &\text{ such that } q \leq p_1, \ldots, p_n\}. \end{aligned}$$

Then the embeddability of A into B is equivalent to the existence of $\mathfrak{D}$-generic filter over P. It follows that A is not embeddable into B. But we have that $\Vdash_P$ "A is embeddable into B". Hence $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ does not hold. This completes the proof of Theorem 3.3.

From Theorem 3.3 and Lemma 3.1 it follows that $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ implies $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ for a regular condition $\mathcal{E}$ on p.o.-sets and every infintie cardinal κ.

Problem 3.5 Is $\mathrm{PI}^{\mathrm{ccc}}_{\aleph_1}$ equivalent to $\mathrm{PE}^{\mathrm{ccc}}_{\aleph_1}$?

Some structures constructed in [7] and [4] exemplify that $\mathrm{PI}^{\omega_1\,\mathrm{preserving}}_{\aleph_1}$ is inconsistent. These examples also show that, just as for MM, the axiom $\mathrm{PI}^{\mathrm{stat.preserving}}_{\aleph_1}$ is maximally (possibly) consistent in the corresponding family of axioms.

Problem 3.6 Is $\mathrm{PI}^{\mathrm{stat.preserving}}_{\aleph_1}$ equivalent to MM?

A subset Y of a p.o.-set P is said to be centered if for every $u \in [Y]^{<\omega}$ there exists $x \in P$ such that $x \leq y$ for all $y \in u$. A p.o.-set P has precaliber κ if for $X \subseteq P$ of size κ there exists $Y \subseteq X$ of size κ such that Y is centered.

Theorem 3.7
(a) $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ *is equivalent to* $\mathrm{MA}_{\aleph_1}$.
(b) *Assume that every* ccc *p.o.-set is productively* ccc (*this is true, e.g., under some weak version of* $\mathrm{MA}_{\aleph_1}$). *Then for any cardinal* κ *of uncountable cofinality,* $\mathrm{PPE}^{\mathrm{ccc}}_{\kappa}$ *is equivalent to the assertion*:

($\mathcal{K}_\kappa$) *Every* ccc *p.o.-set has precaliber* κ.

Proof:
(a) In [9] it is proved that $\mathrm{MA}_{\aleph_1}$ is equivalent to the assertion:

($\mathcal{K}$) Every ccc p.o.-set has precaliber $\aleph_1$.

So it is enough to show the equivalence of $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ to this assertion. First let us assume $\mathcal{K}$. Let A and B be structures in some relational language L. Suppose that there exists a ccc p.o.-set P and P-names $\dot{C}, \dot{h}$ such that $\Vdash_P$ "$\dot{C}$ is a substructure of A of size $\aleph_1$ and $\dot{h}$ is an embedding of $\dot{C}$ into B". Let $\dot{g}$ be a P-name of injective enumeration of $\dot{C}$ of length ω_1 (i.e., $\Vdash_P$ "$\dot{g} : \omega_1 \to \dot{C}$ is 1-1 onto"). For each $\alpha \in \omega_1$ let $p_\alpha \in P$, $a_\alpha \in A$ and $b_\alpha \in B$ be such that $p_\alpha \Vdash_P$ "$\dot{g}(\alpha) = a_\alpha$ and $\dot{h} \circ \dot{g}(\alpha) = b_\alpha$". By the assumption there exists an uncountable $X \subseteq \omega_1$ such that $\{p_\alpha : \alpha \in X\}$ is centered. Let C be the substructure of A with the underlining set $\{a_\alpha : \alpha \in X\}$ and let $h : C \to B$ be defined by $h(a_\alpha) = b_\alpha$ for $\alpha \in X$. Then h is an embedding of C into B.

Now assume that there exists a ccc p.o.-set Q which does not have precaliber $\aleph_1$. So there exists an uncountable subset X of P which does not have any uncountable centered subset. Let P be a ccc p.o.-set forcing $\mathrm{MA}_{\aleph_1}$. If $\Vdash_P$ "Q does not have the ccc" then, as in Example 1.1a, we can construct a counterexample to $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$. So let us assume $\Vdash_P$ "Q satisfies the ccc". Then we have $\Vdash_P$ "every uncountable subset of Q has an uncountable centered subset".

Let $A = (\aleph_1, \aleph_1, \aleph_1^n)_{n \in \omega}$ and $B = (Q, X, R_n)_{n \in \omega}$ where

$$R_n = \{(q_1, \ldots, q_n) : q_1, \ldots, q_n \in Q \text{ and there exists } r \in Q \text{ such that } r \leq q_1, \ldots, q_n\}.$$

Then any uncountable substructure of A is not embeddable into B but $\Vdash_P$ "A is embeddable into B". This shows that $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ does not hold.

(b) is proved similarly.

4 Forcing axioms for homogeneous covering of structures Let $\mathcal{E}$ be a condition on p.o.-sets. A partition $[S]^{<\omega} = K_0 \cup K_1$ for a set S is said to be $\mathcal{E}$-destructible if there exist a p.o.-set P satisfying the condition $\mathcal{E}$ and a P-name $\dot{X}$ of a 0-homogeneous subset of S with respect to the partition (i.e., $\Vdash_P$ "$[\dot{X}]^{<\omega} \subseteq K_0$") such that for all $s \in S$ there exists $p \in P$ such that $p \Vdash_P s \in \dot{X}$. In [9] the following characterization of MA_κ is given.

Theorem 4.1 ([9], see also Todorčević [8]) *MA_κ is equivalent to the following assertion*:

($\mathcal{L}_\kappa$) *Let S be a set of size $\leq \kappa$. Suppose that a partition $[S]^{<\omega} = K_0 \cup K_1$ is* ccc-*destructible. Then there exists a σ-covering $S = \bigcup_{i\in\omega} S_i$ of S such that each S_i is 0-homogeneous with respect to this partition (i.e., $[S_i]^{<\omega} \subseteq K_0$).*

In this section, we give a similar theorem in terms of potential embedding. Let A, B be structures in a language L which contains a unary relation symbol S. We say that S in B has an A-homogeneous σ-covering if there exist embeddings f_i of A into B for $i \in \omega$ such that $\mathsf{S}^B = \bigcup_{i\in\omega} f_i[\mathsf{S}^A]$.

Following the above terminology of Todorčević and Veličković, let us say that S in B is $\mathcal{E}$-destructible by A if there exist a p.o.-set P with the property $\mathcal{E}$ and a P-name $\dot{f}$ of embedding of A into B such that for every $b \in \mathsf{S}^B$ there is a $p \in P$ such that $p \Vdash_P$ "$b \in \dot{f}[\mathsf{S}^A]$".

Let $\mathrm{HC}^{\mathcal{E}}_\kappa$ denote the following assertion:

Let A and B be structures in a language L which contains a unary relation symbol S such that $|A|, |\mathsf{S}^B| \leq \kappa$. If S in B is $\mathcal{E}$-destructible by A then S in B has an A-homogeneous σ-covering.

The following lemma shows that $\mathrm{HC}^{\mathrm{ccc}}_\kappa$ is a generalization of $\mathcal{L}_\kappa$.

Lemma 4.2 *Let S be any infinite set and let $[S]^{<\omega} = K_0 \cup K_1$ be a partition. Then there exist structures A, B in a language L which contains a unary relation symbol S such that*

(a) $|A| = |B| = |S|$,

(b) *$[S]^{<\omega} = K_0 \cup K_1$ is $\mathcal{E}$-destructible if and only if S in B is $\mathcal{E}$-destructible and*

(c) *S has a σ-covering $\{S_i : i \in \omega\}$ of 0-homogeneous sets with respect to the partition $[S]^{<\omega} = K_0 \cup K_1$ such that $|S_i| = |S|$ for all $i \in \omega$ if and only if S in B has an A-homogeneous σ-covering.*

Proof: Let

$$A = ([S]^{<\omega}, [S]^1, [S]^{<\omega}, g_n)_{n\in\omega},$$
$$B = ([S]^{<\omega}, [S]^1, K_0, g_n)_{n\in\omega},$$

where $[S]^1$ is supposed to be the interpretation of S in both the structures and g_n is an n-place function defined by

$$g_n(a_0, \ldots, a_{n-1}) = \begin{cases} \{s_0, \ldots, s_{n-1}\}; & \text{if } a_i \text{ is the singleton } \{s_i\} \text{ for } i < n \\ \varnothing; & \text{otherwise.} \end{cases}$$

Proposition 4.3 *Let $\mathcal{E}$ be a regular condition on p.o.-sets and κ an infinite cardinal. Suppose that $\mathrm{MA}^{\mathcal{E}}_\kappa$ implies that, for any p.o.-set P satisfying the condition $\mathcal{E}$, the finite support product of ω copies of P still satisfies the condition $\mathcal{E}$. Then $\mathrm{HC}^{\mathcal{E}}_\kappa$ is equivalent to $\mathrm{MA}^{\mathcal{E}}_\kappa$.*

Proof: First assume $\mathrm{MA}^{\mathcal{E}}_\kappa$. Let A, B be structures in a language L which contains a unary relation symbol S such that $|A|, |\mathsf{S}^B| \leq \kappa$ and S in B is $\mathcal{E}$-destructible by A. Let P and $\dot{f}$ be as in the definition of $\mathcal{E}$-destructibility. Let Q be the finite support product of ω copies of P; i.e., let

$$Q = \{q \in {}^{\omega}P : \{i \in \omega : q(i) \neq 1_P\} \text{ is finite}\}$$

with the ordering

$$q \leq r \text{ if and only if } q(i) \leq r(i) \text{ for all } i \in \omega.$$

By the assumption, Q satisfies the condition $\mathcal{E}$.

For each $b \in \mathsf{S}^B$ let

$$D_b = \{q \in Q : q(i) \Vdash_P \text{“}b \in \dot{f}[\mathsf{S}^A]\text{”} \text{ for some } i \in \omega\}.$$

Then D_b is dense in Q. For $a \in A$ and $i \in \omega$ let

$$E_{a,i} = \{q \in Q : q(i) \Vdash_P \text{“}\dot{f}(a) = b\text{”} \text{ for some } b \in B\}.$$

Then $E_{a,i}$ is dense in Q. Let

$$\mathfrak{D} = \{D_b : b \in \mathsf{S}^B\} \cup \{E_{a,i} : a \in A,\ i \in \omega\}.$$

By $\mathrm{MA}_\kappa^{\mathcal{E}}$ there is a $\mathfrak{D}$-generic filter G over Q. For $i \in \omega$ let $f_i : A \to B$ be defined by $f_i(a) = b$ for $a \in A$ where $b \in B$ is such that $q(i) \Vdash_P$ “$\dot{f}(a) = b$” for some $q \in Q$. Then each f_i is well-defined embedding of A into B and $\bigcup_{i\in\omega} f_i[\mathsf{S}^A] = \mathsf{S}^B$.

Assume now that $\mathrm{MA}_\kappa^{\mathcal{E}}$ does not hold. Then as in the proof of Theorem 3.3 there are p.o.-set P of size κ satisfying the condition $\mathcal{E}$ and a family $\mathfrak{D} = \{D_\alpha : \alpha < \kappa\}$ of dense subsets of P such that there exists no $\mathfrak{D}$-generic filter over P and for every $\alpha < \kappa$ and $p \in D_\alpha$ there exits $q \geq p$ such that

$$q \in D_\alpha \Big\backslash \bigcup_{\beta \in \kappa \setminus \{\alpha\}} D_\beta.$$

Now let X be any set of size $\leq \kappa$ disjoint from P. Let $A = (\kappa \,\dot{\cup}\, X, X, \{\alpha\}, \kappa^n)_{\alpha<\kappa, n<\omega}$ and $B = (P \,\dot{\cup}\, X, X, D_\alpha, C_n)_{\alpha<\kappa, n<\omega}$ where X is supposed to be the interpretation of S in both the structures and $C_n, n \in \omega$ is defined as in the proof of Theorem 3.3. Then S in B is $\mathcal{E}$-destructible by A but there is no A-homogeneous σ-partition of S in B since A is not embeddable in B.

5 Some other axioms In this section we shall consider the following weakenings of the axioms defined in the introduction. For $n \geq 1$, $n \in \omega$, a language L is said to be n-ary if each function symbol of L is k-ary for some $k < n$ and each relation symbol of L is l-ary for some $l \leq n$.

For a condition $\mathcal{E}$ on p.o.-sets and a cardinal κ let $\mathrm{PE}_{\kappa,n}^{\mathcal{E}}$ ($\mathrm{PI}_{\kappa,n}^{\mathcal{E}}$) be the following axiom:

> For all structures A, B in some n-ary language L such that $|A| = \kappa$, if A is $\mathcal{E}$-potentially embeddable into B (if A is $\mathcal{E}$-potentially isomorphic to B and $|A| = |B|$) then A is embeddable into (isomorphic to) B.

Similarly let $\mathrm{PPE}_{\kappa,n}^{\mathcal{E}}$ be the axiom saying:

> For all structures A, B in some n-ary relational language L, if $\Vdash_P$ “there exists a substructure C of A of size κ which is embeddable into B” for some p.o.-set P with the property $\mathcal{E}$ then there exists a substructure C of A of size κ which is embeddable into B.

$\mathrm{MA}^{\mathcal{E}}_{\kappa,n}$ is the axiom:

For every p.o.-set P satisfying the condition $\mathcal{E}$, if D_α is a dense subset of P for $\alpha < \kappa$, then there exists an n-linked subset G of P such that $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$.

Lemma 5.1
(a) $\mathrm{PE}^{\mathrm{ccc}}_{\kappa,1}$, $\mathrm{PI}^{\mathrm{ccc}}_{\kappa,1}$ *and* $\mathrm{PPE}^{\mathrm{ccc}}_{\kappa,1}$ *hold in* ZFC *for any cardinal* κ.
(b) $\mathrm{PE}^{\omega_1\,\mathrm{preserving}}_{\aleph_1,1}$, $\mathrm{PI}^{\omega_1\,\mathrm{preserving}}_{\aleph_1,1}$, *and* $\mathrm{PPE}^{\omega_1\,\mathrm{preserving}}_{\aleph_1,1}$ *hold in* ZFC.

Proof: (a) Immediate from the fact that a generic extension with a ccc p.o.-set preserves every cardinal.

For $n \geq 2$, everything proved in Section 3 can be rewritten to form corresponding assertions for n-indexed axioms:

Lemma 5.2 *For a condition $\mathcal{E}$ on p.o.-sets, an infinite cardinal κ, and $n \geq 2$,* $\mathrm{MA}^{\mathcal{E}}_{\kappa,n}$ *implies* $\mathrm{PE}^{\mathcal{E}}_{\kappa,n}$, $\mathrm{PI}^{\mathcal{E}}_{\kappa,n}$, *and* $\mathrm{PPE}^{\mathcal{E}}_{\kappa,n}$.

Proof: Similar to Lemma 3.1.

Lemma 5.3 *For a regular condition $\mathcal{E}$ on p.o.-sets, an infinite cardinal κ and $n \geq 2$,* $\mathrm{MA}^{\mathcal{E}}_{\kappa,n}$ *is equivalent to* $\mathrm{PE}^{\mathcal{E}}_{\kappa,n}$.

Proof: Similar to Theorem 3.3.

Theorem 5.4
(a) *For $n \geq 2$,* $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1,n}$ *is equivalent to the assertion*:

($\mathcal{K}_n$) *For every* ccc *p.o.-set P, every uncountable subset X of P has an uncountable subset Y which is n-linked.*

(b) *Assume that every* ccc *p.o.-set is productively* ccc. *Then for any cardinal κ of uncountable cofinality,* $\mathrm{PPE}^{\mathrm{ccc}}_{\kappa,n}$ *is equivalent to the assertion*:

($\mathcal{K}_{\kappa,n}$) *For every* ccc *p.o.-set P, every subset X of P of size κ has a subset Y of size κ such that Y is n-linked.*

Proof: Similar to Theorem 3.6.

Lemma 5.5 *Let $\mathcal{E}$ be any condition on p.o.-sets and κ a cardinal. Then* (a) $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ *is equivalent to* $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ *and* (b) $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ *is equivalent to* $\mathrm{PI}^{\mathcal{E}}_{\kappa,2}$.

Proof:

(a) We shall prove that $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ implies $\mathrm{PE}^{\mathcal{E}}_{\kappa}$. Assume that $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ holds. Let A be a structure of the form $A = (A, g_i, R_j)_{i\in I, j\in J}$ where g_i is a k_i-ary function on A for $i \in I$ and R_j is a k_j-ary relation on A for $j \in J$. Let $\tilde{A}$ be the structure defined by

$$\tilde{A} = (A^{<\omega}, \emptyset, \tilde{g}_i, p_l, A^1, \tilde{R}_j)_{i\in I, j\in J, l\in\omega}$$

where

$$\tilde{g}_i((a_0,\ldots,a_{k-1})) = \begin{cases} (g_i(a_0,\ldots,a_{k-1})), & \text{if } k = k_i, \\ \emptyset, & \text{otherwise,} \end{cases}$$

for $i \in I$,

$$p_l((a_0, \ldots, a_{k-1})) = \begin{cases} (a_l), & \text{if } l < k, \\ \varnothing, & \text{otherwise,} \end{cases}$$

for $l \in \omega$ and

$$\tilde{R}_j = \{((a_0, \ldots, a_{k_j-1})) : (a_0, \ldots, a_{k_j-1}) \in A^{<\omega} \text{ and } R_j(a_0, \ldots, a_{k_j-1})\}$$

for $j \in J$. $\tilde{A}$ is a structure in a binary language. For any structures A and B we have that A is embeddable into B if and only if $\tilde{A}$ is embeddable into $\tilde{B}$. Now if $\Vdash_P$ "A is embeddable into B" then $\Vdash_P$ "$\tilde{A}$ is embeddable into $\tilde{B}$". By $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ it follows that $\tilde{A}$ is embeddable into $\tilde{B}$. Hence A is embeddable into B.

(b) is proved similarly.

By Theorem 3.3 and Lemma 5.3, we can also prove Lemma 5.5a without model theoretic arguments: It is easy to prove that $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ is equivalent to $\mathrm{MA}^{\mathcal{E}}_{\kappa,2}$ for arbitrary condition $\mathcal{E}$.

Let φ be the empty condition on p.o.-sets.

Problem 5.6 Is $\mathrm{PPE}^{\varphi}_{\aleph_1,2}$ *consistent?*

This problem is also connected with the problem of consistency of the axiom RFA considered in [9] and [8], since $\mathrm{PPE}^{\varphi}_{\aleph_1,n}$ implies RFA^n where RFA is equal to RFA^2. In [9] it is proved that RFA^n is inconsistent for every $n \geq 3$. From this we obtain:

Proposition 5.7 $\mathrm{PPE}^{\varphi}_{\aleph_1,n}$ *is inconsistent for all* $n \geq 3$. *In particular* $\mathrm{PPE}^{\varphi}_{\aleph_1}$ *is inconsistent.*

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