On a Quasi-Set Theory

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Abstract The main features of a theory that enables us to deal, in terms of a set theory, with collections of indistinguishable objects are presented. The fundamental idea is to restrict the concept of "identity" in the underlying logical apparatus. The basic entities of the theory are *Urelemente* of two sorts; to those called *m*-atoms, the usual concept of identity, in a precise sense, does not apply, but there exists a primitive equivalence relation called "the indistinguishability relation" that holds among them. The other sort of atoms (*M*-atoms) are treated as *Urelemente stricto sensu*. The underlying logic is a kind of "nonreflexive logic" and reflects formally this situation. The intuitive motivation is twofold: seeking agreement with Schrödinger's dictum that "identity" lacks sense with respect to the elementary particles of modern physics, and building Weyl's "effective aggregates" "directly", that is, dealing *ab initio* with indistinguishable objects; hence, their collection must not be considered a "set". Despite these motivations, in this paper quasi-set theory is delineated as a set theory, independently of its possible applications to other domains.

1 The intuitive idea of a quasi-set   To understand intuitively what we mean by a *quasi-set* (*qset* for short), the reader may think of a *classical* set with atoms (in the sense of Zermelo–Fraenkel with *Urelemente* – ZFU). Suppose now that the atoms are of two sorts. In the first category we have the *M-atoms*, which can be thought of as the *macroscopic* objects of our environment. They will be treated as *Urelemente* of ZFU *stricto sensu*; hence, we will admit that classical logic is valid with respect to them in all its aspects. The atoms of the other kind (*m-atoms*) may be intuitively thought of as elementary particles of modern physics, and we will suppose, following Schrödinger's ideas, that *identity* is meaningless with respect to them ([10], pp. 16–18).[1] Then we will admit that the Traditional Theory of Identity (TTI) does not apply to the *m*-atoms. These facts enable us to hold, with regard to the *m*-atoms, that the concepts of *indistinguishability* and *identity* may not be equivalent.[2] Therefore, roughly speaking we can say that a *qset* is a collection of objects (called *elements*) such that to the elements

of one of the species (the *m*-atoms), the notion of identity (ascribed by classical logic and mathematics) lacks sense.

This intuitive sketch is in a sense supported by certain philosophical views concerning the notions of *identity, indistinguishability,* and *individuality* in classical and quantum physics (see French [5], French and Redhead [6], and [10]). We will not discuss these philosophical questions here (but see Krause [8]); in this paper, we will study the quasi-set theory as a *set-theory,* independently of its philosophical motivations or its (possible) applications to other domains.

In order to provide a motivation, we note only that in some domains of knowledge, such as quantum mechanics, chemistry, biology, or genetics (cf. Weyl [12], App. B), it is necessary to consider collections of entities that are capable of being in certain *states,* but such that it is impossible to say *what* elements belong to each particular state. Only the quantity of elements in each state may be known. Weyl called such collections *effective aggregates* of individuals ([12], p. 239). The idea is that it is not possible to distinguish among the elements that belong to the same state of an effective aggregate. It is important to note that such aggregates cannot be considered *sets* in the usual sense (ZFU, say), since in a set the elements are always distinguishable.[3] This point was recently observed by Dalla Chiara and Toraldo di Francia [3].[4] In this paper, we present the main features of a theory of quasi-sets which intends to provide adequate mathematical tools for dealing with effective aggregates (in Weyl's sense) *directly,* that is, without using the subterfuge of *distinguishing first* (that is, considering their collection as a *set*), *and then abstracting the distinction previously made* (by the underlying mathematical apparatus), keeping only the *quantity* of the elements in each particular state.

2 The quasi-set theory   We will denote the *quasi-set theory* by S^*; the language of S^* has the following primitive symbols: (i) connectives: $\neg$ (not), $\vee$ (or); the symbols $\wedge$ (and), $\Rightarrow$ (implication), and $\Leftrightarrow$ (equivalence) are introduced as usual; (ii) Universal quantifier $\forall$ (for all) and the existential quantifier $\exists$ (there exists) are defined as usual; (iii) Three unary predicate symbols: m (*m*-object), $\mathcal{M}$ (*M*-object), and Z (set), and two binary predicate symbols: $\in$ (membership) and $\equiv$ (indistinguishability); (iv) a unary functional symbol *qcard* (quasi-cardinality); (v) Parenthesis and comma; (vi) Individual variables: a denumerably infinite collection of variables.

The concepts of term and formula, of bound variable, of closed formula, etc. are defined as usual. We observe that if x is a variable, $m(x)$, $\mathcal{M}(x)$, and $Z(x)$ may be read "*x is an m-object*", "*x is an M-object*", and "*x is a set*", respectively. Then, if x is a variable, the term $qcard(x)$ means "*the quasi-cardinality of the qset x*". We will use the following abbreviations: $\forall_\Box x(\ldots)$ and $\exists_\Box x(\ldots)$ for $\forall x(\Box(x) \Rightarrow (\ldots))$ and $\exists x(\Box(x) \wedge (\ldots))$ respectively, where $\Box$ stands for a predicate of the language or some of the ones defined below.

The Postulates (axiom schemata and inference rules) for Propositional and Predicate levels are the standard ones, based on the primitive connectives we considered (as for instance, those of Hilbert and Ackermann [7]). The only difference from the systems of classical first order logic is that instead of the Axioms of Identity, we have the *Axioms of Indistinguishability*:

$(\equiv_1)$ $\forall x(x \equiv x)$
$(\equiv_2)$ $\forall x \forall y(x \equiv y \rightarrow y \equiv x)$
$(\equiv_3)$ $\forall x \forall y \forall z(x \equiv y \wedge y \equiv z \Rightarrow x \equiv z)$
$(\equiv_4)$ $\forall x \forall y(\neg m(x) \wedge \neg m(y) \Rightarrow (x \equiv y \Rightarrow (A(x,x) \Rightarrow A(x,y))))$

where $A(x,x)$ is a formula and $A(x,y)$ arises from $A(x,x)$ by replacing some, but not necessarily all, free occurrences of x by y, provided that y is free from x in $A(x,x)$.

Definition 1
(a) "quasi-set": $Q(x) =_{df} \neg (m(x) \vee \mathcal{M}(x))$
(b) "identity": $x = y =_{df} \neg m(x) \wedge \neg m(y) \wedge x \equiv y$
(c) "pure qset": $\mathcal{P}(x) =_{df} Q(x) \wedge \forall y(y \in x \Rightarrow m(y))$
(d) $\mathcal{D}(x) =_{df} \mathcal{M}(x) \vee Z(x)$
(e) $\mathcal{E}(x) =_{df} Q(x) \wedge \forall y(y \in x \Rightarrow Q(y))$.

According to the above definition, nothing is at the same time a qset and an atom. The identity relation is valid only to those entities which are not m-atoms and, in this case, indistinguishability and identity coincide, as we will prove below.

Theorem 1   *The defined relation = has all the usual properties of classical equality.*

Proof: By Definition 1(b) the defined identity may be applied only to those entities that are not m-atoms. In this case, the substitutivity principle (Axiom $(\equiv_4)$) is valid; then, Axiom $(\equiv_1)$ and the Schema $(\equiv_4)$ are exactly the corresponding Axioms of Identity (Mendelson [9], p. 74).

Theorem 2 $\forall x \forall y(\neg m(x) \wedge \neg m(y) \Rightarrow (x \equiv y \Leftrightarrow x = y))$.

The proof is immediate. From this result, if we consider only the M-atoms, the axioms $(\equiv_2)$ and $(\equiv_3)$ may be proved as theorems.

Now we will present the specific postulates together with some brief comments about them:

(A1) $\forall x(\neg (m(x) \wedge \mathcal{M}(x)))$
(A2) $\forall x \forall y(x \in y \Rightarrow Q(y))$
(A3) $\forall x(Z(x) \Rightarrow Q(x))$
(A4) $\forall_Q x(\exists_m y(y \in x) \Rightarrow \neg Z(x))$
(A5) $\forall_Q x(\forall y(y \in x \Rightarrow \mathcal{D}(y)) \Rightarrow Z(x))$
(A6) $\forall x \forall y(m(x) \wedge x \equiv y \Rightarrow m(y))$.

Intuitively, these axioms have the following interpretation: by A1, for every x, x cannot be an m- and an M-atom simultaneously. A2 says that the atoms are empty, A3 says that all sets are qsets, and A4 says that if a qset has an m-atom as an element, then it is not a set, A5 says that if the transitive closure (see below) of a qset x has no m-atoms, then x is a set. As we will see, the converse also holds. A6 is introduced by the following motive: if x is an m-atom, the antecedent of the main conditional in $(\equiv_4)$ is false. Then, taking $A(x,y)$ as $m(x)$, it may be possible to admit $x \equiv y$ and $\neg m(y)$. A6 says that it is not so: if x is an m-atom, every y indistinguishable from x is also an m-atom.

(A7) Extensionality: $\forall_Q x \forall_Q y (\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y)$

(A8) Null Qset: $\exists_Q x \forall y (\neg (y \in x))$

(A9) Unordered pair: $\forall x \forall y \exists_Q z \forall t (t \in z \Leftrightarrow t \equiv x \vee t \equiv y)$

(A10) Separation: $\forall_Q x \exists_Q y \forall t (t \in y \Leftrightarrow t \in x \wedge A(t))$ with the usual restrictions.

It is easy to prove that the qset postulated by A8 is a set and that it is unique. We will call it the *empty set* and denote it by $\varnothing$. The qset given by A9 will be denoted by $z = [x, y]$ and by $z = \{x, y\}$ when x and y are M-atoms or sets, as is usual. It is important to note that $[x, y]$ denotes the qset that has as elements *all* the indistinguishable elements from x or y, and not only x and y as in the classical set theory. The qset postulated by A10 will be represented by $y = [t \in x : A(t)]$.

Definition 2

(a) $[x] =_{df} [x, x]$

(b) $\langle x, y \rangle =_{df} [[x], [x, y]]$

(c) If x and y are qsets, then: $x \subseteq y =_{df} \forall t (t \in x \Rightarrow t \in y)$.
If $x \subseteq y$, x is said to be a *subqset* of y, and $x \subset y$ stands for $x \subseteq y$ and $x \neq y$.

Theorem 3

(a) *the usual properties of $\subseteq$ are valid in S^**

(b) *for all x and y, $[x] = [y]$ if and only if $x \equiv y$*

(c) *for all x, y, z, and w, $\langle x, y \rangle = \langle z, w \rangle$ iff $x \equiv z$ and $y \equiv w$.*

(d) $\forall x (\neg m(x) \Rightarrow \forall y (y \in [x] \Rightarrow y = x))$.

Item (d) justifies the usual notation $\{x\}$ for the singleton of x, if x is not an m-atom.

In what follows, we will refer mainly to the m-atoms with the aim of covering the most problematic case. The axioms and definitions above permit us to consider the qset $[x]$ of the indistinguishable elements from x, but we are not able to form a qset with, say, a previously obtained quantity n of objects which are indistinguishable from x. We may have either all of them or none. But, if some device (perhaps nonlogical) teaches us what the expression "n indistinguishable elements from x" means[5] (and if by hypothesis this concept coincides with the intuitive one), then it is consistent with the axioms of S^* to suppose that a qset with only n elements indistinguishable from x may be considered in S^*. Of course, this fact follows from $(\equiv_4)$, as is easy to see if we take $A(x, x)$ as $\exists_Q w (x \in w)$ where w is a variable other than either x or y. In this case, we have the following instance of $(\equiv_4)$:

$$\forall x \forall y \exists_Q w (\neg m(x) \wedge \neg m(y) \Rightarrow (x \equiv y \Rightarrow (x \in w \Rightarrow y \in w))).$$

Then, if x and y are m-objects, the antecedent of the conditional is false, therefore the consequent *might* be false too. It may occur that, despite $x \equiv y$ and $x \in w$, it is the case that $y \notin w$. This possibility justifies the following definition:

Definition 3

(a) $S_x(w) =_{df} Q(w) \wedge w \subseteq [x] \wedge x \in w$.

(b) If x is an m-atom, then we define:

$$SBNA_x(w) =_{df} S_x(w) \wedge w \neq [x].$$

This definition introduces the qset that contains *some* elements indistinguishable from x and the qset which has *some, but not all* elements indistinguishable from the m-atom x. As we just have observed (Theorem 3(d)), if x is not an m-atom, then w is the unitary $\{x\}$. The above definition is interesting when x is an m-atom; in this case, we can deal, in the scope of S^*, with a qset which has some, but not all, elements indistinguishable from x without the danger of deriving inconsistencies.

(A11)   Union: $\forall_Q x(\mathcal{E}(x) \Rightarrow \exists_Q y(\forall z(z \in y \Leftrightarrow \exists t(z \in t \wedge t \in x))))$.

As usual, this qset will be denoted by $\bigcup x$. It follows immediately that if all the elements of x are sets, $\bigcup x$ is a set. As usual, we can introduce the concepts of $x \cup y$, $x \cap y$, and $x - y$ based on the previous definitions and axioms.

(A12)   Power Qset: $\forall_Q x \exists_Q y \forall t(t \in y \Leftrightarrow t \subseteq x)$.

This qset will be denoted by $\mathcal{P}(x)$. It is important to note that, keeping only with the language of S^*, one is not able to follow the underlying intuition concerning the subqsets of a "pure" qset (cf. Def. 1(c)). In fact, if we think of a qset x whose elements are n indistinguishable m-atoms, even then it seems intuitive to imagine that we are capable of considering n "singletons" $[y]$ with $y \in x$. But, if all elements of x are indistinguishable one from another, it follows from the above (Theorem 3 (b)) that all these "singletons" are identical. The vindication of this situation is that the qsets $[y]$ are not singletons in the usual sense, once they have as elements *all* the elements indistinguishable from y. To express the singletons in the usual intuitive sense, we need the concept of quasi-cardinality.

Now we will consider the relations and functions that can be defined among qsets.

Definition 4
(a) If x and y are qsets, then:

$$x \times y =_{df} [\langle t, w \rangle \in \mathcal{P}(\mathcal{P}(x \cup y)) : t \in x \wedge w \in y]$$

(b) a *relation* between two qsets x and y is a subqset of $x \times y$. That is, R is a relation between x and y iff $Q(x)$ and $Q(y)$ and

$$Q(R) \wedge \forall z(z \in R \Rightarrow \exists w \exists t(w \in x \wedge t \in y \wedge z = \langle w, t \rangle)).$$

If $x = y$, then R is a relation *on* x.

(c) if R is a relation between the qsets x and y, then R is an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Once $\equiv$ is an equivalence relation on x, we can consider a *partition* of a qset x into *classes of indistinguishable* elements. In other words, the quotient qset $x/\equiv$ has as elements exactly the classes of elements of x that are indistinguishable one from another. If we know how many elements there are in each class, we have Weyl's *effective aggregates*. That is, in this situation we know (by hypothesis) the cardinality of x and the cardinality of each class of $x/\equiv$ (such that the sum of the cardinals of the classes is equal to the cardinal of x), but (in the case where the elements are m-atoms) we cannot distinguish one of them from the others if they belong to a same class. As Weyl said (adapting the terminology):

If . . . no artificial differences between elements are introduced . . . and merely the intrinsic differences of state are made use of, then the aggregate is completely characterized by assigning to each class $\mathcal{C}_i$ ($i = 1, \ldots, k$) the number n_i of elements that belong to $\mathcal{C}_i$. These numbers, the sum of which equals $\underline{n}$ (where $\underline{n}$ is the number of elements of x) describe what may be conveniently called the *visible or effective state* of the system. . . . ([12], p. 239)

This fact justifies Definition 9 below. Now we are faced with another problem: if x and y are qsets that have *m*-atoms as elements, it is not possible to define a *function* F between x and y as usual, once we cannot say that, if $\langle w, t \rangle \in F$ and $\langle w, t' \rangle \in F$, then $t = t'$. We introduce the following definition:

Definition 5   Let x and y be qsets. Then F is a *q-function* from x to y iff $Q(F) \land \forall z(z \in F \Rightarrow \exists w \exists t (w \in x \land t \in y \land z = \langle w, t \rangle)) \land \forall w(w \in x \Rightarrow \exists t (t \in y \land \langle w, t \rangle \in F)) \land \forall w \forall w' \forall t \forall t'(\langle w, t \rangle \in F \land \langle w', t' \rangle \in F \land w \equiv w' \Rightarrow t \equiv t')$.

This fact will be denoted by $F : x \to y$. Intuitively, a q–function maps indistinguishable things into indistinguishable things. If there are no *m*-atoms involved, the definition stands for the usual definition of a *function*.

Definition 6   Let F be a q-function from x to y. Then:
(a) F is a *q-injection* iff $\forall w \forall t \forall s \forall r(\langle w, s \rangle \in F \land \langle t, r \rangle \in F \land s \equiv r \Rightarrow w \equiv t)$.
(b) F is *onto* (or a *q-surjection*) iff $\forall t(t \in y \Rightarrow \exists\ w(w \in x \land \langle w, t \rangle \in F))$.
(c) F is a *q-bijection* iff it is a q-injection and a q-surjection.

By means of the concept of q-function, we can formulate the quasi-set version of the Schema of Replacement. To begin with, we will introduce some previous notation; let $A(x, y)$ be a formula where x and y are free variables. We will say that $A(x, y)$ defines a *y-* (*q-functional*) condition on the qset t iff:

$$\forall w(w \in t \Rightarrow \exists s A(w, s)) \land \forall w \forall w'(w \in t \land w' \in t$$

$$\Rightarrow \forall s \forall s'(A(w, s) \land A(w', s') \land w \equiv w' \Rightarrow s \equiv s')).$$

This fact will be denoted by $\forall x \exists_i y A(x, y)$. Then, we can postulate that the images of qsets by q-functions are qsets:

(A13) $\forall x \exists_i y A(x, y) \Rightarrow \forall_Q u \exists_Q v(\forall z(z \in v \Rightarrow \exists w(w \in u \land A(w, z))))$.

The next postulates are the quasi-set versions of the Axioms of Infinity and Regularity, which do not differ essentially from those of ZFU (see Brignole and da Costa [1]).

(A14) $\exists_Q x(\varnothing \in x \land \forall_Q y(y \in x \Rightarrow y \cup [y] \in x))$
(A15) $\forall_Q x(\mathcal{E}(x) \land x \neq \varnothing \Rightarrow \exists_Q y(y \in x \land y \cap x = \varnothing))$.

Concerning the quasi-set version of the Axiom of Choice, we use the following one:

(A16) $\forall_Q x(\mathcal{E}(x) \Rightarrow \forall y \forall z(y \in x \land z \in x \Rightarrow y \cap z = \varnothing \land y \neq \varnothing) \Rightarrow \exists_Q u \forall y$
$\exists v \exists_Q w(y \in x \land v \in y \Rightarrow y \cap u = w \land S_v(w) \land \forall t(t \in w \Rightarrow t \in y)))$.

The axiom says that, if the mentioned restrictions on x are obeyed, there exists a qset whose members are those elements indistinguishable from the elements of the elements of x. If there are no *m*-atoms involved, that is, if all elements of the

elements of x are qsets or M-atoms, the axiom is equivalent to the usual one of ZFU (see [1]). This fact, in a certain sense, can be assured by the following definition, where we define a *translation* from the formulas of ZFU to formulas of S^* (we will admit that in the language of ZFU there exists a unary predicate $\mathcal{C}$. Intuitively, $\mathcal{C}(x)$ says, in ZFU, that x is a set).

Definition 7   Let A be a formula of ZFU. The *translation* A^q of A to the language of S^* is defined in the following sense:
(a) if A is $\mathcal{C}(x)$, then A^q is $\mathcal{Z}(x)$
(b) if A is $x = y$, then A^q is $(\mathcal{M}(x) \vee \mathcal{Z}(x)) \wedge (\mathcal{M}(y) \vee \mathcal{Z}(y)) \wedge x \equiv y$.
(c) if A is $x \in y$, then A^q is $(\mathcal{M}(x) \vee \mathcal{Z}(x)) \wedge \mathcal{Z}(y) \wedge x \in y$.
(d) if A is $\neg B$, then A^q is $\neg B^q$.
(e) if A is $B \vee C$, then A^q is $B^q \vee C^q$.
(f) if A is $\forall x B$, then A^q is $\forall x(\mathcal{M}(x) \vee \mathcal{Z}(x) \Rightarrow B)$.

Theorem 4   *For any formula A of ZFU, ZFU $\vdash A$ iff $S^* \vdash A^q$.*

Proof: If A is an axiom of ZFU, its translation is an instance of an axiom of S^*, hence, it is a theorem of S^*. Since the translation schema given by the previous definition preserves the inference rules, it is possible to translate a proof of A in ZFU into a proof of A^q in S^*. The converse is easy to verify by dropping the m-atoms: in this case, all the formulas A^q express facts which can be proved in ZFU (for instance, in the system presented in [1]).

The theorem implies that we have a copy of ZFU in S^*. Notwithstanding this fact, we will continue to talk in terms of ZFU instead of its copy. So we may suppose that all the concepts that are defined in ZFU can also be defined in S^* (in reality, these definitions are made in the copy of ZFU); in particular, we can define the concepts of *natural number, cardinality, finite set,* and so on. We will use the following terminology concerning these concepts: $Cd(x)$ says that x is a cardinal; $card(x)$ stands for the cardinal of the set x; $N(x)$ says that x is a natural number (that is, a finite ordinal) and $Fin(x)$ says that x is a finite qset.[6] By means of these concepts, we can present the postulates concerning quasi-cardinality.

(A17) $\forall x(\neg Q(x) \Rightarrow qcard(x) = 0)$
(A18) $\forall_Q x \exists ! y (Cd(y) \wedge y = qcard(x) \wedge (\mathcal{Z}(x) \Rightarrow y = card(x)))$[7]
(A19) $\forall_Q x(x \neq \varnothing \Rightarrow qcard(x) \neq 0)$

If α and β are cardinals,

(A20) $\forall_Q x(qcard(x) = \alpha \Rightarrow \forall \beta(\beta \leq \alpha \Rightarrow \exists_Q y(y \subseteq x \wedge qcard(y) = \beta))$
(A21) $\forall_Q x \forall_Q y(y \subseteq x \Rightarrow qcard(y) \leq qcard(x))$
(A22) $\forall_Q x \forall_Q y(Fin(x) \wedge x \subset y \Rightarrow qcard(x) < qcard(y))$
(A23) $\forall_Q x(qcard(\mathcal{P}(x)) = 2^{qcard(x)})$
(A24) $\forall x(\neg m(x) \Rightarrow qcard([x]) = 1)$.

Intuitively, A18 says that the quasi-cardinal of a quasi-set x is a cardinal and it coincides with the cardinal of x *stricto sensu* if x is a set. A12–A22 are obvious. A23 reflects the intuitive idea that, if for instance we consider n "identical" elementary particles, it is reasonable to suppose that we can *think* that there exist n "singletons", $\frac{1}{2}(n(n-1))$ aggregates with two of them, etc.; but the lan-

guage does not permit us to determine these aggregates in a precise way, as we have mentioned.

By means of the quasi-cardinality notion, we can introduce the concept of Weyl's effective aggregates:

Definition 9   If x is a pure qset such that $qcard(x) = n$ (n finite), then $x/\equiv$ is the *effective aggregate* associated with x.

In fact, the elements of $x/\equiv$ are classes of indistinguishable elements and, once each class has a quasi-cardinality k_i and $\sum_i k_i = n$ (as is easy to prove), $x/\equiv$ plays the role of the effective aggregates in Weyl's sense. Of course, in these qsets, we can know *how many* elements there are in each class, but not *what* elements belong to each one of them.

Now we will mention some results that can be introduced in S^*. As we have mentioned, we will not develop quasi-set theory in all its details in this paper. If we define $Trans(x)$ iff $Q(x)$ and $\forall y \forall z (y \in z \wedge z \in x \Rightarrow y \in x)$, to each qset x we may define the *transitive closure* of x as follows: $x^0 = x$, $x^1 = \bigcup x, \ldots, x^n = \bigcup x^{n-1}, \ldots$ and $TC(x) = \bigcup_{n \in N} x^n$. Then we have the following theorem:

Theorem 5 $\forall_Q x (\forall y (y \in TC(x) \Rightarrow \neg m(y)) \Rightarrow \mathcal{Z}(x))$.

This theorem stands for a kind of "converse" of Axiom A5. It results from the characterization of *sets* given by the axioms of S^*. That is, they are those qsets whose transitive closure has no *m*-atoms, and they are exactly those entities that can be obtained in the copy of ZFU. Other results can also be derived. For instance, we can introduce the *hierarchy of qsets*: if A is the qset of atoms (*m*- and *M*-atoms), then we can introduce the *qset-universe* Q as follows:

$$Q_0 = A$$

$$Q_{\alpha+1} = Q_\alpha \cup \mathcal{P}(Q_\alpha))$$

$$Q_\lambda = \bigcup_{\alpha < \lambda} Q_\alpha \cup \mathcal{P}(Q_\alpha) \text{ if } \lambda \text{ is a limit ordinal, and}$$

$$Q = \bigcup_{\alpha \in On} Q_\alpha \cup \mathcal{P}(Q_\alpha).$$

With the help of the notion of *qcard*, we can define some peculiar qsets, such as, for instance, the *strong singleton*:

Definition 10   If y is a qset, then we say that y is a *strong singleton*, and denote it by $y = \lceil x \rfloor$ iff $y \subseteq [x] \wedge qcard(y) = 1$.

We will say that y is a *strong singleton* from x. Note that in y there is *one* element indistinguishable from x. If $x_1, \ldots, x_k$ are *m*-atoms such that $\neg(x_i \equiv x_j)$ for $i \neq j$, then if we consider the union of strong singletons $\lceil x_1 \rfloor, \ldots, \lceil x_k \rfloor$, we can talk about a qset that has k "distinguishable" *m*-atoms, but there is no sense in talking about the identity or about the diversity of the elements of this qset.

From the above results, the most we can say about S^* and ZFU is that $Cons(S^*) \Rightarrow Cons(ZFU)$. The converse apparently holds, but we will postpone the proof to another paper.

3 Final remarks   In this paper we have delineated only some of the features of quasi–set theory. Of course it can be shown that a lot of classical results of ZFU can be proved in S^*, such as, for instance, Cantor's theorem and the Schröder–Bernstein theorem. Perhaps it will be possible to investigate the relations between the capacity of the language to express the strong singleton of x and the concept of indefinite descriptions: that is to say, perhaps the strong singleton of x may be described as $\epsilon_y(y \in x \wedge m(y))$ and, if it is so, then indefinite descriptions (that is, Hilbert's ϵ symbol) can be expressed in some way in the language of S^*.

Another topic concerns the underlying logic of S^*; it is obvious that the *Traditional* notion of identity (see [9]) is not valid with respect to m-atoms. The underlying logic is a kind of *nonreflexive* logic. There are logical systems that offer some kind of *deviance* from the traditional notion of identity (see [8]). In a future paper, we intend to present other characteristics of the theory delineated here.

NOTES

1. More or less in the same sense, in his book "Le cose e i loro nomi" Toraldo di Francia [11] emphasizes that the usual languages we use to "talk" about things like elementary particles gave rise to some problems concerning semantics, once the elementary particles are "nomological". In particular, he says, ". . . mentre posso dire che per ogni x deve valere '$x = x$', non ha senso dire che 'un elettron = un elettron' (*Microphysics . . .*). È veramente un mondo nuovo e diverso" ([11], p. 193).

2. Intuitively speaking, by *identity* we mean a relation that exists between the entities of some domain of discourse such that, when we say '$a = b$', we mean that there are not in reality two distinct entities, but only one, which may be referred to indifferently as either a or b. We say that two entities are *indistinguishable* if they have identical attributes in common (French [5]). According to TTI, these concepts are equivalent, once identity is defined (in second order logics with identity) by Leibniz's law: $a = b \Leftrightarrow \forall F(F(a) \Leftrightarrow F(b))$ where a and b are individual terms and F is a predicate variable ranging over the possible attributes of the individuals.

3. Cantor said that "a set is a collection into a whole of definite, distinct elements of our intuition or of our thought" (Fraenkel [4], p. 9).

4. Dalla Chiara and Toraldo di Francia present a theory of quasi-sets with the aim of discussing the "intensions" and "extensions" in microphysics [Dalla Chiara and Toraldo di Francia [3]; see also Toraldo di Francia [11], p. 191]. Despite the close relation between their theory and the one developed here (we intend to analyze this fact in a future work), the approaches are distinct. Our use of the expression "quasi-set" follows da Costa ([2], p. 117).

5. To define this concept precisely, we need the axioms of quasi-cardinality (see below). The natural numbers are used at this point as metalinguistic entities.

6. The meaning of the concept of quasi-cardinality is expressed by the specific axioms below. Intuitively, we can say that a qset x is finite if $qcard(x)$ is a natural number.

7. An equivalent form of this axiom is used also in [3].

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