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A Certain Conception of the Calculus of Rough Sets*

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Abstract We consider the family of rough sets in the present paper. In this family we define, by means of a minimal upper sample, the operations of rough addition, rough multiplication, and pseudocomplement. We prove that the family of rough sets with the above operations is a complete atomic Stone algebra. We prove that the family of rough sets, determined by the unions of equivalence classes of the relation R with the operations of rough addition, rough multiplication, and complement, is a complete atomic Boolean algebra. If the relation R determines a partition of set U into one-element equivalence classes, then the family of rough sets with the above operations is a Boolean algebra that is isomorphic with a Boolean algebra of subsets of universum U.

1 Introduction The rough set concept was introduced by Pawlak [4], [5]. A certain generalization of his conception was offered by Iwiński [3]. Both formulations were then extended by Janusz and Jacek Pomykała [7]. Janusz Pomykała also proposed another definition of approximation space [6]. This definition was modified by Bryniarski [1], who also proposed a different formulation of rough set theory.

The aim of this paper is to prove some algebraic properties of rough sets and to show that the algebra of classes is a particular case of the algebra of rough sets.

2 Approximation space and approximations of set Let U be a finite non-empty set and let R be an equivalence relation in U. The set U is called the *universum* and the relation R is called the *indiscernibility relation*. We will call the pair $\mathfrak{A} = (U, R)$ the *approximation space*. As U is a finite set, the relation R determines a partition of U into a finite number of equivalence classes. The equiva-

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lence classes of the relation R will be called the *atoms of relation R* (the *elementary sets of relation R*). We assume that the empty set is also an atom of relation R. Every union of elementary sets of relation R will be called a *composed set*. It follows from this assumption that every composed set is a union of a finite number of elementary sets of relation R. We denote the family of all composed sets by Com$R$. If the relation R determines n equivalence classes, Com$R$ is composed of 2^n elements.

It is easily noticed that:

Theorem 2.1 *The algebra $\mathcal{B} = (\mathrm{Com}R, \cap, \cup, ', \varnothing, U)$ is the atomic, complete Boolean algebra. The inclusion relation $\subseteq$ is its natural order.*

Let us observe that the equivalence classes of relation R are atoms of the algebra $\mathcal{B}$. Every nonempty atom of relation R is an atom of $\mathcal{B}$.

Let X be any fixed subset of set U.

Definition 2.2

(a) The *lower approximation of set X* is the set

$$\underline{\mathrm{P}}X = \bigcup\{Y : Y \subseteq X \wedge Y \in U/R\}.$$

(b) The *upper approximation of set X* is the set

$$\bar{\mathrm{P}}X = \bigcup\{Y : X \cap Y \neq \varnothing \wedge Y \in U/R\}.$$

(c) The *boundary of set X* is the set

$$\mathrm{BN}(X) = \bar{\mathrm{P}}X \backslash \mathrm{P}X.$$

Definition 2.2 implies

Conclusion 2.3

(a) $\underline{\mathrm{P}}X = \bigcup\{Y : Y \subseteq X \wedge Y \in \mathrm{Com}R\}$.
(b) $\bar{\mathrm{P}}X = \bigcap\{Y : X \subseteq Y \wedge Y \in \mathrm{Com}R\}$.

Notice also that

Conclusion 2.4 *The following conditions are equivalent*:

(a) $X = \bar{\mathrm{P}}X$,
(b) $X = \underline{\mathrm{P}}X$,
(c) $X \in \mathrm{Com}R$.

One can easily prove the following

Theorem 2.5 *The operations of the lower approximation and the upper approximation have the following properties*:

W1	$\underline{\mathrm{P}}X \subseteq X \subseteq \bar{\mathrm{P}}X$	W2	$\underline{\mathrm{P}}U = \bar{\mathrm{P}}U = U$
W3	$\underline{\mathrm{P}}\varnothing = \bar{\mathrm{P}}\varnothing = \varnothing$	W4	$\bar{\mathrm{P}}(\bar{\mathrm{P}}X) = \underline{\mathrm{P}}(\bar{\mathrm{P}}X) = \bar{\mathrm{P}}X$
W5	$\underline{\mathrm{P}}(\underline{\mathrm{P}}X) = \bar{\mathrm{P}}(\underline{\mathrm{P}}X) = \underline{\mathrm{P}}X$	W6	$\bar{\mathrm{P}}(X \cup Y) = \bar{\mathrm{P}}X \cup \bar{\mathrm{P}}Y$
W7	$\underline{\mathrm{P}}(X \cap Y) = \underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y$	W8	$\bar{\mathrm{P}}X = (\underline{\mathrm{P}}(X'))'$
W9	$\underline{\mathrm{P}}X = (\bar{\mathrm{P}}(X'))'$	W10	$\bar{\mathrm{P}}(X \cap Y) \subseteq \bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y$
W11	$\underline{\mathrm{P}}X \cup \underline{\mathrm{P}}Y \subseteq \underline{\mathrm{P}}(X \cup Y)$	W12	$\bar{\mathrm{P}}X \backslash \bar{\mathrm{P}}Y \subseteq \bar{\mathrm{P}}(X \backslash Y)$
W13	$\underline{\mathrm{P}}(X \backslash Y) \subseteq \underline{\mathrm{P}}X \backslash \underline{\mathrm{P}}Y$	W14	$X \subseteq Y \Rightarrow \bar{\mathrm{P}}X \subseteq \bar{\mathrm{P}}Y$
W15	$X \subseteq Y \Rightarrow \underline{\mathrm{P}}X \subseteq \underline{\mathrm{P}}Y$.		

Theorem 2.6 *If* $X \in \mathrm{Com}R$ *or* $Y \in \mathrm{Com}R$ *then*
(a) $\underline{\mathrm{P}}(X \cup Y) = \underline{\mathrm{P}}X \cup \underline{\mathrm{P}}Y$,
(b) $\bar{\mathrm{P}}(X \cap Y) = \bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y$.

Proof: (a) Let us assume, without loss of generality, that $X \in \mathrm{Com}R$. Let us denote the lower approximation of set $X \cup Y$ by W. It follows from Conclusion 2.3a that $X \subseteq W$. Hence $W = X \cup (W \backslash X)$. Since $W \subseteq X \cup Y$, so $W \backslash X \subseteq Y$ and $W \backslash X \in \mathrm{Com}R$. This and Conclusion 2.3a imply that $W \backslash X \subseteq \underline{\mathrm{P}}Y$. It follows from this that $W \subseteq X \cup \underline{\mathrm{P}}Y$, and, hence $W = \underline{\mathrm{P}}(X \cup Y) \subseteq \underline{\mathrm{P}}X \cup \underline{\mathrm{P}}Y$. The inverse inclusion is satisfied by property W11 of Theorem 2.5.

(b) Let $X \in \mathrm{Com}R$. Let W denote the upper approximation of set $X \cap Y$. This and Conclusion 2.3b imply that $W \subseteq X$. Hence $W = X \cap (X' \cup W)$. Since $X' \cup W \in \mathrm{Com}R$ and $Y \subseteq X' \cup W$, $\bar{\mathrm{P}}Y \subseteq X' \cup W$. Hence $X \cap \bar{\mathrm{P}}Y \subseteq X \cap (X' \cup W) = W$. This and Conclusion 2.4 imply that $\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y \subseteq \bar{\mathrm{P}}(X \cap Y)$. The inverse inclusion holds by property W10 of Theorem 2.5.

3 *Sample of a set*

Let $\mathfrak{A} = (U, R)$ be an approximation space.

Definition 3.1
(a) The set Y is called the *lower sample of set* X iff $Y \subseteq X$ and $\bar{\mathrm{P}}Y = \underline{\mathrm{P}}X$.
(b) The set Y is called the *upper sample of set* X iff $Y \subseteq X$ and $\bar{\mathrm{P}}Y = \bar{\mathrm{P}}X$.

Definition 3.2
(a) The set Y is called the *minimal lower sample of set* X ($Y = \mathrm{mls}(X)$) iff Y is the lower sample of set X and there is no lower sample Z of set X such that $|Z| < |Y|$.
(b) The set Y is called the *minimal upper sample of set* X ($Y = \mathrm{mus}(X)$) iff Y is the upper sample of set X and there is no upper sample Z of set X such that $|Z| < |Y|$.

One can easily prove the following.

Conclusion 3.3

$$X \in \mathrm{Com}R \Rightarrow (Y = \mathrm{mls}(X) \Leftrightarrow Y = \mathrm{mus}(X)).$$

Theorem 3.4 *Let X be any subset of U.*
(a) *Every nonempty lower* (*upper*) *sample of set X has a nonempty intersection with every nonempty elementary set of relation R from* $\underline{\mathrm{P}}X$ ($\bar{\mathrm{P}}X$).
(b) *Every nonempty minimal lower* (*upper*) *sample of set X has exactly one common element with every nonempty elementary set of relation R from* $\underline{\mathrm{P}}X$ ($\bar{\mathrm{P}}X$).

Proof: (a) If $\underline{\mathrm{P}}X = \varnothing$, then there exists exactly one lower sample of set X, namely the empty set. If $\underline{\mathrm{P}}X \neq \varnothing$ then $\underline{\mathrm{P}}X$ is a union of a finite number of elementary sets of relation R. Let Y be a nonempty lower sample of set X. To generate a contradiction, let us assume that there exists a nonempty elementary set $Z \subseteq \underline{\mathrm{P}}X$, such that $Z \cap Y = \varnothing$. So Z is not a subset of $\bar{\mathrm{P}}Y$. Hence $\bar{\mathrm{P}}Y \neq \underline{\mathrm{P}}X$, which contradicts the assumption that Y is the lower sample of set X. The truthfulness of the theorem for an upper sample of set X is shown analogously.

(b) Let Y be a nonempty minimal lower sample of set X. Y has a nonempty intersection with every nonempty elementary set from $\underline{\mathrm{P}}X$ because Y is a nonempty lower sample of set X. To generate a contradiction, let us assume that there exists an elementary set $Z \subseteq \underline{P}X$ such that $|Y \cap Z| > 1$. Let x and y be different elements of set $Y \cap Z$. Hence set $Y \setminus \{y\}$ is also the lower sample of set X and $|Y \setminus \{y\}| < |Y|$. This contradicts the assumption that Y is the minimal lower sample of set X. That the theorem is true for the minimal upper sample of set X is proved analogously.

Theorem 3.5 *Let X, Y be any subset of U.*
(a) *If P is the minimal upper (lower) sample of set $\bar{P}X \cap \bar{P}Y$ then $\underline{P}P \subseteq \underline{P}X \cap \underline{P}Y$.*
(b) *If P is the minimal upper (lower) sample of set $\bar{P}X \cup \bar{P}Y$ then $\underline{P}P \subseteq \underline{P}X \cup \underline{P}Y$.*

Proof: (a) Let P be a minimal upper sample of set $\bar{P}X \cap \bar{P}Y$. If $\bar{P}X \cap \bar{P}Y = \varnothing$ then $P = \varnothing$ and $\underline{P}P \subseteq \underline{P}X \cap \underline{P}Y$. Let us suppose then that $\bar{P}X \cap \bar{P}Y \neq \varnothing$. Since $\bar{P}X \cap \bar{P}Y \in \mathrm{Com}R$, then from Conclusion 2.4 we have $\bar{P}X \cap \bar{P}Y = \bar{P}(\bar{P}X \cap \bar{P}Y)) \neq \varnothing$. Hence $P \neq \varnothing$. Definition 2.2b and the fact that P is the upper sample of set $\bar{P}X \cap \bar{P}Y$ imply that $P \subseteq \bar{P}X \cap \bar{P}Y$. From this and property W1 of Theorem 2.5 we have $\underline{P}P \subseteq \bar{P}X \cap \bar{P}Y$. Since $\bar{P}X \cap \bar{P}Y$ is the nonempty composed set, so $\bar{P}X \cap \bar{P}Y = \bigcup_{i=1}^{k} A_i$, where A_i are nonempty elementary sets of relation R. If $|A_i| > 1$ for every $1 \leq i \leq k$, then the assumption of the theorem, Theorem 3.4b and Definition 2.2a imply that $\underline{P}P = \varnothing$. Hence $\underline{P}P \subseteq \underline{P}X \cap \underline{P}Y$. Now suppose that there exists at least one one-element elementary set of relation R included in $\bar{P}X \cap \bar{P}Y$. From Theorem 3.4b it follows that P has exactly one common element with every set A_i, $1 \leq i \leq k$. We obtain from this and Definition 2.2a that $\varnothing \neq \underline{P}P = \{x \in P : \exists i \in \{1, \ldots, k\} \{x\} = A_i\}$. Let $x \in \underline{P}P$; then $x \in \bar{P}X \cap \bar{P}Y$. Because $x \in \bar{P}X$ and $\{x\}$ is the elementary set of relation R, it follows from Definition 2.2b that $x \in X$. This, Definition 2.2a, and the fact that $\{x\} \in U/R$ imply that $x \in \underline{P}X$. It is shown analogously that $x \in \underline{P}Y$. So $x \in \underline{P}X \cap \underline{P}Y$. It follows from this that $\underline{P}P \subseteq \underline{P}X \cap \underline{P}Y$.

If P is the minimal lower sample of set $\bar{P}X \cap \bar{P}Y$, then Conclusion 3.3 implies that P is the minimal upper sample of set $\bar{P}X \cap \bar{P}Y$. Hence $\underline{P}P \subseteq \underline{P}X \cap \underline{P}Y$.

Likewise one can prove that $\underline{P}P \subseteq \underline{P}X \cup \underline{P}Y$, where P is a minimal upper (lower) sample of set $\bar{P}X \cup \bar{P}Y$.

4 Rough sets, relations and operations Let $\mathfrak{A} = (U, R)$ be an approximation space and let X, Y, Z be subsets of U. Let us define the rough inclusion relation and the rough equality relation.

Definition 4.1 The set X is *roughly included in* Y ($X \subseteq_R Y$) iff $\underline{P}X \subseteq \underline{P}Y$ and $\bar{P}X \subseteq \bar{P}Y$.

Definition 4.2 The sets X, Y are *roughly equal* ($X \approx Y$) iff $\underline{P}X = \underline{P}Y$ and $\bar{P}X = \bar{P}Y$.

These definitions of the inclusion and equality relations imply the following conclusions:

Conclusion 4.3 *The rough inclusion relation ($\subseteq_R$) is a quasi-ordering relation in P(U).*

Conclusion 4.4 *The rough equality relation ($\approx$) is an equivalence relation in $P(U)$.*

From the assumption that set U is finite, it follows that relation R determines a partition of U into a finite number of equivalence classes. Let us suppose that relation R determines a partition of U into n equivalence classes. Then the rough equality relation determines as many equivalence classes as there are ordered pairs of composed sets (X, Y) such that $X \subseteq Y$. One can show easily that there exist 3^n equivalence classes of relation $\subseteq_R$.

Since $P(U)$ is a nonempty finite set and the rough equality relation is an equivalence relation in $P(U)$, we can define the following approximation space:

Definition 4.5 The approximation space $\mathfrak{A}^* = (P(U), \approx)$ is called an *extension of space* $\mathfrak{A} = (U, R)$.

Definition 4.6 Equivalence classes of the rough equality relation are called *rough sets in* $\mathfrak{A}^*$.

Let us introduce inclusion of rough sets.

Definition 4.7 $[X]_\approx \leq_\approx [Y]_\approx \Leftrightarrow X \subseteq_R Y$.

The above definition and Conclusion 4.1 imply

Theorem 4.8 *The relation $\leq_\approx$ is a partial ordering in $P(U)/\approx$.*

From Theorem 4.8 there follows:

Conclusion 4.9 *The ordered pair $(P(U)/\approx, \leq_\approx)$ is a partially ordered set.*

It follows from the above conclusion that there can exist the least element and the greatest element. It turns out that $[U]_\approx = \{U\}$ is the greatest element and $[\varnothing]_\approx = \{\varnothing\}$ is the least element. This follows from properties W14 and W15 of Theorem 2.5 and the fact that $\varnothing \subseteq X \subseteq U$ holds for every subset X of set U.

Let us notice that a rough set determined by a composed set is a one-element set composed only of this composed set. Therefore rough sets determined by composed sets will be called *exact sets*.

Let us take two rough sets $[X]_\approx$ and $[Y]_\approx$. We will investigate whether the set $\{[X]_\approx, [Y]_\approx\}$ has an infimum and a supremum.

Let us denote a fixed minimal upper sample of the set $\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y$ by P. We have from Definition 3.1b

$$\bar{\mathrm{P}}P = \bar{\mathrm{P}}(\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y) = \bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y.$$

Let $Z = \underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \cup P$. We shall prove that $[Z]_\approx$ is the infimum of set $\{[X]_\approx, [Y]_\approx\}$. Since $\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \in \mathrm{Com}R$, so by Theorems 2.6 and 3.5 we have:

$$\underline{\mathrm{P}}Z = \underline{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y) \cup \underline{\mathrm{P}}P = \underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y.$$

Hence $\underline{\mathrm{P}}Z \subseteq \underline{\mathrm{P}}X$ and $\underline{\mathrm{P}}Z \subseteq \underline{\mathrm{P}}Y$. From property W6 of Theorem 2.5 and the assumption it follows that

$$\bar{\mathrm{P}}Z = \bar{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y) \cup \bar{\mathrm{P}}P = \underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \cup \bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y = \bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y.$$

Hence $\bar{\mathrm{P}}Z \subseteq \bar{\mathrm{P}}X$ and $\bar{\mathrm{P}}Z \subseteq \bar{\mathrm{P}}Y$.

The above remarks and Definition 4.1 imply that $Z \subseteq_R X$ and $Z \subseteq_R Y$. This and Definition 4.7 imply that $[Z]_{\approx} \leq_{\approx} [X]_{\approx}$ and $[Z]_{\approx} \leq_{\approx} [Y]_{\approx}$.

Now let us suppose that $[W]_{\approx} \leq_{\approx} [X]_{\approx}$ and $[W]_{\approx} \leq_{\approx} [Y]_{\approx}$. From Definition 4.7 and Definition 4.1 we obtain

$$\underline{\mathrm{P}}W \subseteq \underline{\mathrm{P}}X \text{ and } \underline{\mathrm{P}}W \subseteq \underline{\mathrm{P}}Y \text{ and also } \bar{\mathrm{P}}W \subseteq \bar{\mathrm{P}}X \text{ and } \bar{\mathrm{P}}W \subseteq \bar{\mathrm{P}}Y.$$

The above inclusions, Theorems 2.6, 2.5, and 3.5 imply

$$\begin{aligned} \underline{\mathrm{P}}W \subseteq \underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y = \underline{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y) &= \underline{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y) \cup \underline{\mathrm{P}}P \\ &= \underline{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \cup P) = \underline{\mathrm{P}}Z, \end{aligned}$$

$$\bar{\mathrm{P}}W \subseteq \bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y = \bar{\mathrm{P}}P \subseteq \bar{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y) \cup \bar{\mathrm{P}}P = \bar{\mathrm{P}}(\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \cup P) = \bar{\mathrm{P}}Z,$$

where P is the earlier fixed minimal upper sample of the set $\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y$. Hence $W \subseteq_R Z$ which implies that $[W]_{\approx} \leq_{\approx} [Z]_{\approx}$. We have proved in this way that $[Z]_{\approx}$ is the infimum of set $\{[X]_{\approx}, [Y]_{\approx}\}$.

Now let us denote the minimal upper sample of the set $\mathrm{P}X \cup \bar{\mathrm{P}}Y$ by P. Definition 3.2b implies that

$$\bar{\mathrm{P}}P = \bar{\mathrm{P}}(\bar{\mathrm{P}}X \cup \bar{\mathrm{P}}Y) = \bar{\mathrm{P}}X \cup \bar{\mathrm{P}}Y.$$

Let $Z = \underline{\mathrm{P}}X \cup \underline{\mathrm{P}}Y \cup P$. One can prove, analogously to the above, that $[Z]_{\approx}$ is the supremum of the set $\{[X]_{\approx}, [Y]_{\approx}\}$.

We have shown in this way that every pair of rough sets possesses an infimum and a supremum. Therefore we can define an operation **u** of rough addition and operation **n** of rough multiplication in $P(U)/\approx$, namely

$$[X]_{\approx}\ \mathbf{u}\ [Y]_{\approx} = \sup(\{[X]_{\approx}, [Y]_{\approx}\}),$$

$$[X]_{\approx}\ \mathbf{n}\ [Y]_{\approx} = \inf(\{[X]_{\approx}, [Y]_{\approx}\}),$$

for any $[X]_{\approx}$ and $[Y]_{\approx}$.

Definition 4.10 Let X, Y be any subset of U.

(a) $[X]_{\approx}\ \mathbf{u}\ [Y]_{\approx} = [\underline{\mathrm{P}}X \cup \underline{\mathrm{P}}Y \cup P]_{\approx}$, where P is a minimal upper sample of set $\bar{\mathrm{P}}X \cup \bar{\mathrm{P}}Y$.

(b) $[X]_{\approx}\ \mathbf{n}\ [Y]_{\approx} = [\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \cup P]_{\approx}$, where P is a minimal upper sample of set $\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y$.

From the above remarks there follows immediately:

Lemma 4.11 *The algebra $P^*(U) = (P(U)/\approx, \mathbf{n}, \mathbf{u})$ is the distributive lattice with identity element* $([\varnothing]_{\approx})$ *and unit element* $([U]_{\approx})$.

5 The Stone algebra of rough sets Let $\mathfrak{A} = (U, R)$ be an approximation space and $P(U)/\approx$ be a family of rough sets.

Lemma 5.1 *The lattice $P^*(U) = (P(U)/\approx, \mathbf{n}, \mathbf{u})$ is the complete atomic lattice where atoms are determined by proper subsets of elementary sets of relation R or by one-element elementary sets of relation R.*

Proof: $P^*(U)$ is the complete atomic lattice because $P^*(U)$ is the finite lattice (see Grätzer [2]).

Let us remember that the expression "elementary set of relation R" is equivalent to the expression "equivalence class of relation R". To show that sets $[X]_{\approx}$ are atoms in $P^{*}(U)$, where X is a one-element elementary set of relation R or proper subset of an elementary set of relation R, it is sufficient to prove that:

a. these sets are atoms in $P^{*}(U)$,
b. only these sets are atoms in $P^{*}(U)$, namely if $[X]_{\approx}$ is an atom in $P^{*}(U)$, then X is a one-element elementary set of relation R or X is a proper subset of a certain elementary set of relation R.

Consider a rough set $[X]_{\approx}$, where X is a proper subset of a certain elementary set Z. Of course $\bar{\mathrm{P}}X = \varnothing$ and $\bar{\mathrm{P}}X = Z$. We notice also that $[X]_{\approx}$ is a family of all proper subsets of elementary set Z. Let us assume that $[X]_{\approx}$ is not an atom. Then there exists rough set $[W]_{\approx}$ such that

$$[W]_{\approx} \neq [\varnothing]_{\approx} \quad \text{and} \tag{1}$$

$$[W]_{\approx} \neq [X]_{\approx} \quad \text{and} \tag{2}$$

$$[W]_{\approx} \leq_{\approx} [X]_{\approx}. \tag{3}$$

From (1) and (2) we have respectively

$$\underline{\mathrm{P}}W \neq \underline{\mathrm{P}}\varnothing = \varnothing \vee \bar{\mathrm{P}}W \neq \bar{\mathrm{P}}\varnothing = \varnothing, \tag{4}$$

$$\underline{\mathrm{P}}W \neq \underline{\mathrm{P}}X = \varnothing \vee \bar{\mathrm{P}}W \neq \bar{\mathrm{P}}X = Z. \tag{5}$$

From (3) we obtain

$$\underline{\mathrm{P}}W \subseteq \underline{\mathrm{P}}X = \varnothing \quad \text{and} \tag{6}$$

$$\bar{\mathrm{P}}W \subseteq \bar{\mathrm{P}}X = Z. \tag{7}$$

(6) implies that $\underline{\mathrm{P}}W = \varnothing$. From this and (4) we have $\bar{\mathrm{P}}W \neq \varnothing$. Since Z is an elementary set, (7) implies that $\bar{\mathrm{P}}W = Z$. From this and (5) we have $\underline{\mathrm{P}}W \neq \varnothing$. We have obtained a contradiction, and in this way we have proved that $[X]_{\approx}$ is an atom in $P^{*}(U)$.

Let us take rough set $[X]_{\approx}$, where X is a one-element elementary set. Hence there is no set $Y \subseteq X$ such that $\underline{\mathrm{P}}Y = \varnothing$ and $\bar{\mathrm{P}}Y = X$. Of course $\underline{\mathrm{P}}X = \bar{\mathrm{P}}X = X$. Let us assume that $[X]_{\approx}$ is not an atom. Therefore there exists rough set $[W]_{\approx}$ satisfying the properties (1)–(3). (1) and (3) imply, respectively, that

$$\underline{\mathrm{P}}W \neq \varnothing \vee \bar{\mathrm{P}}W \neq \varnothing, \tag{8}$$

$$\underline{\mathrm{P}}W \neq X \vee \bar{\mathrm{P}}W \neq X. \tag{9}$$

From (3) we obtain

$$\underline{\mathrm{P}}W \subseteq X. \tag{10}$$

$$\bar{\mathrm{P}}W \subseteq X. \tag{11}$$

Since X is a one-element set, either $\underline{\mathrm{P}}W = \varnothing$ or $\underline{\mathrm{P}}W = X$. $\underline{\mathrm{P}}W = \varnothing$ implies by (8) that $\bar{\mathrm{P}}W \neq \varnothing$. Then by (11), $\bar{\mathrm{P}}W = X$. We have obtained a contradiction, because there is no set $W \subseteq X$ such that $\underline{\mathrm{P}}W = \varnothing$ and $\bar{\mathrm{P}}W = X$. $\underline{\mathrm{P}}W = X$ implies by (9) that $\bar{\mathrm{P}}W \neq X$. Hence by (11) $\bar{\mathrm{P}}W = \varnothing$. We have obtained a contradiction, because an upper approximation of a set cannot be a proper subset of a lower approximation of that set. In both cases we have obtained a contradiction. Hence $[X]_{\approx}$ is an atom in $P^*(U)$.

We will prove now that condition (b) holds. Let $[X]_{\approx}$ be a rough set such that X is not a one-element elementary set and X is not a proper subset of any elementary set of relation R. Two cases are possible: either X is an elementary set of relation R (where, of course, $|X| > 1$) or X has a nonempty intersection with at least two elementary sets.

Let us suppose that X is an elementary set and $|X| > 1$. Of course $\underline{P}X = \bar{P}X = X$. Let Y be a proper subset of X. Hence $\underline{P}Y = \emptyset$ and $\bar{P}Y = X$. Therefore inclusions $\underline{P}Y \subseteq \underline{P}X$ and $\bar{P}Y \subseteq \bar{P}X$ are true. By Definition 4.7, $[Y]_{\approx} \leq_{\approx} [X]_{\approx}$. Because also $[Y]_{\approx} \neq \emptyset$ and $[Y]_{\approx} \neq [X]_{\approx}$, then $[X]_{\approx}$ cannot be an atom in $P^*(U)$.

Let X therefore have a nonempty intersection with at least two elementary sets. Let Z be such set. Now let us suppose that $|Z| > 1$. Let us denote a proper subset of Z included in X by W. Of course $\underline{P}W = \emptyset \subseteq \underline{P}X$. Since $\bar{P}W = Z$ is the proper subset of $\bar{P}X$, $[W]_{\approx} \leq_{\approx} [X]_{\approx}$ and $[W]_{\approx} \neq [X]_{\approx}$. Because also $[W]_{\approx} \neq [\emptyset]_{\approx}$, $[X]_{\approx}$ cannot be an atom in $P^*(U)$. If Z is a one-element elementary set, then $Z = \underline{P}Z \subseteq \underline{P}X$ and $Z = \bar{P}Z \subseteq \bar{P}X$. Hence $[Z]_{\approx} \leq_{\approx} [X]_{\approx}$. Since $\underline{P}Z \neq \emptyset$, $[Z]_{\approx} \neq [\emptyset]_{\approx}$. And $[Z]_{\approx} \neq [X]_{\approx}$, because $\bar{P}Z \neq \bar{P}X$. It follows from this that $[X]_{\approx}$ cannot be an atom in $P^*(U)$.

Since $P^*(U)$ is a distributive complete lattice, its every element possesses a pseudocomplement. One can easily prove that the rough set $[U\backslash\bar{P}X]_{\approx}$ is the greatest rough set such that $[U\backslash\bar{P}X]_{\approx}\,\mathbf{n}\,[X]_{\approx} = [\emptyset]_{\approx}$. Hence we have the following result.

Theorem 5.2 *A rough set* $[X]^*_{\approx} = [U\backslash\bar{P}X]_{\approx}$ *is a pseudocomplement of set* $[X]_{\approx}$.

We will now prove the following theorem.

Theorem 5.3 *The algebra* $P^*(U) = (P(U)/{\approx}, \mathbf{n}, \mathbf{u}, {}^*, [\emptyset]_{\approx}, [U]_{\approx})$ *is the complete atomic Stone algebra.*

Proof: Let us recall that algebra $\mathcal{B} = (V, \wedge, \vee, {}^*, 0, 1)$ is called the *Stone algebra* only if the following conditions hold (see [2]):

(i) $\mathcal{B} = (V, \wedge, \vee)$ is the distributive lattice with the identity element 0 and the unit element 1,
(ii) for any element $x \in V$ element x^* is its pseudo-complement,
(iii) the Stone identity $(x^* \vee x^{**} = 1)$ holds for every element $x \in V$.

The truth of condition (i) follows from Lemma 4.11. Condition (ii) holds by Theorem 5.2. Let $[X]_{\approx}$ be any rough set. Hence

$$[X]^*_{\approx} = [U\backslash\bar{P}X]_{\approx} \quad \text{and}$$

$$[X]^{**}_{\approx} = [U\backslash\bar{P}(U\backslash\bar{P}X)]_{\approx} = [U\backslash(U\backslash\bar{P}X)]_{\approx} = [\bar{P}X]_{\approx}.$$

This implies that

$$[X]^*_{\approx}\ \mathbf{u}\ [X]^{**}_{\approx} = [P(U\backslash\bar{P}X) \cup \underline{P}(\bar{P}X) \cup P]_{\approx},$$

$$\text{where } P = \mathrm{mus}(\bar{P}(U\backslash\bar{P}X) \cup \bar{P}(\bar{P}X)).$$

Hence we obtain

$$[X]^*_{\approx} \mathbf{u} [X]^{**}_{\approx} = [(U \backslash \bar{\mathrm{P}}X) \cup \bar{\mathrm{P}}X \cup P]_{\approx} = [U \cup P]_{\approx} = [U]_{\approx}.$$

This means that the Stone identity holds. Hence $P^*(U)$ is the Stone algebra. Lemma 5.1 implies that $P^*(U)$ is complete and atomic.

6 *The Boolean algebra of exact sets* Let $\mathfrak{A} = (U, R)$ be an approximation space and $P(U)/\approx$ be a family of rough sets. Let us denote the family of exact sets by E. Let $[X]_{\approx}, [Y]_{\approx} \in E$. Hence $X, Y \in \mathrm{Com}R$. Of course, $\underline{\mathrm{P}}X = \bar{\mathrm{P}}X = X$ and $\underline{\mathrm{P}}Y = \bar{\mathrm{P}}Y = Y$. Hence

$$[X]_{\approx} \mathbf{u} [Y]_{\approx} = [\underline{\mathrm{P}}X \cup \underline{\mathrm{P}}Y \cup P]_{\approx}, \quad \text{where } P = \mathrm{mus}(\bar{\mathrm{P}}X \cup \bar{\mathrm{P}}Y) \text{ and}$$

$$[X]_{\approx} \mathbf{n} [Y]_{\approx} = [\underline{\mathrm{P}}X \cap \underline{\mathrm{P}}Y \cup S]_{\approx}, \quad \text{where } S = \mathrm{mus}(\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y).$$

Because $\bar{\mathrm{P}}X \cup \bar{\mathrm{P}}Y = X \cup Y$ and $\bar{\mathrm{P}}X \cap \bar{\mathrm{P}}Y = X \cap Y$, $P \subseteq X \cup Y$ and $S \subseteq X \cap Y$. Hence

$$[X]_{\approx} \mathbf{u} [Y]_{\approx} = [X \cup Y]_{\approx} \in E \quad \text{and} \quad [X]_{\approx} \mathbf{n} [Y]_{\approx} = [X \cap Y]_{\approx} \in E.$$

This means that the algebra $\mathcal{E} = (E, \mathbf{n}, \mathbf{u})$ is the sublattice of lattice $P^*(U)$. This, together with Lemmas 4.11 and 5.1, implies the following.

Lemma 6.1 *The algebra $\mathcal{E} = (E, \mathbf{n}, \mathbf{u})$ is the complete, atomic, distributive lattice with identity element $[\varnothing]_{\approx}$ and unit element $[U]_{\approx}$. The atoms are determined by the equivalence classes of relation R.*

One can easily prove that in the family of exact sets the following holds.

Lemma 6.2 *An exact set $[X]'_{\approx} = [U \backslash X]_{\approx}$ is a complement of the exact set $[X]_{\approx}$.*

Let us notice that a pseudocomplement of an exact set is also its complement ($[X]^*_{\approx} = [X]'_{\approx}$, if $X \in \mathrm{Com}R$). From the above lemmas there follows immediately:

Theorem 6.3 *The algebra $\mathcal{E} = (E, \mathbf{n}, \mathbf{u}, ', [\varnothing]_{\approx}, [U]_{\approx})$ is the complete, atomic Boolean algebra.*

Theorem 6.4 *The Boolean algebras $\mathcal{B}$ and $\mathcal{E}$ are isomorphic.*

Proof: Let $\psi : \mathcal{B} \to \mathcal{E}$ be a function such that $\psi(X) = [X]_{\approx}$, $X \in \mathrm{Com}R$. One can easily prove that the function ψ is an isomorphism.

Let us consider now a particular case, when the relation R determines the partition of set U into one-element equivalence classes. Then the family $\mathrm{Com}R$ of composed sets and the family $P(U)$ of subsets of set U are equal. It follows from this that the family $P(U)/\approx$ of rough sets and the family E of exact sets are equal. One can easily prove that the family of rough sets together with the operations of rough addition, rough multiplication, and complement is the Boolean algebra, which is isomorphic with the Boolean algebra of subsets of universum U. This implies that the rough sets algebra can be treated as an extension of the algebra of classes.

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