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# Relative Separation Theorems for $\mathcal{L}_{\kappa+\kappa}$

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**Abstract** Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. We prove a strong form of a separation theorem for the language $\mathcal{L}_{\kappa+\kappa}$, where the separant is in $\mathcal{M}_{\lambda+\lambda}$. We also prove that $\mathcal{M}_{\lambda+\lambda}$ allows Lyndon and Malitz interpolation for $\mathcal{L}_{\kappa+\kappa}$. This implies that every sentence of $\mathcal{L}_{\kappa+\kappa}$ preserved under submodels is equivalent to a determined universal sentence of $\mathcal{M}_{\lambda+\lambda}$. From the separation theorem we obtain the corollary that if a sentence $\varphi$ *of* $\mathcal{M}_{\kappa+\kappa}$ has a negation in $\mathcal{M}_{\kappa+\kappa}$, then there is a determined sentence $\psi \in \mathcal{M}_{\lambda+\lambda}$ equivalent to $\varphi$. Using a result of Mekler and Väänänen we show it consistent that the $\Delta$-closure of $\mathcal{L}_{\kappa+\kappa}$ does not allow separation for $\mathcal{L}_{\kappa+\kappa}$, if $\kappa = \mu^+$, $\mu$ a regular cardinal.

***1 Introduction*** Hyttinen [3] and Oikkonen [7] have proved a separation theorem for $\mathcal{L}_{\kappa+\kappa}$, where the separant is in the infinitely deep language $\mathcal{M}_{\kappa+\kappa}$, assuming $\kappa$ regular and $\kappa^{<\kappa} = \kappa$. (For the definition of $\mathcal{M}_{\kappa+\kappa}$, see Definition 1.7.) They have also shown that $\mathcal{M}_{\kappa+\kappa}$ allows Beth definability for $\mathcal{L}_{\kappa+\kappa}$. In this work we prove a stronger form of the separation theorem for $\mathcal{L}_{\kappa+\kappa}$ (Theorem 3.5):

**Separation Theorem for $\mathcal{L}_{\kappa+\kappa}$** *Let $\tau$ be a vocabulary. Assume $\kappa$ is regular and $\lambda = \kappa^{<\kappa}$. If $\varphi$ and $\psi$ are sentences of $\mathcal{L}_{\kappa+\kappa}(\tau)$, they have vocabularies $\mu$ and $\nu$, and $\varphi \wedge \psi$ has no $\tau$-model, then there is $\theta \in \mathcal{M}_{\lambda+\lambda}(\tau)$, such that for all $\tau$-models $\mathfrak{M}$:*
(i) *the vocabulary of $\theta$ is $\mu \cap \nu$;*
(ii) *if $\mathfrak{M} \models \varphi$ then $\mathfrak{M} \models \theta$;*
(iii) *if $\mathfrak{M} \models \psi$ then $\mathfrak{M} \models {\sim}\theta$.*

${\sim}\theta$ denotes the dual of $\theta$ (Definition 1.9). Since sentences in $\mathfrak{M}_{\lambda+\lambda}$ are not always determined, $\mathfrak{M} \not\models \theta$ does not always imply $\mathfrak{M} \models {\sim}\theta$. Thus our theorem is stronger than Hyttinen's, because in Hyttinen's formulation (iii) above is replaced by:

(iii′) *if $\mathfrak{M} \models \psi$ then $\mathfrak{M} \not\models \theta$.*

The separation theorem above implies that $\mathfrak{M}_{\lambda+\lambda}$ allows separation also for $\mathcal{M}_{\kappa+\kappa}$, and assuming $\kappa^{<\kappa} = \kappa$, $\mathcal{M}_{\kappa+\kappa}$ allows separation for itself.

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The proof of the theorem is roughly the following: let $\Phi$ and $\Psi$ be the Vaught game sentences which code the Henkin constructions for $\varphi$ and $\psi$, respectively. Now $\Phi$ is a separant for $\varphi$ and $\psi$. By playing the Henkin construction games simultaneously for $\varphi$ and $\psi$, we find an approximation of $\Phi$, $\theta = \Phi^t \in \mathcal{M}_{\lambda+\lambda}$, which separates $\varphi$ and $\psi$.

We prove two variants of the separation theorem, which are used to obtain Lyndon and Malitz interpolation theorems for $\mathcal{L}_{\kappa+\kappa}$, where the interpolant is in $\mathcal{M}_{\lambda+\lambda}$. Keisler [4] contains the proofs for these results in the simplest case $\kappa = \omega$, that is, $\mathcal{L}_{\omega_1\omega}$ allows Lyndon and Malitz interpolation for itself. These classical results are obtained as a special case in this paper. We apply our Malitz theorem to show that the sentences of $\mathcal{L}_{\kappa+\kappa}$ preserved under submodels are equivalent to determined universal sentences of $\mathcal{M}_{\lambda+\lambda}$. From the separation theorem it also follows that if $\varphi \in \mathcal{M}_{\kappa+\kappa}$ has a negation in $\mathcal{M}_{\kappa+\kappa}$, then there is a determined $\psi \in \mathcal{M}_{\lambda+\lambda}$ equivalent to $\varphi$. We apply our results to generalized Borel sets in the space $\mathfrak{N}_\kappa = \kappa^\kappa$.

Using a result of Mekler and Väänänen [6] we show it consistent that the determined part of $\mathcal{M}_{\kappa+\kappa}$, which, assuming $\kappa^{<\kappa} = \kappa$ is the $\Delta$-closure of $\mathcal{L}_{\kappa+\kappa}$, does not allow separation for $\mathcal{L}_{\kappa+\kappa}$, where $\kappa$ is a successor of a regular cardinal.

**Notation 1.1** We denote by $\|\mathfrak{M}\|$ the universe of a model $\mathfrak{M}$, by $|\mathfrak{M}|$ the cardinality of $\|\mathfrak{M}\|$ and by $\tau(\mathfrak{M})$ the vocabulary of $\mathfrak{M}$. If $\varphi$ is a formula, then $\tau(\varphi)$ is the set of all function, constant, and relation symbols that occur in $\varphi$. By $\#(R)$ we denote the arity of a relation symbol $R$, which may also be infinite. If $C$ is a set and $\bar{c}$ a sequence, then $\bar{c} \subseteq C$ means $\operatorname{ran}(\bar{c}) \subseteq C$.

If $\tau$ is a vocabulary, by $\operatorname{Mod}^\tau(\varphi)$ we denote the class of $\tau$-models of $\varphi$ and by $\operatorname{Str}(\tau)$ the class of all $\tau$-models.

In the definitions of concepts of abstract model theory we mostly follow Ebbinghaus [1]. One exception is that when Ebbinghaus says $\mathbf{L}'$ allows interpolation for $\mathbf{L}$, we say $\mathbf{L}'$ allows separation for $\mathbf{L}$.

**Definition 1.2** (i) We define a *logic* as a pair $(\mathbf{L},\models)$ which fulfills Definition 1.1.1 of [1]. (1.1.1 is a rather minimal definition for a logic.) Here $\mathbf{L}$ is a mapping defined on vocabularies $\tau$ and $\mathbf{L}(\tau)$ is the class of $\tau$-sentences.

(ii) Let $\mathbf{L}$ be a logic and $M$ a class of $\tau$-models.

We say that $M$ is an *elementary class* (EC) in $\mathbf{L}$ iff there is $\varphi \in \mathbf{L}(\tau)$ such that $M = \operatorname{Mod}^\tau(\varphi)$.

We say that $M$ is a *projective class* (PC) in $\mathbf{L}$ iff there is $\tau' \supseteq \tau$ and a class $M'$ of $\tau'$-models EC in $\mathbf{L}$, such that $M = \{\mathfrak{A} \restriction \tau \mid \mathfrak{A} \in M'\}$.

We say that $M$ is a *relativized projective class* (RPC) in $\mathbf{L}$ iff there is $\tau' \supseteq \tau$, a unary relation symbol $U \in \tau' - \tau$, and a class $M'$ of $\tau'$-models EC in $\mathbf{L}$, such that $M = \{(\mathfrak{A} \restriction \tau) \restriction U^{\mathfrak{A}} \mid \mathfrak{A} \in M'\}$.

We say that $M$ is $\Delta$ in $\mathbf{L}$ iff $M$ and $\operatorname{Str}(\tau) - M$ are PC in $\mathbf{L}$.

(iii) Let $\mathbf{L}$ and $\mathbf{L}'$ be logics. We say that $\mathbf{L}$ and $\mathbf{L}'$ are equivalent, in symbols $\mathbf{L} \equiv \mathbf{L}'$, iff any class of models is EC in $\mathbf{L}$ iff it is EC in $\mathbf{L}'$.

**Definition 1.3** (i) The logic $\Sigma_1^1\mathbf{L}$ is the logic which has as elementary classes just the classes which are PC in $\mathbf{L}$.

We define a canonical version of $\Sigma_1^1\mathbf{L}$. Let $\Sigma_1^1\mathbf{L}(\tau)$ consist of all sentences $\exists\bar{R}\varphi$, where $\bar{R}$ is a set of symbols, $\bar{R} \cap \tau = \varnothing$, and $\varphi \in \mathbf{L}(\tau \cup \bar{R})$.

If $\mathfrak{A}$ is a $\tau$-model, then we let $\mathfrak{A} \models \exists \bar{R}\varphi$ iff there is a $\tau \cup \bar{R}$-model $\mathfrak{A}'$, such that $\mathfrak{A} = \mathfrak{A}' \upharpoonright \tau$ and $\mathfrak{A}' \models \varphi$.

(ii) The $\Delta$-closure of $\mathbf{L}$, denoted by $\Delta\mathbf{L}$ or $\Delta_1^1\mathbf{L}$, is the logic which has as elementary classes just the classes that are $\Delta$ in $\mathbf{L}$.

We define a canonical version of $\Delta\mathbf{L}$. Let $\Delta\mathbf{L}(\tau)$ consist of all sentences $\exists\bar{R}\varphi$ of $\Sigma_1^1\mathbf{L}(\tau)$ for which $\mathrm{Mod}^\tau(\exists\bar{R}\varphi)$ is $\Delta$ in $\mathbf{L}$.

**Definition 1.4** (i) We say that $\mathbf{L}$ is closed under *negation* if for all $\tau$ and $\varphi \in \mathbf{L}(\tau)$ there is $\psi$ *e* $\mathbf{L}(\tau)$ such that $\mathrm{Mod}^\tau(\psi) = \mathrm{Str}(\tau) - \mathrm{Mod}^\tau(\varphi)$. We say that $\psi$ is a negation of $\varphi$ in $\mathbf{L}(\tau)$.

(ii) If $\varphi \in \mathbf{L}(\tau)$ and $\psi \in \mathbf{L}'(\tau)$, then we say that $\varphi$ and $\psi$ are *equivalent* iff $\mathrm{Mod}^\tau(\varphi) = \mathrm{Mod}^\tau(\psi)$.

**Definition 1.5** (i) If $M_1$, $M_2$, and $M_3$ are classes of $\tau$-models, $M_1 \cap M_2 = \varnothing$, $M_1 \subseteq M_3$ and $M_3 \cap M_2 = \varnothing$, then we say that $M_3$ *separates* $M_1$ and $M_2$.

(ii) If $\varphi, \psi \in \mathbf{L}(\tau)$, $\theta \in \mathbf{L}'(\tau)$ and the class $\mathrm{Mod}^\tau(\theta)$ separates $\mathrm{Mod}^\tau(\varphi)$ and $\mathrm{Mod}^\tau(\psi)$, then we call $\theta$ a *separant* of $\varphi$ and $\psi$.

(iii) Let $\mathbf{L}$ and $\mathbf{L}'$ be logics. We say that $\mathbf{L}'$ *allows separation for* $\mathbf{L}$ iff for any $\tau$ any two disjoint classes of $\tau$-models PC in $\mathbf{L}$ can be separated by a class of $\tau$-models EC in $\mathbf{L}'$.

In Definition 1.5(iii) we do not say "interpolation" because if $\mathbf{L}$ is not closed under negation then separation and interpolation theorems are not necessarily equivalent (see the remark after Theorem 3.11).

We shall next define the logics (or languages) $\mathcal{L}_{\lambda\kappa}$ and $\mathcal{M}_{\lambda\kappa}$. To avoid confusion with vocabularies, in most of our results we fix a vocabulary $\tau$ and work with $\mathcal{L}_{\lambda\kappa}(\tau)$, $\mathcal{M}_{\lambda\kappa}(\tau)$, and $\tau$-models.

**Definition 1.6** Let $\kappa$ and $\lambda$ be cardinals. A tree $t$ is a $\lambda, \kappa$*-tree*, if $t$ does not contain branches of length $\geq \kappa$, each node $x \in t$ has $< \lambda$ immediate successors, and for all $x, y \in t$ the following holds: if $\{z \in t \mid z < x\} = \{z \in t \mid z < y\}$ and $x$ and $y$ have no immediate predecessors, then $x = y$.

**Definition 1.7** Let $\kappa$ and $\lambda$ be cardinals. A formula of $\mathcal{M}_{\lambda\kappa}$ is a pair $(t, l)$, where $t$ is a $\lambda, \kappa$-tree and $l$ is a labeling function. The pair $(t, l)$ must fulfill:

(1) $t$ does not contain branches of a limit ordinal length;
(2) if $x \in t$ does not have any successors, then $l(x)$ is either an atomic or negated atomic formula;
(3) if $x \in t$ has exactly one immediate successor, then $l(x)$ is of the form $\exists u$ or $\forall u$, $u$ a variable;
(4) if $x \in t$ has more than one immediate successor, then $l(x)$ is either $\bigvee$ or $\bigwedge$;
(5) if $x, y \in t$ and $x < y$, then $l(x)$ and $l(y)$ must not quantify over the same variable.

By $\mathcal{M}_{\lambda\kappa}(\tau)$ we denote the set of those sentences $\varphi \in \mathcal{M}_{\lambda\kappa}$ for which $\tau(\varphi) \subseteq \tau$.

We define $\mathcal{L}_{\lambda\kappa}$ in the usual way, i.e., conjunctions and disjunctions of size $< \lambda$ and quantification over $< \kappa$ variables are allowed.

We have the following assumption: in $\mathcal{L}_{\lambda\kappa}$ and $\mathcal{M}_{\lambda\kappa}$ functions and relations may have $< \kappa$ arguments.

**Definition 1.8** Let $\mathfrak{A}$ be a $\tau$-model, $\varphi \in \mathcal{M}_{\lambda\kappa}(\tau)$ a sentence and $\varphi = (t, l)$. The semantic game $S(\mathfrak{A}, \varphi)$ is a game of two players, $\forall$ and $\exists$. When the game begins, the players are in the root of $t$, and during the game the players go up the tree $t$. In each round the players are in some node $x \in t$, and it depends on $l(x)$ how they continue the game. In a limit round the players start from the supremum of the nodes chosen before.

(i) If $l(x) = \bigvee\ (\bigwedge)$, then $\exists$ ($\forall$) chooses one immediate successor of $x$ to be the node where the players go next.

(ii) If $l(x) = \exists u\ (\forall u)$, then $\exists$ ($\forall$) chooses an element $u^{\mathfrak{A}}$ in $\|\mathfrak{A}\|$ to be the interpretation of $u$. The players go to the immediate successor of $x$.

(iii) If $l(x) = \psi(\bar{u})$, then the game is over and $\exists$ has won if $\mathfrak{A} \models \psi(\bar{u}^{\mathfrak{A}})$.

We write $\mathfrak{A} \models \varphi$ if $\exists$ has a winning strategy for $S(\mathfrak{A}, \varphi)$.

**Definition 1.9** (i) We say that $\varphi \in \mathcal{M}_{\lambda\kappa}(\tau)$ is determined if for every $\tau$-model $\mathfrak{A}$, $\exists$ or $\forall$ has a winning strategy in $S(\mathfrak{A}, \varphi)$. We define $\mathcal{M}^{n}_{\lambda\kappa}(\tau) = \{\varphi \in \mathcal{M}_{\lambda\kappa}(\tau) \mid \varphi$ has a negation in $\mathcal{M}_{\lambda\kappa}(\tau)\}$ and $\mathcal{M}^{d}_{\lambda\kappa}(\tau) = \{\varphi \in \mathcal{M}_{\lambda\kappa}(\tau) \mid \varphi$ is determined$\}$.

(ii) If $\varphi = (t, l) \in \mathcal{M}_{\lambda\kappa}$, then the *dual* of $\varphi$ is $\sim\varphi = (t, l')$, where for each $x \in t$:

(a) $l'(x) = \exists(\forall)$ if $l(x) = \forall(\exists)$;

(b) $l'(x) = \bigwedge\ (\bigvee)$ if $l(x) = \bigvee\ (\bigwedge)$;

(c) $l'(x) = \psi\ (\neg\psi)$ if $l(x) = \neg\psi\ (\psi)$.

Obviously, $\exists$ ($\forall$) has a winning strategy in $S(\mathfrak{M}, \sim\varphi)$ iff $\forall(\exists)$ has a winning strategy in $S(\mathfrak{M}, \varphi)$. Thus $\mathfrak{M} \models \sim\varphi \Rightarrow \mathfrak{M} \not\models \varphi$, but the converse implication does not hold, if $S(\mathfrak{M}, \varphi)$ is nondetermined.

**Definition 1.10** (i) *Conjunctive* $\lambda\kappa$*-Vaught sentences* are of the form

$$\Phi = \forall u_0 \bigvee_{i_0 \in I_0} \bigwedge_{j_0 \in J_0} \exists v_0 \ldots \forall u_\alpha \bigvee_{i_\alpha \in I_\alpha} \bigwedge_{j_\alpha \in J_\alpha} \exists v_\alpha \ldots$$

$$\bigwedge_{\alpha < \kappa} \varphi_{i_0 j_0 \ldots i_\alpha j_\alpha}(u_0, v_0, \ldots, u_\alpha, v_\alpha),$$

where $\varphi_{i_0 j_0 \ldots i_\alpha j_\alpha}$ are conjunctions of atomic and negated atomic formulas and $|I_\alpha|, |J_\alpha| < \lambda$. The semantic game $S(\mathfrak{A}, \Phi)$ is defined like for $\mathcal{M}_{\lambda\kappa}$, and it consists of $\kappa$ rounds, where in round $\alpha$ the truth of $\varphi_{i_0 j_0 \ldots i_\alpha j_\alpha}$ is tested. If $\exists$ can play all $\kappa$ rounds without losing, then he wins the game. We denote the logic of conjunctive $\lambda\kappa$-Vaught sentences by $V_{\lambda\kappa}$.

(ii) If $G$ is a game and $t$ a tree, then by $G^t$ we denote a game which is like $G$, except that before each round $\alpha$, $\forall$ must choose some $x_\alpha \in t$. The elements $x_\alpha$ must form a strictly increasing sequence in $t$ and if $\forall$ runs out of $t$ then $\forall$ loses. If $\Phi$ is the conjunctive $\lambda\kappa$-Vaught sentence from (i) and $t$ a $\lambda, \kappa$-tree, then by $\Phi^t$ we denote the $\mathcal{M}_{\lambda\kappa}$-sentence defined from $\Phi$ in the obvious way so that the game $S^t(\mathfrak{A}, \Phi)$ is essentially the same as $S(\mathfrak{A}, \Phi^t)$.

**Definition 1.11** (i) We say that a formula of $\mathcal{L}_{\lambda\kappa}$ or $\mathcal{M}_{\lambda\kappa}$ is in the *negation normal form* (NNF) if all negations in the syntax tree of $\varphi$ occur immediately before atomic formulas. (In $\mathcal{M}_{\lambda\kappa}$ all formulas are in NNF.) If $\varphi$ is in NNF, by $n$-subformulas of $\varphi$ we mean the smallest set $S$ such that:

(a) $\varphi \in S$;
(b) if $\forall \bar{u}\psi \in S$ or $\exists \bar{u}\psi \in S$ then $\psi$ in $S$;
(c) if $\bigwedge \Psi \in S$ or $\bigvee \Psi \in S$, then $\Psi \subseteq S$.

(ii) If $\varphi \in \mathcal{L}_{\lambda\mu}$ is a sentence, then we define $\mathrm{sub}(\varphi, \kappa) = \kappa + |\{\psi(\bar{c}) \mid \psi(\bar{u})$ a subformula of $\varphi$ and $\bar{c} \subseteq C\}|$, where $C$ is a set of cardinality $\kappa$ of new constants.

## 2 A Henkin construction

In this section we apply a Henkin construction also known as the Hintikka game to derive a separation theorem. To simplify the proofs we consider in this chapter only relational vocabularies.

**Definition 2.1** (Modified from Makkai [5].) Let $\kappa$ be an infinite cardinal. Let $\exists \bar{R}\varphi$ be a $\Sigma_1^1\mathcal{L}_{\kappa^+\kappa}(\tau)$-sentence where $\tau$ and $\bar{R}$ are relational and $\varphi$ is in NNF.

Let $C = \{c_\alpha \mid \alpha < \kappa\}$ be a set of new constants. Let $\Delta_\varphi(C)$ be the smallest such that:
(i) $\varphi \in \Delta_\varphi(C)$;
(ii) if $\psi(\bar{u})$ is an $n$-subformula of $\varphi$ with at most $\bar{u}$ free and $\bar{c} \subseteq C$, then $\psi(\bar{c}) \in \Delta_\varphi(C)$;
(iii) if $c_\alpha, c_\beta \in C$, then $(c_\alpha = c_\beta) \in \Delta_\varphi(C)$ and $(\neg c_\alpha = c_\beta) \in \Delta_\varphi(C)$.

By the definition of an $n$-subformula, $R$ occurs positively (negatively) in $\varphi$ iff it occurs positively (negatively) in $\Delta_\varphi(C)$. Clearly, $|\Delta_\varphi(C)| = \mathrm{sub}(\varphi, \kappa)$. Let $\xi = |\Delta_\varphi(C)|$.

Let $\Phi$ be the following $V_{\xi+\kappa}(\tau)$-sentence:

$$\Phi = \forall u_0 \bigvee_{d_0 \in C} \bigwedge_{e_0 \in C} \exists v_0 \bigwedge_{\delta_0 \in \Delta_\varphi(C)} \bigvee_{\theta_0 \in \Delta_\varphi(C)} \forall u_1 \ldots \left( \bigwedge_{\alpha < \kappa} N^{d_0 e_0 \delta_0 \theta_0 \ldots \theta_\alpha}(u_0, v_0, \ldots, u_\alpha, v_\alpha) \right).$$

Denote $H_\alpha = \{\varphi, \theta_0, \ldots, \theta_\beta, \ldots\}_{\beta<\alpha}$. Suppose:

(1) if $\pi(\bar{u})$ is an atomic formula with $\bar{u}$ free, $\bar{c} = (c_{\beta_\gamma})_{\gamma<\delta}$ and $\bar{c}' = (c_{\epsilon_\gamma})_{\gamma<\delta}$ are constants of $C$, $\pi(\bar{c}) \in H_{\alpha+1}$ and $c_{\beta_\gamma} = c_{\epsilon_\gamma} \in H_{\alpha+1}$ for all $\gamma < \delta$, then $\neg\pi(\bar{c}') \notin H_{\alpha+1}$.
(2) if $\delta_\alpha \in H_\alpha$ and $\delta_\alpha = \bigvee \Psi$, then $\theta_\alpha = \psi$ for some $\psi \in \Psi$;
(3) if $\delta_\alpha \in H_\alpha$ and $\delta_\alpha = \exists \bar{u}\psi(\bar{u})$, then $\theta_\alpha = \psi(\bar{c})$ for some $\bar{c} \subseteq C$;
(4) if $\bigwedge \Psi \in H_\alpha$ and $\delta_\alpha \in \Psi$, then $\theta_\alpha = \delta_\alpha$;
(5) if $\forall \bar{u}\psi(\bar{u}) \in H_\alpha$ and $\delta_\alpha = \psi(\bar{c})$ for some $\bar{c} \subseteq C$, then $\theta_\alpha = \delta_\alpha$;
(6) if $\delta_\alpha$ is of the form $c = c'$, then $\theta_\alpha = (c = c')$ or $\theta_\alpha = (\neg c = c')$.

If (1)–(6) hold, then

$$N^{d_0 \ldots \theta_\alpha}(u_0, v_0, \ldots, u_\alpha, v_\alpha) = \bigwedge \{\pi(u_0, v_0, \ldots, u_\alpha, v_\alpha) \mid \pi \text{ is an atomic or negated atomic formula of } \tau \text{ and } \pi(d_0, e_0, \ldots, d_\alpha, e_\alpha) \in H_{\alpha+1}\}.$$

If (1)–(6) do not hold, then $N^{d_0 \ldots \theta_\alpha}$ is identically false.

Let $\Phi^e$ be the following existential $V_{\xi+\kappa}(\tau)$-sentence:

$$\Phi^e = \bigwedge_{e_0\in C} \exists v_0 \bigwedge_{\delta_0\in\Delta_\varphi(C)} \bigvee_{\theta_0\in\Delta_\varphi(C)} \bigwedge_{e_1\in C} \cdots \left(\bigwedge_{\alpha<\kappa} N^{e_0\delta_0\theta_0\cdots\theta_\alpha}(v_0, v_1, \ldots, v_\alpha)\right).$$

Here $N^{e_0\cdots\theta_\alpha}$ is defined like $N^{d_0\cdots\theta_\alpha}$ above with the following modification: if (1)–(6) hold, then

$$N^{e_0\cdots\theta_\alpha}(v_0,\ldots,v_\alpha) = \bigwedge\{\pi(v_0,\ldots,v_\alpha)\,|\,\pi \text{ is an atomic or negated atomic formula of vocabulary } \tau \text{ and } \pi(e_0,\ldots,e_\alpha)\in H_{\alpha+1}\}.$$

**Theorem 2.2** *Let $\exists\bar{R}\varphi \in \Sigma^1_1\mathcal{L}_{\kappa+\kappa}(\tau)$ and $\Phi \in V_{\xi+\kappa}(\tau)$ be as in Definition* 2.1. *Let $\mathfrak{M}$ be a $\tau$-model.*

(i) *Assume $\kappa$ is regular, or $\kappa$ is singular and there is $\lambda < \kappa$ such that $\varphi \in \mathcal{L}_{\kappa+\lambda}$. If $\mathfrak{M} \models \exists\bar{R}\varphi$, then $\mathfrak{M} \models \Phi$.*

(ii) *Assume* $\mathrm{sub}(\varphi,\kappa) = \kappa$. *If $|\mathfrak{M}| \le \kappa$ and $\mathfrak{M} \not\models \exists\bar{R}\varphi$, then $\mathfrak{M} \models {\sim}\Phi$.*

*Proof:* As in [5].

**Theorem 2.3** *Let $\exists\bar{R}\varphi \in \Sigma^1_1\mathcal{L}_{\kappa+\kappa}(\tau)$ and $\Phi^e \in V_{\xi+\kappa}(\tau)$ be as in Definition* 2.1. *Let $\mathfrak{M}$ be a $\tau$-model.*

(i) *If $\kappa$ is singular, we assume there is $\lambda < \kappa$ such that $\varphi \in \mathcal{L}_{\kappa+\lambda}$; if $\kappa$ is regular we do not assume anything. If $\mathfrak{M}$ has a submodel $\mathfrak{M}_0$ such that $\mathfrak{M}_0 \models \exists\bar{R}\varphi$, then $\mathfrak{M} \models \Phi^e$.*

(ii) *Assume* $\mathrm{sub}(\varphi,\kappa) = \kappa$. *If $\mathfrak{M}$ has no submodel $\mathfrak{M}_0$ such that $\mathfrak{M}_0 \models \exists\bar{R}\varphi$, then $\mathfrak{M} \models {\sim}\Phi^e$.*

*Proof:* (i) Suppose first that $\mathfrak{M}$ has such a submodel $\mathfrak{M}_0$. The proof that $\mathfrak{M} \models \Phi^e$ is exactly as in Theorem 2.2(i): $\exists$ just lets $\mathfrak{M}'$ in the proof to be $\mathfrak{M}_0$ completed to a model of $\varphi$.

(ii) Let $\forall$ play $S(\mathfrak{M},\Phi^e)$ according to the strategy defined in the proof of 2.2(ii) ((S1) is not needed). If $\exists$ can play all $\kappa$ moves against this strategy, then exactly as in 2.2(ii) we can prove that there is a submodel $\mathfrak{M}_0 \subseteq \mathfrak{M}$ such that $\mathfrak{M}_0 \models \exists\bar{R}\varphi$, a contradiction.

**Definition 2.4** Let $\Phi$ and $\Psi$ be conjunctive $\lambda\kappa$-Vaught sentences and $\mathfrak{M}$ a model. We define a combined semantic game $S_2(\mathfrak{M},\Phi,\Psi)$, in which $\exists$ and $\forall$ play the semantic games $S(\mathfrak{M},\Phi)$ and $S(\mathfrak{M},\Psi)$ at the same time. In round $\alpha$ of $S_2$

(i) players $\forall$ and $\exists$ first make the moves of round $\alpha$ in $S(\mathfrak{M},\Phi)$,

(ii) then $\forall$ and $\exists$ make the moves of round $\alpha$ in $S(\mathfrak{M},\Psi)$.

$\forall$ wins $S_2$ in round $\alpha$ if he wins either $S(\mathfrak{M},\Phi)$ or $S(\mathfrak{M},\Psi)$ in round $\alpha$.

**Definition 2.5** Let $\varphi$ and $\psi$ be $\mathcal{L}_{\kappa+\kappa}(\tau)$-sentences in NNF, where $\tau$ is relational. They are also $\Sigma^1_1\mathcal{L}_{\kappa+\kappa}(\tau)$, where the prefix $\exists\bar{R}$ is empty. Let $C = \{c_\alpha\,|\,\alpha<\kappa\}$ and $C' = \{c'_\alpha\,|\,\alpha<\kappa\}$ be disjoint sets of new constants. Let (see Definition 2.1)

$$\Phi_* = \forall u_0 \bigvee_{d_0 \in C} \bigwedge_{e_0 \in C} \exists v_0 \bigwedge_{\delta_0 \in \Delta_\varphi(C)} \bigvee_{\theta_0 \in \Delta_\varphi(C)} \forall u_1 \ldots$$
$$\left( \bigwedge_{\alpha<\kappa} N_*^{d_0 e_0 \delta_0 \theta_0 \ldots \theta_\alpha}(u_0, v_0, \ldots, u_\alpha, v_\alpha) \right)$$

and

$$\Psi = \forall u_0' \bigvee_{d_0' \in C'} \bigwedge_{e_0' \in C'} \exists v_0' \bigwedge_{\delta_0' \in \Delta_\psi(C')} \bigvee_{\theta_0' \in \Delta_\psi(C')} \forall u_1' \ldots$$
$$\left( \bigwedge_{\alpha<\kappa} N^{d_0' e_0' \delta_0' \theta_0' \ldots \theta_\alpha'}(u_0', v_0', \ldots, u_\alpha', v_\alpha') \right).$$

$\Psi$ is defined from $\psi$ as in Definition 2.1. In the definition of $\Phi_*$ there is a small difference. Here $N_*^{d_0 \cdots \theta_\alpha}$ is defined like $N^{d_0 \cdots \theta_\alpha}$ in 2.1 with the following exception:

(e) if $R$ is a relation symbol that does not occur negatively (positively) in $\psi$, then all positive (negative) occurrences of $R$ are deleted from $N_*^{d_0 \cdots \theta_\alpha}$.

We define $\Phi_*^e$ like $\Phi^e$ with the exception (e).

Note that in the following theorem and many others we have replaced a cardinal assumption ($\kappa^{<\kappa} = \kappa$) by an assumption on the number of subformulas of $\varphi$ and $\psi$.

**Theorem 2.6** *Let $\varphi, \psi \in \mathfrak{L}_{\kappa^+\kappa}(\tau)$. Assume* $\operatorname{sub}(\varphi, \kappa) = \operatorname{sub}(\psi, \kappa) = \kappa$. *Let $\Phi_*, \Psi \in V_{\kappa^+\kappa}(\tau)$ be as in Definition 2.5. If $\varphi \wedge \psi$ does not have a $\tau$-model $\mathfrak{A}$, then there is a $\kappa^+$, $\kappa$-tree $t$ such that $\forall$ has a winning strategy in $S_2^t(\mathfrak{A}, \Phi_*, \Psi)$ for all $\tau$-models $\mathfrak{A}$.*

*Proof:* Note that $\operatorname{sub}(\varphi, \kappa) = \operatorname{sub}(\psi, \kappa) = \kappa$ implies $\#(R) < \operatorname{cf}(\kappa)$ for any $R \in \tau(\varphi) \cup \tau(\psi)$. Let $(\rho_\alpha)_{\alpha<\kappa}$ be such that $\rho_\alpha \in \Delta_\varphi(C)$, $\alpha < \kappa$, and $\sup\{\alpha \mid \rho_\alpha = \theta\} = \kappa$ for all $\theta \in \Delta_\varphi(C)$. Here we need the assumption $\operatorname{sub}(\varphi, \kappa) = |\Delta_\varphi(C)| = \kappa$. We define $\rho_\alpha' \in \Delta_\psi(C')$, $\alpha < \kappa$, in a similar way.

Let $a_0$ be an arbitrary fixed set (e.g. $\varnothing$). Without loss of generality we may consider only models $\mathfrak{A}$ such that $a_0 \in \|\mathfrak{A}\|$. We describe $\forall$'s strategy $S_\forall$ in $S_2(\mathfrak{A}, \Phi_*, \Psi)$. For all $\alpha < \kappa$, $\forall$ chooses:

(S1) $u_\alpha^{\mathfrak{A}} = (v_{\alpha-1}')^{\mathfrak{A}}$, if $\alpha$ is a successor, else $u_\alpha^{\mathfrak{A}} = a_0$;
(S2) $e_\alpha = c_\alpha$;
(S3) $\delta_\alpha = \rho_\alpha$;
(S4) $(u_\alpha')^{\mathfrak{A}} = v_\alpha^{\mathfrak{A}}$;
(S5) $e_\alpha' = c_\alpha'$;
(S6) $\delta_\alpha' = \rho_\alpha'$.

Suppose $\mathfrak{A}$ is a model and in $\mathfrak{A}$ $\exists$ plays against $S_\forall$ all rounds before round $\alpha$ without losing. From this play we get a sequence

$$d_0 e_0 \delta_0 \theta_0 d_0' e_0' \delta_0' \theta_0' \ldots d_\beta e_\beta \delta_\beta \theta_\beta d_\beta' e_\beta' \delta_\beta' \theta_\beta' \ldots, \beta < \alpha.$$

We denote by $t_{\mathfrak{A}}$ the set of all such sequences where $\exists$ has not yet lost. Let $t = \bigcup\{t_{\mathfrak{A}} \mid \mathfrak{A} \text{ a } \tau\text{-model}\}$. We order $t$ into a tree by the initial segment relation.

Next we prove that if there is a branch of length $\kappa$ in $t$ then $\varphi \wedge \psi$ has a model. Assume

$$B = d_0 \dots d_\alpha e_\alpha \delta_\alpha \theta_\alpha d'_\alpha e'_\alpha \delta'_\alpha \theta'_\alpha \dots, \quad \alpha < \kappa$$

gives such a branch. Let $H_\varphi = \{\varphi, \theta_0, \theta_1, \dots\}$, $H_\psi = \{\psi, \theta'_0, \theta'_1, \dots\}$, and $H = H_\varphi \cup H_\psi$. We define a relation $\sim$ in the following way:

(r1) $c_\alpha \sim c_\beta$ iff $(c_\alpha = c_\alpha) \in H$;
(r2) $c'_\alpha \sim c'_\beta$ iff $(c'_\alpha = c'_\beta) \in H$;
(r3) $c_\alpha \sim c'_\beta$ and $c'_\beta \sim c_\alpha$ iff there are $\gamma, \delta$, such that $(c_\alpha = c_\gamma) \in H$, $(c'_\delta = c'_\beta) \in H$ and $d'_\gamma = c'_\delta$.

Note that in case (r3) for some $\xi < \kappa$, $N_*^{d_0 \cdots \theta_\xi}$ contains the formula

$$v_\alpha = v_\gamma$$

(from $c_\alpha = c_\gamma$) and $N^{d'_0 \cdots \theta'_\xi}$ contains

$$u'_\gamma = v'_\beta$$

(from $c'_\delta = c'_\beta$).

**Lemma A** *The relation $\sim$ is an equivalence relation.*

*Proof:*

*Reflexivity.* Let $\alpha$ be arbitrary. By the choice of $S_\forall$ either $(c_\alpha = c_\alpha) \in H$ or $(\neg c_\alpha = c_\alpha) \in H$. But, if $(\neg c_\alpha = c_\alpha) \in H$, then for some $\xi$, $N_*^{d_0 \cdots \theta_\xi}$ contains $\neg v_\alpha = v_\alpha$, which is identically false. Thus $\exists$ would lose all plays of length $\xi + 1$ associated with the branch $B$. This contradicts our assumption about $B$. Case $c'_\alpha = c'_\alpha$ is similar.

*Symmetry.* Suppose $c_\alpha \sim c_\beta$, i.e., $(c_\alpha = c_\beta) \in H$. If $(\neg c_\beta = c_\alpha) \in H$, then for some $\xi$, $N_*^{d_0 \cdots \theta_\xi}$ contains $v_\alpha = v_\beta \wedge \neg v_\beta = v_\alpha$, a contradiction. Thus $(c_\beta = c_\alpha) \in H$ and $c_\beta \sim c_\alpha$. Case $c'_\alpha \sim c'_\beta$ is similar, and the others are trivial.

*Transitivity.* Suppose $c_\alpha \sim c_\beta$ and $c_\beta \sim c_\gamma$. As before we see $(c_\alpha = c_\gamma) \in H$ and $c_\alpha \sim c_\gamma$.

Suppose $c_\alpha \sim c'_\beta$ and $c'_\beta \sim c'_\epsilon$. Let $c'_\delta$ be as in (r3). Now $(c'_\delta = c'_\epsilon) \in H$, and thus $c_\alpha \sim c'_\epsilon$.

Suppose $c_{\alpha_1} \sim c'_\beta$ and $c'_\beta \sim c_{\alpha_2}$. Let $c_{\gamma_1}, c'_{\delta_1}, c_{\gamma_2}, c'_{\delta_2}$ be as in (r3). Assume for a contradiction $(\neg c_{\alpha_1} = c_{\alpha_2}) \in H$. Then for some $\xi < \kappa$, $N_*^{d_0 \cdots \theta_\xi}$ and $N^{d'_0 \cdots \theta'_\xi}$ contain the formulas:

(f1) $v_{\alpha_1} = v_{\gamma_1}$, $u'_{\gamma_1} = v'_\beta$ (from $c_{\alpha_1} \sim c'_\beta$);
(f2) $v_{\alpha_2} = v_{\gamma_2}$, $u'_{\gamma_2} = v'_\beta$ (from $c_{\alpha_2} \sim c'_\beta$);
(f3) $\neg v_{\alpha_1} = v_{\alpha_2}$.

Suppose $\exists$ has played $\xi$ rounds without losing in some model $\mathfrak{A}$. Then $(u'_{\gamma_1})^{\mathfrak{A}} = (u'_{\gamma_2})^{\mathfrak{A}}$ (from (f1)–(f2)), and $v_{\gamma_1}^{\mathfrak{A}} \neq v_{\gamma_2}^{\mathfrak{A}}$ (from (f1)–(f3)). But this is a contradiction, because $\forall$ always plays so that $(u'_\alpha)^{\mathfrak{A}} = v_\alpha^{\mathfrak{A}}$.

Suppose then $c'_{\beta_1} \sim c_\alpha$, $c_\alpha \sim c'_{\beta_2}$, and $\neg c'_{\beta_1} \sim c'_{\beta_2}$. Then we get the formulas:

(f1) $v_\alpha = v_{\gamma_1}$, $u'_{\gamma_1} = v'_{\beta_1}$;
(f2) $v_\alpha = v_{\gamma_2}$, $u'_{\gamma_2} = v'_{\beta_2}$;
(f3) $\neg v'_{\beta_1} = v'_{\beta_2}$.

Again we get a contradiction. This proves Lemma A.

We are now ready to define our model $\mathfrak{M}$ of vocabulary $\tau \cup C \cup C'$.

(M1) $\|\mathfrak{M}\|$ = equivalence classes of $\sim$.
(M2) If $c \in C$ and $c' \in C'$, then $c^{\mathfrak{M}} = [c]$ and $(c')^{\mathfrak{M}} = [c']$.
(M3) If $R \in \tau$ and $a_\gamma \in \mathfrak{M}$, $\gamma < \delta$, then $\mathfrak{M} \models R(a_0, \ldots, a_{\gamma<\delta}, \ldots)$ if for some $(c_{\alpha_\gamma})_{\gamma<\delta}$, where $c^{\mathfrak{M}}_{\alpha_\gamma} = a_\gamma$, $\gamma < \delta$,

$$R(c_{\alpha_0}, \ldots, c_{\alpha_{\gamma<\delta}}, \ldots) \in H$$

or for some $(c'_{\alpha_\gamma})_{\gamma<\delta}$, where $(c'_{\alpha_\gamma})^{\mathfrak{M}} = a_\gamma$, $\gamma < \delta$,

$$R(c'_{\alpha_0}, \ldots, c'_{\alpha_{\gamma<\delta}}, \ldots) \in H.$$

Let $\mathfrak{M}' = \mathfrak{M} \restriction \{c^{\mathfrak{M}}_\alpha \mid \alpha < \kappa\}$.

**Lemma B** *$\mathfrak{M} \models \theta$ for all $\theta \in H_\psi$ and $\mathfrak{M}' \models \theta$ for all $\theta \in H_\varphi$.*

*Proof:* By induction. We prove first $\mathfrak{M} \models \theta$ for all $\theta \in H$ (negated) atomic.

(a1) If $\theta = (c_\alpha = c_\beta)$ then by definition $c_\alpha \sim c_\beta$ and $\mathfrak{M} \models \theta$. Case $\theta = (c'_\alpha = c'_\beta)$ similar.

(a2) Suppose $\theta = (\neg c_\alpha = c_\beta)$. Then as before we see $(c_\alpha = c_\beta) \notin H$. Case $\theta = (\neg c'_\alpha = c'_\beta)$ is similar.

(a3) Suppose $\theta = R(c_{\alpha_0}, \ldots, c_{\alpha_{\gamma<\delta}}, \ldots)$. Then by definition $\mathfrak{M} \models \theta$.

(a4) Suppose $\theta = \neg R(c_{\alpha_0}, \ldots, c_{\alpha_{\epsilon<\zeta}}, \ldots) \in H_\varphi$. Assume for a contradiction $\mathfrak{M} \models \neg\theta$. There are two cases. Suppose first there are $(c_{\beta_\epsilon})_{\epsilon<\zeta}$, where $c^{\mathfrak{M}}_{\alpha_\epsilon} = c^{\mathfrak{M}}_{\beta_\epsilon}$, $R(c_{\beta_0}, \ldots, c_{\beta_{\epsilon<\zeta}}, \ldots) \in H_\varphi$. This means $(c_{\alpha_\epsilon} = c_{\beta_\epsilon}) \in H_\varphi$ for all $\epsilon < \zeta$. But now we have a contradiction with Definition 2.1(1). Here we need $\#(R) < \mathrm{cf}(\kappa)$.

Suppose then there are some $(c'_{\beta_\epsilon})_{\epsilon<\zeta}$, such that $(c'_{\alpha_\epsilon})^{\mathfrak{M}} = c^{\mathfrak{M}}_{\beta_\epsilon}$ and $R(c'_{\beta_0}, \ldots, c'_{\beta_{\epsilon<\zeta}}, \ldots) \in H_\psi$. Thus $c'_{\beta_\epsilon} \sim c_{\alpha_\epsilon}$. Let $\gamma_\epsilon$, $\epsilon < \zeta$, be as in (r3). Then for some $\xi$, $N_*^{d_0 \cdots \theta_\xi}$ and $N^{d'_0 \cdots \theta'_\xi}$ contain formulas:

(f1) $\neg R(v_{\alpha_0}, \ldots, v_{\alpha_{\epsilon<\zeta}}, \ldots)$ (remember Definition 2.5(e) and that $R$ occurs positively in $\psi$ because it occurs positively in $H_\psi$);
(f2) $v_{\alpha_\epsilon} = v_{\gamma_\epsilon}$, $\epsilon < \zeta$;
(f3) $u'_{\gamma_\epsilon} = v'_{\beta_\epsilon}$, $\epsilon < \zeta$;
(f4) $R(v'_{\beta_0}, \ldots, v'_{\beta_{\epsilon<\zeta}}, \ldots)$.

As before we get a contradiction, since $v^{\mathfrak{A}}_{\gamma_\epsilon} = (u'_{\gamma_\epsilon})^{\mathfrak{A}}$.

(a5) Case $\theta = \neg R(\ldots) \in H_\psi$ is similar (in (f1)–(f4) above $\neg R$ and $R$ are just exchanged).

Now we have treated the case $\theta$ (negated) atomic. Suppose then, for example, $\theta = \forall \bar{u} \rho(\bar{u})$, $\theta \in H_\varphi$. By our assumption $\mathfrak{M}' \models \rho(\bar{c})$ for all $\bar{c} \subseteq C$. This implies $\mathfrak{M}' \models \forall \bar{u} \rho(\bar{u})$. Note that every equivalence class of $\sim$ contains an element from $C'$ (by (r3)). All other steps are similar. This proves Lemma B.

**Lemma C** *$\mathfrak{M}' = \mathfrak{M}$, i.e., every equivalence class of $\sim$ contains an element from $C$.*

*Proof:* Let $c'_\alpha \in C'$ be arbitrary. Let $c_\beta = d_{\alpha+1}$ and $c'_\gamma = d'_\beta$. Then $c_\beta \sim c'_\gamma$. We show $c'_\gamma \sim c'_\alpha$, which implies $c_\beta \sim c'_\alpha$. Assume for a contradiction $(\neg c'_\gamma = c'_\alpha) \in H$. Then for some $\xi < \kappa$, $N_*^{d_0 \cdots \theta_\xi}$ and $N^{d'_0 \cdots \theta'_\xi}$ contain the formulas:

(f1) $u_{\alpha+1} = v_\beta$ (from $c_\beta = c_\beta$);
(f2) $\neg u'_\beta = v'_\alpha$ (from $\neg c'_\gamma = c'_\alpha$).

This is a contradiction, because $u^{\mathfrak{A}}_{\alpha+1} = (v'_\alpha)^{\mathfrak{A}}$ and $(u'_\beta)^{\mathfrak{A}} = v^{\mathfrak{A}}_\beta$. This proves Lemma C.

This ends the proof that $\mathfrak{M} \models \varphi \wedge \psi$. Thus there cannot be branches of length $\kappa$ in the tree $t$. We describe $\forall$'s winning strategy for $S_2^t(\mathfrak{A}, \Phi_*, \Psi)$. Except for the moves in $t$, $\forall$ just follows his winning strategy $S_\forall$. If $\forall$ has not yet won in round $\alpha$, then he moves $d_0 \ldots \theta'_{\beta<\alpha} \ldots$ in $t$ and makes his other moves according to $S_\forall$. This proves the theorem.

**Theorem 2.7** *Let $\varphi, \psi \in \mathcal{L}_{\kappa+\kappa}(\tau)$. Assume* $\mathrm{sub}(\varphi, \kappa) = \mathrm{sub}(\psi, \kappa) = \kappa$. *Let $\Phi_*^e, \Psi \in V_{\kappa+\kappa}(\tau)$ be as in Definition* 2.5. *If there do not exist $\tau$-models $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $\mathfrak{M}' \models \varphi$ and $\mathfrak{M} \models \psi$, then there is a $\kappa^+, \kappa$-tree $t$ such that $\forall$ has a winning strategy in $S_2^t(\mathfrak{A}, \Phi_*^e, \Psi)$ for all $\tau$-models $\mathfrak{A}$.*

*Proof:* If we look at the proof of Theorem 2.6, we see that $u_\alpha$ and $d_\alpha$ are needed in Lemma C only to prove $\mathfrak{M}' = \mathfrak{M}$.

***3 Lyndon separation*** In this section we apply the results of the previous section to derive Lyndon separation theorems for $\mathcal{L}_{\kappa+\kappa}$ and $\mathcal{M}_{\kappa+\kappa}$.

From now on we consider arbitrary vocabularies, not just relational ones. To simplify notation we consider constants as functions without arguments.

**Definition 3.1** Let $\tau$ be a vocabulary, let $\tau_f$ contain exactly the function symbols in $\tau$, and let $\varphi$ be a formula of $\mathcal{L}_{\lambda\kappa}(\tau)$ or $\mathcal{M}_{\lambda\kappa}(\tau)$. We say that $\varphi$ is in a *function normal form* (FNF) if $\varphi$ is in NNF and function symbols occur only in atomic formulas of the form

$$u_0 = F(u_1, u_2, \ldots),$$

where $u_0, u_1, \ldots$ are variables.

We define an operation that canonically transforms functions to relations. Let $\tau' = R_{\tau_f}(\tau)$ be a vocabulary such that $\tau'$ is exactly like $\tau$, except that if $F \in \tau_f$ is an $\alpha$-place function symbol in $\tau$, then $F$ is a $1 + \alpha$-place relation symbol in $\tau'$.

If $\mathfrak{M}$ is a $\tau$-model, then we define $\mathfrak{M}' = R_{\tau_f}(\mathfrak{M})$ as a $\tau'$-model such that $\mathfrak{M}' \upharpoonright (\tau - \tau_f) = \mathfrak{M} \upharpoonright (\tau - \tau_f)$ and if $F \in \tau_f$, then $\mathfrak{M}' \models F(a_0, a_1, \ldots)$ iff $\mathfrak{M} \models a_0 = F(a_1, \ldots)$.

If $\varphi$ is in FNF, then we define $\varphi' = R_{\tau_f}(\varphi)$ as a formula where each atomic formula of the form $u_0 = F(u_1, \ldots)$, $F \in \tau_f$, is replaced by $F(u_0, u_1, \ldots)$.

If $\tau_0$ is a set of relation symbols, then by $\rho_{\tau_0}$ we denote a sentence which says that the relations in $\tau_0$ determine functions in the canonical way.

**Lemma 3.2** *Let $\tau$, $\tau_f$ and $\varphi$ be as in Definition* 3.1.
(i) *If $\mathfrak{M}$ is a $\tau$-model and $\varphi$ is in* FNF, *then $\mathfrak{M} \models \varphi \Leftrightarrow R_{\tau_f}(\mathfrak{M}) \models R_{\tau_f}(\varphi)$.*
(ii) *If $\mathfrak{M}'$ is an $R_{\tau_f}(\tau)$-model and $\mathfrak{M}' \models \rho_{\tau_f}$, then $R^{-1}_{\tau_f}(\mathfrak{M}')$ is defined.*
(iii) *If $\varphi'$ is any $R_{\tau_f}(\tau)$-formula in* NNF, *then $R^{-1}_{\tau_f}(\varphi')$ is defined.*

**Lemma 3.3** *If $\varphi \in \mathcal{L}_{\lambda\kappa}(\tau)$, $\lambda \geq \kappa$, then there is $\varphi' \in \mathcal{L}_{\lambda\kappa}(\tau)$ in* FNF *such that $\varphi \Leftrightarrow \varphi'$ and for every relation symbol $R$, $R$ occurs positively (negatively) in $\varphi$ iff it occurs positively (negatively) in $\varphi'$.*

*Proof:* Suppose $t(u_1, u_2, \dots)$ is a $\tau$-term. We prove by induction that for the formula $u_0 = t(u_1, u_2, \dots)$ there is $\varphi_t(u_0, u_1, \dots)$ which is equivalent to it and in FNF. Suppose

$$t(u_1, u_2, \dots) = F(t_0(u_1, u_2, \dots), t_1(u_1, u_2, \dots), \dots).$$

Then we let $\varphi_t$ be

$$\exists v_0, v_1, \dots (u_0 = F(v_0, v_1, \dots) \wedge \varphi_{t_0}(v_0, u_1, u_2, \dots) \wedge \cdots).$$

Now it is obvious how we can construct $\varphi'$ by replacing atomic formulas in $\varphi$.

**Lyndon Separation Theorem 3.4** *Let $\kappa$ be infinite. Suppose $\varphi$ and $\psi$ are sentences of $\mathcal{L}_{\kappa^+\kappa}(\tau)$, they are in* FNF, *and $\varphi \wedge \psi$ has no $\tau$-model. Assume* $\mathrm{sub}(\varphi, \kappa) = \mathrm{sub}(\psi, \kappa) = \kappa$. *Then there is a sentence $\theta$ of $\mathcal{M}_{\kappa^+\kappa}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*
(i) $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \psi \Rightarrow \mathfrak{M} \models \sim\theta$;
(iii) $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$;
(iv) *if a relation symbol $R$ occurs positively (negatively) in $\theta$, then it occurs positively (negatively) in $\varphi$ and negatively (positively) in $\psi$.*

*Proof:* We prove the claim first for relational vocabularies. Let $\Phi_*$ and $\Psi$ be as in Theorem 2.6. For some $\kappa^+, \kappa$-tree $t$, $\forall$ has a winning strategy in $S_2^t(\mathfrak{M}, \Phi_*, \Psi)$ for all $\mathfrak{M}$. Let $\theta = \Phi_*^t$.

Let $\mathfrak{M}$ be arbitrary. If $\mathfrak{M} \models \varphi$, then by Theorem 2.2(i) $\mathfrak{M} \models \Phi$. Note that if $\kappa$ is singular, we can apply 2.2(i) because $\mathrm{sub}(\varphi, \kappa) = \kappa$ implies that $\varphi \in \mathcal{L}_{\kappa^+\lambda}$, where $\lambda = \mathrm{cf}(\kappa)$ (if we remove from $\varphi$ quantification over variables not occurring in the scope of the quantifier). Since $\Phi_*$ is a weaker sentence than $\Phi$ (see Definition 2.5), $\mathfrak{M} \models \Phi_*$. This implies $\mathfrak{M} \models \theta$.

Suppose then $\mathfrak{M} \models \psi$. Then $\exists$ has a winning strategy in $S(\mathfrak{M}, \Psi)$. Since $\forall$ has a winning strategy in $S_2^t(\mathfrak{M}, \Phi_*, \Psi)$, $\forall$ must obviously have a winning strategy in $S^t(\mathfrak{M}, \Phi_*)$. This means $\mathfrak{M} \models \sim\theta$.

If a relation symbol occurs positively (negatively) in $\Phi_*$, then it occurs positively (negatively) in $\Delta_\varphi(C)$ and thus in $\varphi$. By Definition 2.5 it must occur negatively (positively) in $\psi$.

Suppose then $\tau$ is not relational. Let $\mu = \tau(\varphi)$ and $\nu = \tau(\psi)$. Let $\tau_f, \mu_f, \nu_f$ contain the function symbols in $\tau, \mu, \nu$, respectively. Let $\tau' = R_{\tau_f}(\tau)$, $\varphi' = R_{\tau_f}(\varphi)$ and $\psi' = R_{\tau_f}(\psi)$. Assume for a contradiction $\mathfrak{M}'$ is a $\tau'$-model of $(\varphi' \wedge \rho_{\mu_f}) \wedge (\psi' \wedge \rho_{\nu_f})$. We redefine the relations $F^{\mathfrak{M}'}$, $F \in \tau_f - (\mu_f \cup \nu_f)$, so that $\mathfrak{M} = R_{\tau_f}^{-1}(\mathfrak{M}')$ is defined. Then $\mathfrak{M} \models \varphi \wedge \psi$, a contradiction.

Clearly, $\mathrm{sub}(\varphi', \kappa) = \mathrm{sub}(\varphi, \kappa) = \kappa$ and $\mathrm{sub}(\rho_{\mu_f}, \kappa) \leq \mathrm{sub}(\varphi', \kappa) = \kappa$, and similarly for $\psi'$. Let $\theta'$ be the separant of $\varphi' \wedge \rho_{\mu_f}$ and $\psi' \wedge \rho_{\nu_f}$. Let $\theta = R_{\tau_f}^{-1}(\theta')$.

Suppose $\mathfrak{M}$ is a $\tau$-model and $\mathfrak{M} \models \psi$. Then $R_{\tau_f}(\mathfrak{M}) \models \psi' \wedge \rho_{\mu_f}$ and $R_{\tau_f}(\mathfrak{M}) \models \sim\theta'$. By Lemma 3.2(i) $\mathfrak{M} \models R_{\tau_f}^{-1}(\sim\theta')$, and obviously $R_{\tau_f}^{-1}(\sim\theta') = \sim\theta$. Similarly we get $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$.

**Lyndon Separation Theorem for $\mathfrak{L}_{\kappa+\kappa}$ 3.5** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Suppose $\varphi$ and $\psi$ are sentences of $\mathcal{L}_{\kappa+\kappa}(\tau)$ and $\varphi \wedge \psi$ has no $\tau$-model. Then there is a sentence $\theta$ of $\mathcal{M}_{\lambda+\lambda}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*

(i) $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \psi \Rightarrow \mathfrak{M} \models \sim\theta$;
(iii) $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$;
(iv) *if a relation symbol $R$ occurs positively (negatively) in $\theta$, then it occurs positively (negatively) in $\varphi$ and negatively (positively) in $\psi$.*

*Proof:* Note that $\text{sub}(\varphi,\lambda) \leq \lambda^{<\kappa} = \lambda$. Thus the claim follows from Theorem 3.4.

If Theorem 3.5 holds with $\mathcal{L}_{\kappa+\kappa}$ and $\mathcal{M}_{\lambda+\lambda}$ replaced by $\mathbf{L}_1$ and $\mathbf{L}_2$, then we say that $\mathbf{L}_2$ allows *Lyndon separation* for $\mathbf{L}_1$.

**Lyndon Separation Theorem for $\mathfrak{L}_{\kappa+\omega}$ 3.6** *If $\kappa$ is infinite, then $\mathcal{M}_{\kappa+\kappa}$ allows Lyndon separation for $\mathcal{L}_{\kappa+\omega}$.*

**Lemma 3.7** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Let $\varphi \in V_{\kappa+\kappa}(\tau)$ or $\varphi \in \mathcal{M}_{\kappa+\kappa}(\tau)$. Then there is a $\Sigma_1^1\mathcal{L}_{\lambda+\kappa}(\tau)$-sentence $\exists\bar{P}\varphi'$ which is equivalent to $\varphi$ and such that a relation symbol $R \in \tau$ occurs positively (negatively) in $\varphi'$ iff it occurs positively (negatively) in $\varphi$.*

*Proof:* It is enough to treat the case $\varphi \in V_{\kappa+\kappa}(\tau)$ because essentially $\mathcal{M}_{\kappa+\kappa}(\tau) \subseteq V_{\kappa+\kappa}(\tau)$. The proof is done by Skolemization, as in Proposition 5.1 of [5]. We just have to add some sentences there to ensure that $\exists$ can move also in rounds $\alpha$, where $\alpha$ is a limit.

**Lyndon Separation Theorem for $\mathcal{M}_{\kappa+\kappa}$ 3.8** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Then $\mathcal{M}_{\lambda+\lambda}$ allows Lyndon separation for $\mathcal{M}_{\kappa+\kappa}$.*

*Proof:* Let $\varphi, \psi \in \mathcal{M}_{\kappa+\kappa}(\tau)$. Let $\exists\bar{R}\varphi', \exists\bar{S}\psi' \in \Sigma_1^1\mathcal{L}_{\lambda+\kappa}(\tau)$ from Lemma 3.7 ($\bar{R} \cap \bar{S} = \emptyset$) be equivalent to $\varphi$ and $\psi$. We may assume that $\varphi'$ and $\psi'$ are in FNF. Now $\varphi' \wedge \psi'$ does not have a $\tau \cup \bar{R} \cup \bar{S}$-model. We can apply Theorem 3.4 because $\text{sub}(\varphi',\lambda) = \text{sub}(\psi',\lambda) = (\kappa^{<\kappa})^{<\kappa} = \lambda$. Let $\theta$ be the separant of $\varphi'$ and $\psi'$. Suppose $\mathfrak{M}$ is a $\tau$-model and $\mathfrak{M} \models \varphi$. Then $\mathfrak{M}$ can be extended to a $\tau \cup \bar{R} \cup \bar{S}$-model $\mathfrak{M}'$, for which $\mathfrak{M}' \models \varphi'$. Thus $\mathfrak{M}' \models \theta$ and $\mathfrak{M} \models \theta$. Case $\mathfrak{M} \models \psi$ is similar.

**Separation Theorem for $\Sigma_1^1\mathcal{M}_{\kappa+\kappa}$ 3.9** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. If $\exists\bar{R}\varphi$ and $\exists\bar{S}\psi$ are $\Sigma_1^1\mathcal{M}_{\kappa+\kappa}(\tau)$-sentences and $\exists\bar{R}\varphi \wedge \exists\bar{S}\psi$ has no $\tau$-model, then there is $\theta \in \mathcal{M}_{\lambda+\lambda}(\tau)$ such that for all $\tau$-models $\mathfrak{M}$:*

(i) $\mathfrak{M} \models \exists\bar{R}\varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \exists\bar{S}\psi \Rightarrow \mathfrak{M} \models \sim\theta$;

**Corollary 3.10**

(i) *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. If $\exists\bar{R}\varphi$ is in $\Delta_1^1 M_{\kappa+\kappa}(\tau)$, then there is determined $\theta \in M_{\lambda+\lambda}(\tau)$ which is equivalent to $\exists\bar{R}_\varphi$.*
(ii) *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Then $\mathcal{M}_{\lambda+\lambda}$ allows separation for $\mathcal{M}_{\kappa+\kappa}$ and $\mathcal{L}_{\kappa+\kappa}$.*
(iii) *Assume $\kappa$ regular and $\kappa^{<\kappa} = \kappa$. Then $\Delta\mathcal{L}_{\kappa+\kappa} \equiv \Delta\mathcal{M}_{\kappa+\kappa} \equiv \mathcal{M}^n_{\kappa+\kappa} \equiv \mathcal{M}^d_{\kappa+\kappa}$.*

**Lyndon Interpolation Theorem for $\mathcal{L}_{\kappa+\kappa}$ 3.11** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Suppose $\varphi, \psi \in \mathcal{L}_{\kappa+\kappa}(\tau)$ and for all $\tau$-models $\mathfrak{M}$, $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \psi$. Then there is a sentence $\theta \in \mathcal{M}_{\lambda+\lambda}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*

(i) $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \theta \Rightarrow \mathfrak{M} \models \psi$;
(iii) $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$;
(iv) *if a relation symbol R occurs positively (negatively) in $\theta$ then it occurs positively (negatively) in both $\varphi$ and $\psi$.*

In the proof of the interpolation theorem 3.11 above we need the fact that $\psi$ has a negation in $\mathcal{L}_{\kappa+\kappa}(\tau)$. We cannot prove 3.11 this way for $\mathcal{M}_{\kappa+\kappa}(\tau)$ because it is consistent that there are sentences of $\mathcal{M}_{\kappa+\kappa}(\tau)$ with no negation in $\mathcal{M}_{\kappa+\kappa}(\tau)$ (see Corollary 6.6). The problem whether Theorem 3.11 holds with $\mathcal{L}_{\kappa+\kappa}$ replaced by $\mathcal{M}_{\kappa+\kappa}$ is open.

**Beth's Theorem for $\mathcal{M}_{\kappa+\kappa}$ 3.12** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Suppose that $\varphi(P) \in \mathcal{M}_{\kappa+\kappa}(\tau \cup \{P\})$ and for all $\mathfrak{M}$,*

$$\mathfrak{M} \models \varphi(P) \wedge \varphi(P') \Rightarrow \mathfrak{M} \models \forall \bar{u}(P(\bar{u}) \Leftrightarrow P'(\bar{u})).$$

*Then there is a formula $\theta \in \mathcal{M}_{\lambda+\lambda}(\tau)$ such that if $\mathfrak{M} \models \varphi(P)$, then*
(i) $\mathfrak{M} \models \forall \bar{u}(P(\bar{u}) \Leftrightarrow \theta(\bar{u}))$,
(ii) $\mathfrak{M} \models \forall \bar{u}(\neg P(\bar{u}) \Leftrightarrow \sim\theta(\bar{u}))$.

*Proof:* Let $\bar{c}$ be new constants. Then

$$(\varphi(P) \wedge P(\bar{c})) \wedge (\varphi(P') \wedge \neg P'(\bar{c}))$$

does not have a model. Let $\theta(\bar{c})$ be the separant of the conjuncts.

***4 Malitz separation*** In this section we apply the results of Section 2 to derive Malitz separation theorems for $\mathcal{L}_{\kappa+\kappa}$ and $\mathcal{M}_{\kappa+\kappa}$.

**Malitz Separation Theorem 4.1** *Suppose $\varphi, \psi \in \mathcal{L}_{\kappa+\kappa}(\tau)$ are in* FNF, *$\tau(\varphi) = \mu$, $\tau(\psi) = \nu$ and $\mu \cap \nu = \eta$. Assume* $\mathrm{sub}(\varphi, \kappa) = \mathrm{sub}(\psi, \kappa) = \kappa$.

*Suppose there do not exist $\tau$-models $\mathfrak{M}'$ and $\mathfrak{M}$ such that $\mathfrak{M}' \restriction \eta \subseteq \mathfrak{M} \restriction \eta$, $\mathfrak{M}' \models \varphi$ and $\mathfrak{M} \models \psi$. Then there is a sentence $\theta$ in $\mathcal{M}_{\kappa+\kappa}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*

(i) $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \psi \Rightarrow \mathfrak{M} \models \sim\theta$;
(iii) *$\theta$ is existential*;
(iv) $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$;
(v) *if a relation symbol R occurs positively (negatively) in $\theta$, then it occurs positively (negatively) in $\varphi$ and negatively (positively) in $\psi$.*

*Proof:* Consider first relational vocabularies. Then the assumptions imply that the conditions in Theorem 2.7 hold. Let $t$, $\Phi^e_*$, and $\Psi$ be as in Theorem 2.7. Let $\theta = (\Phi^e_*)^t$. If $\mathfrak{M} \models \varphi$, then by Theorem 2.3(i) $\mathfrak{M} \models \Phi^e$, $\mathfrak{M} \models \Phi^e_*$, and $\mathfrak{M} \models \theta$. If $\mathfrak{M} \models \psi$, then $\mathfrak{M} \models \Psi$ and $\forall$ must have a winning strategy in $S^t(\mathfrak{M}, \Phi^e_*)$. This means $\mathfrak{M} \models \sim\theta$.

Consider then arbitrary vocabularies. Let $\tau_f, \mu_f, \nu_f$ contain the function symbols in $\tau, \mu, \nu$, respectively. Let $\tau' = R_{\tau_f}(\tau)$, $\varphi' = R_{\tau_f}(\varphi)$, and $\psi' = R_{\tau_f}(\psi)$.

Assume for a contradiction $\mathfrak{M}_0'$ and $\mathfrak{M}_0$ are $\tau'$-models, $\mathfrak{M}_0' \upharpoonright \eta \subseteq \mathfrak{M}_0 \upharpoonright \eta$, $\mathfrak{M}_0' \models \varphi' \wedge \rho_{\mu_f}$, and $\mathfrak{M}_0 \models \psi' \wedge \rho_{\nu_f}$. We may redefine the relations $F^{\mathfrak{M}_0'}$, $F \in \tau_f - \mu_f$, so that $\mathfrak{M}' = R_{\tau_f}^{-1}(\mathfrak{M}_0')$ is defined. Similarly, we can make $\mathfrak{M} = R_{\tau_f}^{-1}(\mathfrak{M}_0)$ defined. Then $\mathfrak{M}' \models \varphi$ and $\mathfrak{M} \models \psi$. Obviously $\mathfrak{M}' \upharpoonright \eta \subseteq \mathfrak{M} \upharpoonright \eta$, a contradiction. Let $\theta'$ be the separant and $\theta = R_{\tau_f}^{-1}(\theta')$. Suppose $\mathfrak{M}$ is a $\tau$-model and $\mathfrak{M} \models \psi$. Then $R_{\tau_f}(\mathfrak{M}) \models \psi' \wedge \rho_{\nu_f}$, $R_{\tau_f}(\mathfrak{M}) \models \sim\theta'$, and $\mathfrak{M} \models \sim\theta$. Case $\mathfrak{M} \models \varphi$ is similar.

The restriction to $\eta$ in $\mathfrak{M}' \upharpoonright \eta \subseteq \mathfrak{M} \upharpoonright \eta$ above is necessary if we allow function (or constant) symbols, as the following example shows. Let $\tau = \{c_0, c_1, c_2\}$. Let $\varphi = \forall u(u = c_0)$ and $\psi = (c_1 \neq c_2)$. Then there are no $\tau$-models $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $\mathfrak{M}' \models \varphi$ and $\mathfrak{M} \models \psi$. Assume $\theta$ is existential, $\tau(\theta) = \varnothing$, $\mathfrak{M} \models \varphi \Rightarrow \theta$ and $\mathfrak{M} \models \psi \Rightarrow \sim\theta$ for every $\tau$-model $\mathfrak{M}$. Then $\theta$ is true in every model of power 1, and since $\theta$ is existential, also in every model of power 2, a contradiction.

**Malitz Separation Theorem for $\mathfrak{L}_{\kappa+\kappa}$ 4.2** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Suppose $\varphi$ and $\psi$ are sentences of $\mathfrak{L}_{\kappa+\kappa}(\tau)$, $\tau(\varphi) = \mu$, $\tau(\psi) = \nu$, and $\mu \cap \nu = \eta$. Suppose there do not exist $\tau$-models $\mathfrak{M}'$ and $\mathfrak{M}$ such that $\mathfrak{M}' \upharpoonright \eta \subseteq \mathfrak{M} \upharpoonright \eta$, $\mathfrak{M}' \models \varphi$ and $\mathfrak{M} \models \psi$. Then there is a sentence $\theta$ in $\mathcal{M}_{\lambda+\lambda}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*
(i) $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \psi \Rightarrow \mathfrak{M} \models \sim\theta$;
(iii) *$\theta$ is existential;*
(iv) $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$

If Theorem 4.2 holds with $\mathfrak{L}_{\kappa+\kappa}$ and $\mathcal{M}_{\lambda+\lambda}$ replaced by $\mathbf{L}_1$ and $\mathbf{L}_2$, then we say that $\mathbf{L}_2$ allows *Malitz separation* for $\mathbf{L}_1$.

**Malitz Interpolation Theorem for $\mathfrak{L}_{\kappa+\kappa}$ 4.3** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Suppose $\varphi, \psi \in \mathfrak{L}_{\kappa+\kappa}(\tau)$, where $\tau$ is relational, $\varphi$ is preserved to extensions, and for every $\tau$-model $\mathfrak{M}$, $\mathfrak{M} \models \varphi \Rightarrow \psi$. Then there is $\theta \in \mathcal{M}_{\lambda+\lambda}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*
(i) $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \theta \Rightarrow \mathfrak{M} \models \psi$;
(iii) *$\theta$ is existential;*
(iv) $\tau(\theta) \subseteq \tau(\varphi) \cap \tau(\psi)$.

**Malitz Separation Theorem for $\mathcal{M}_{\kappa+\kappa}$ 4.4** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Then $\mathcal{M}_{\lambda+\lambda}$ allows Malitz separation for $\mathcal{M}_{\kappa+\kappa}$.*

**Malitz Separation Theorem for $\Sigma_1^1\mathcal{M}_{\kappa+\kappa}$ 4.5** *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. Suppose $\exists\bar{R}\varphi, \exists\bar{S}\psi \in \Sigma_1^1\mathcal{M}_{\kappa+\kappa}(\tau)$ and there do not exist $\tau$-models $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $\mathfrak{M}' \models \exists\bar{R}\varphi$ and $\mathfrak{M} \models \exists\bar{S}\psi$. Then there is $\theta \in \mathcal{M}_{\lambda+\lambda}(\tau)$ such that for every $\tau$-model $\mathfrak{M}$:*
(i) $\mathfrak{M} \models \exists\bar{R}\varphi \Rightarrow \mathfrak{M} \models \theta$;
(ii) $\mathfrak{M} \models \exists\bar{S}\psi \Rightarrow \mathfrak{M} \models \sim\theta$;
(iii) *$\theta$ is existential.*

*Proof:* We assume that $\bar{R}$ and $\bar{S}$ are disjoint. We may assume that $\tau \subseteq \tau(\varphi) \cup \tau(\psi)$, and by adding dummy subformulas to $\varphi$ and $\psi$, we may extend $\tau(\varphi)$ and $\tau(\psi)$ so that $\tau = \tau(\varphi) \cap \tau(\psi)$. Now we can apply Theorem 4.4 to $\varphi$ and $\psi$ as $\tau \cup \bar{R} \cup \bar{S}$-sentences, yielding $\theta$.

**Corollary 4.6**

(i) *Let $\kappa$ be regular and $\lambda = \kappa^{<\kappa}$. If $\exists\bar{R}\varphi \in \Delta_1^1\mathcal{M}_{\kappa+\kappa}(\tau)$ is preserved to extensions (submodels) then it is equivalent to a determined existential (universal) sentence of $\mathcal{M}_{\lambda+\lambda}(\tau)$.*

(ii) *Let $\kappa$ be infinite. If $\exists\bar{R}\varphi \in \Delta_1^1\mathcal{L}_{\kappa+\omega}(\tau)$ is preserved to extensions (submodels) then it is equivalent to a determined existential (universal) sentence of $\mathcal{M}_{\kappa+\kappa}(\tau)$.*

*Proof:* (i) Let $\exists\bar{S}\psi$ be a negation of $\exists\bar{R}\varphi$. Then we can apply Theorem 4.5 and get the separant $\theta$. The submodels case is dual.

(ii) Follows from Theorem 4.1.

Next we shall give an application of Corollary 4.6.

**Definition 4.7** (i) If $\mathfrak{A}$ and $\mathfrak{B}$ are $\tau$-models and $f$ is a partial injection $\mathfrak{A} \to \mathfrak{B}$, then $f$ is a *partial isomorphism* if for all atomic and negated atomic $\tau$-formulas $\varphi$ holds: $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$ iff $\mathfrak{B} \models \varphi(f(a_1), \ldots, f(a_n))$, where $a_1, \ldots, a_n$ are any elements from $\mathrm{dom}(f)$.

(ii) Let $\lambda, \kappa$ be cardinals and $t$ a $\lambda, \kappa$-tree. The *Ehrenfeucht-Fraïssé game* approximated by $t$ between models $\mathfrak{A}$ and $\mathfrak{B}$, $G^t(\mathfrak{A},\mathfrak{B})$, is the following. At each move $\alpha$:

(a) player $\forall$ chooses $x_\alpha \in t$, and either $a_\alpha \in \mathfrak{A}$ or $b_\alpha \in \mathfrak{B}$;
(b) if $\forall$ chose $a_\alpha \in \mathfrak{A}$ then $\exists$ chooses $b_\alpha \in \mathfrak{B}$ else $\exists$ chooses $a_\alpha \in \mathfrak{A}$.

$\forall$ must move so that $(x_\beta)_{\beta\le\alpha}$ form a strictly increasing sequence in $t$. $\exists$ must move so that $\{(a_\beta, b_\beta) \mid \beta \le \alpha\}$ is a partial isomorphism $\mathfrak{A} \to \mathfrak{B}$. The player who first has to break the rules loses. By $G_1^t(\mathfrak{A},\mathfrak{B})$ we mean a game where $\forall$ is only allowed to choose elements in $\mathfrak{A}$.

**Definition 4.8** Suppose $t$ and $t'$ are trees. We define the game $G_{\le}(t, t')$. In this game in each round player $\forall$ first picks an element in $t$ and then $\exists$ must choose an element in $t'$. The choices of each player must form a strictly increasing sequence. If $\exists$ cannot choose his move according to rules, then $\exists$ loses, and similarly if $\forall$ cannot choose, then $\forall$ loses. We denote $t \le t'$ $(t \gg t')$ if $\exists$ $(\forall)$ has a winning strategy. It is easy to show that $t \gg t' \Rightarrow t' \le t$.

**4.9 Definition.** (i) Let $t, t'$ be trees. For simplicity we assume $t$ and $t'$ are disjoint.

The sum $t \oplus t'$ is defined as the disjoint union of $t$ and $t'$, except that the roots are identified.

The domain of the product $t'' = t \times t'$ is $\{(x,f,y) \mid x \in t', f$ a function from the predecessors of $x$ to the branches of $t, y \in t\}$. Here $(x,f,y) \le (x',f',y')$ iff either

(a) $x = x', f = f'$, and $y \le y'$, or
(b) $x < x', f \subseteq f'$, and $y \in f'(x)$.

(ii) We say that $t$ is *special* if there is a mapping $f: t \to \omega$ such that for all $x, y \in t$, if $x < y$, then $f(x) \ne f(y)$.

(iii) Let $t_Q = \{s \mid s: \alpha \to \omega$, $s$ is an injection and $\alpha < \omega_1$ is successor$\}$. Let $s \le s'$ if $s \subseteq s'$. Then it is very easy to show that $t_Q$ is special and for every special $t$ holds $t \le t_Q$.

(iv) If $t$ is a tree then by $\sigma t$ we denote the tree which consists of all initial segments of branches of $t$. It is quite easy to prove (see [3]) that $\sigma t \gg t$. Thus $\sigma t_Q$ is not a special tree.

(v) Suppose $\varphi = (t, l)$ is a sentence of $\mathcal{M}_{\lambda\kappa}$. Let $t'$ be the restriction of $t$ to those nodes $x$, for which $l(x) = \exists u$ or $\forall u$. We write that the quantifier rank $\mathrm{qr}(\varphi) = t'$. Let $\mathcal{M}^t_{\lambda\kappa} = \{\varphi \in \mathcal{M}_{\lambda\kappa} \mid \mathrm{qr}(\varphi) \leq t\}$.

Note the following easy facts. If $\exists$ has a winning strategy in $G^t(\mathfrak{A},\mathfrak{B})$, then $\mathfrak{A} \equiv \mathfrak{B}$ relative to all $\varphi \in \mathcal{M}^t_{\lambda\kappa}$. If $\theta = (t, l) \in \mathcal{M}_{\lambda\kappa}$, $\mathfrak{A} \models \theta$ and $\mathfrak{B} \models \sim\theta$, then $\forall$ has a winning strategy in $G^t(\mathfrak{A},\mathfrak{B})$. Furthermore, if $\theta$ is existential, then $\forall$ has a winning strategy in $G^t_1(\mathfrak{A},\mathfrak{B})$.

**Example 4.10** Let $\varphi = \forall u_0 \ldots u_{n<\omega} \ldots \exists u_\omega \bigwedge_{n<\omega} u_\omega \neq u_n$. Thus $\varphi$ says that a model is uncountable. Clearly, $\varphi \in \mathcal{L}_{\omega_2\omega_1}(\varnothing)$ is preserved to extensions and $\varphi$ is equivalent to a $\Delta^1_1\mathcal{L}_{\omega_2\omega}(\varnothing)$-sentence.

Let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be models of empty vocabulary, $|\mathfrak{M}_0| = \omega$ and $|\mathfrak{M}_1| = \omega_1$. Using the Ehrenfeucht–Fraïssé game $G^{t_Q}_1$ (Definition 4.7) it is easy to see $\mathfrak{M}_0 \equiv \mathfrak{M}_1$ relative to all existential $\mathcal{L}_{\omega_2\omega_1}(\varnothing)$ sentences and actually relative to all existential sentences of $\mathcal{M}^{t_Q}_{\omega_2\omega_1}(\varnothing)$.

But let $\psi$ be the following existential sentence of $\mathcal{M}^{\sigma t_Q}_{\omega_2\omega_1}(\varnothing)$:

$$\psi = \bigwedge_{x_0 \in \sigma t_Q} \exists u_0 \Bigg( \bigwedge_{x_0 < x_1 \in \sigma t_Q} \exists u_1 \Bigg( u_1 \neq u_0 \wedge \bigwedge_{x_1 < x_2} \cdots \Bigg)\Bigg).$$

It is easy to see that $\psi$ is determined. We show that $\psi$ is equivalent to $\varphi$. Clearly $\mathcal{M}_1 \models \psi$. Assume for a contradiction $\exists$ has winning strategy in $\mathfrak{M}_0 \models \psi$. Then the winning strategy of $\exists$ gives a specializing function $f: \sigma t_Q \to \omega$, a contradiction.

It is an open problem whether there are sentences of $\mathcal{L}_{\omega_2\omega_1}(\tau)$ preserved to extensions but not equivalent to existential sentences of $M^{\sigma t_Q}_{\omega_2\omega_1}(\tau)$, assuming CH.

## *5 Generalized Borel sets*

We apply our results to generalized Borel sets. It is quite straightforward to show that the following definition agrees with Halko's [2] and Väänänen's [9] topological definition of generalized Borel sets, and in the classical case $\kappa = \omega$ it agrees with the usual Borel sets. Väänänen [9] has topological proofs for the results below.

**Definition 5.1** Let $\tau$, $|\tau| \leq \kappa$, be a vocabulary and $C = \{c_\alpha \mid \alpha < \kappa\}$ a set of new constants. Let $\mathfrak{N}_\kappa(\tau) = \{\mathfrak{M} \mid \mathfrak{M}$ a $\tau$-model and $\|\mathfrak{M}\| = \kappa\}$. If $\mathfrak{M} \in \mathfrak{N}_\kappa(\tau)$ then $\mathfrak{M}_C$ is a $\tau \cup C$-model such that $\mathfrak{M}_C \restriction \tau = \mathfrak{M}$ and $c_\alpha^{\mathfrak{M}_C} = \alpha$ for all $\alpha < \kappa$. Suppose $\varphi \in \mathcal{M}_{\kappa+\kappa}(\tau \cup C)$. Let

$$B_\varphi = \{\mathfrak{M} \in \mathfrak{N}_\kappa(\tau) \mid \mathfrak{M}_C \models \varphi\}.$$

We say that $B_\varphi$ is a *Borel set* in $\mathfrak{N}_\kappa(\tau)$. We denote the complement $\mathfrak{N}_\kappa(\tau) - B$ by $\neg B$. Suppose $\exists\bar{R}\varphi$ is a $\Sigma^1_1\mathcal{M}_{\kappa+\kappa}(\tau \cup C)$-sentence. Let

$$A_{\exists\bar{R}_\varphi} = \{\mathfrak{M} \in \mathfrak{N}_\kappa(\tau) \mid \mathfrak{M}_C \models \exists\bar{R}\varphi\}.$$

Then we call $A_{\exists\bar{R}\varphi}$ a $\Sigma^1_1$-set. If $A$ and $\neg A$ are $\Sigma^1_1$, then we say that $A$ is $\Delta^1_1$.

Let $\varphi_C$ denote the sentence $(\forall u \bigvee_{\alpha<\kappa} u = c_\alpha) \wedge (\bigwedge_{\alpha\neq\beta<\kappa} c_\alpha \neq c_\beta)$.

**Separation Theorem for $\Sigma^1_1$-sets 5.2** *Assume $\kappa$ regular and $\kappa^{<\kappa} = \kappa$. If $A_{\exists\bar{R}\varphi} \cap A_{\exists\bar{S}\psi} = \varnothing$, then there is $\theta$ such that $A_{\exists\bar{R}\varphi} \subseteq B_\theta$ and $A_{\exists\bar{S}\psi} \subseteq B_{\sim\theta}$.*

*Proof:* Let $\theta$ be the separant of $\varphi_C \wedge \varphi$ and $\psi$ from Theorem 3.8.

**Corollary 5.3** *Assume $\kappa$ regular and $\kappa^{<\kappa} = \kappa$. If $A_{\exists\bar{R}\varphi}$ is $\Delta^1_1$ then there is $\theta$ such that $A_{\exists\bar{R}\varphi} = B_\theta$ and $\neg A_{\exists\bar{R}\varphi} = B_{\sim\theta}$.*

***6 Counterexamples to separation*** In this section we prove negative results about relative separation of $\mathcal{L}_{\kappa+\kappa}$ in several logics. First we prove an undefinability theorem analogous to the undefinability of well-orderings in $\mathcal{L}_{\omega_1\omega}$.

**Lemma 6.1** *Let $\kappa$ be regular and let $u, t'$ be trees with no $\geq \kappa$-branches (i.e., branches of length $\geq \kappa$). If $\forall$ has a winning strategy $S$ in $G^u_1((\kappa,<), t')$, then $t = (\bigoplus_{\alpha<\kappa} \alpha) \times u \gg t'$.*

*Proof:* We show that $\forall$ has a winning strategy in $G_{\leq}(t, t')$. As $\forall$ plays $G_{\leq}$, he also simulates $G^u_1$. Suppose $S$ gives $\alpha \in \kappa$ and $x \in u$ as $\forall$'s first move in $G^u_1$. Then $\forall$ moves $(x, g, \beta)$, $\beta \leq \alpha$, where $g$ is arbitrary and $\beta$ are in some single branch of $\bigoplus_{\alpha<\kappa} \alpha$, in the first $\alpha + 1$ rounds of $G_{\leq}$.

Suppose $\exists$ does not lose yet in $G_{\leq}$ and let his moves be $y_\beta \in t'$, $\beta \leq \alpha$. We define $f(\beta) = y_\beta$ for $\beta \leq \alpha$. Now $f$ is a partial isomorphism $(\kappa,<) \to t'$ and $\forall$ lets $\exists$ move $f(\alpha) \in t'$ in $G^u_1$. Since $S$ is a winning strategy, $\forall$ can continue this way extending $f$ until $\exists$ loses in $G_{\leq}$.

**Proposition 6.2** *Assume $\kappa$ is regular and $\kappa^{<\kappa} = \kappa$. Assume that $T$ is a class of trees with no $\geq \kappa$-branches and $T$ is* RPC *in $\mathcal{M}_{\kappa+\kappa}$. Then there is a $\kappa^+, \kappa$-tree $t$ such that $t \geq t'$ for every $t' \in T$.*

*Proof:* We denote $\mu = \{<, U\}$. Suppose $T = \{(\mathfrak{A} \restriction \{<\}) \restriction U^{\mathfrak{A}} \mid \mathfrak{A} \models \psi, \mathfrak{A}$ a $\tau$-model$\}$, where $\psi \in \mathcal{M}_{\kappa+\kappa}(\tau)$, $\tau \supseteq \mu$. Let $C = \{c_\alpha \mid \alpha < \kappa\}$ be new constants and

$$\varphi = \bigwedge_{\alpha<\kappa} U(c_\alpha) \wedge \bigwedge_{\alpha<\beta<\kappa} c_\alpha < c_\beta,$$

$\varphi \in \mathcal{L}_{\kappa+\kappa}(\tau \cup C)$. Clearly, we can apply Theorem 4.4 to $\varphi$ and $\psi$ as $\tau \cup C$-sentences. Let $\theta \in \mathcal{M}_{\kappa+\kappa}(\mu)$ be existential and such that for all $\tau \cup C$-models $\mathfrak{M}$, $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \theta$, $\mathfrak{M} \models \psi \Rightarrow \mathfrak{M} \models {\sim}\theta$.

Let $\mathfrak{B}$ be an arbitrary $\tau$-model of $\psi$. Let $\mathfrak{A}$ be a $\tau \cup C$-model of $\varphi$, such that $\|\mathfrak{A}\| = \kappa$, $c^{\mathfrak{A}}_\alpha = \alpha$, $\mathfrak{A} \models \alpha < \beta$ iff $\alpha < \beta$ and $U^{\mathfrak{A}} = \kappa$. Since $\mathfrak{A} \models \theta$ and $\mathfrak{B} \models {\sim}\theta$, we know that $\forall$ has a winning strategy in $G^u_1(\mathfrak{A} \restriction \mu, \mathfrak{B} \restriction \mu)$, where $\theta = (u, l)$. Let $t' = (\mathfrak{B} \restriction \{<\}) \restriction U^{\mathfrak{B}}$. Then $\forall$ has a winning strategy in $G^u_1((\kappa,<), t')$ and by Lemma 6.1 $t = (\bigoplus_{\alpha<\kappa} \alpha) \times u \gg t'$.

Our version (suggested by Oikkonen) of Proposition 6.2 above is slightly stronger than Hyttinen's [3] corresponding result. Hyttinen's version says that there is $t$ such that for all $t' \in T$, $t \not\leq t'$.

**Proposition 6.3**

(i) *Assume that $\kappa$ is regular and $\kappa^{<\kappa} > \kappa$. Then $\mathcal{M}_{\kappa+\kappa}$ does not allow separation for $\mathcal{L}_{\kappa+\kappa}$.*

(ii) *Assume $\kappa$ regular and $\kappa^{<\kappa} = \kappa$. Then for no $\kappa^+, \kappa$-tree $t$, $\mathcal{M}^t_{\kappa+\kappa}$ allows separation for $\mathcal{L}_{\kappa+\omega}$.*

*Proof:* (i) Let $\exists\bar{R}\varphi \in \Sigma_1^1\mathcal{L}_{\kappa+\kappa}(\emptyset)$ be a sentence such that $\mathfrak{M} \models \exists\bar{R}\varphi$ iff $\lambda^{<\kappa} \leq \lambda$, where $|\mathfrak{M}| = \lambda$. Let $\exists\bar{S}\psi \in \Sigma_1^1\mathcal{L}_{\kappa+\kappa}(\emptyset)$ be a sentence such that $\mathfrak{M} \models \exists\bar{S}\psi$ iff $|\mathfrak{M}| = \kappa$. By our assumption $\exists\bar{R}\varphi$ and $\exists\bar{S}\psi$ determine disjoint classes of $\emptyset$-models PC in $\mathcal{L}_{\kappa+\kappa}$, but using Ehrenfeucht–Fraïssé games we trivially see that these cannot be separated by an $\mathcal{M}_{\kappa+\kappa}(\emptyset)$-sentence.

(ii) By Tuuri [8] in this case there exist $\tau$-models $\mathfrak{A},\mathfrak{B}$, such that $|\mathfrak{A}| = |\mathfrak{B}| = \kappa$, $\mathfrak{A} \not\cong \mathfrak{B}$, and $\exists$ has a winning strategy in $G^t(\mathfrak{A},\mathfrak{B})$. Let $\exists\bar{R}\varphi$ and $\exists\bar{S}\psi$ be $\Sigma_1^1\mathcal{L}_{\kappa+\omega}(\tau)$-sentences (describing the diagrams) which characterize $\mathfrak{A}$ and $\mathfrak{B}$ up to isomorphism. If $\theta \in \mathcal{M}_{\kappa+\kappa}^t(\tau)$, then $\mathfrak{A} \models \theta$ iff $\mathfrak{B} \models \theta$. Thus there cannot be a separant in $\mathcal{M}_{\kappa+\kappa}^t$.

Next we prove the consistency of a situation where $\mathcal{M}_{\kappa+\kappa}^n$ (see Definition 1.9) does not allow separation for $\mathcal{L}_{\kappa+\kappa}$, though $\kappa^{<\kappa} = \kappa$.

Let $\kappa > \omega$ be regular. If $A \subseteq \kappa$, then by $t(A)$ we denote the tree of all closed increasing sequences of length $< \kappa$ of elements of $A$. By an $\omega$-cub subset of $\kappa$ we mean a set $A$ which is unbounded and closed under supremums of countable subsets of $A$. These notions are defined in the same way for any well-ordering of type $\kappa$.

Let $\varphi_\kappa$ be a sentence of $\mathcal{L}_{\kappa+\kappa}$ which says that $<$ well-orders the universe of a model and the order type is $\kappa$ and $P$ and $Q$ are complementary unary relations in the $\omega$-cofinal elements of the universe. Let $\rho(P)$ be the sentence

$$\forall u_0 \exists v_0 \ldots \forall u_{n<\omega} \exists v_{n<\omega} \ldots \exists v_\omega$$
$$\left[ \bigwedge_{n<\omega} v_n > u_n \wedge \bigwedge_{n<\omega} v_\omega > v_n \wedge \left( \forall u < v_\omega \bigvee_{n<\omega} v_n > u \right) \wedge P(v_\omega) \right].$$

It is easy to prove that if $\mathfrak{M} \models \varphi_\kappa$, then $\mathfrak{M} \models \rho(P)$ iff $P^{\mathfrak{M}}$ contains an $\omega$-cub subset.

**Theorem 6.4** (see [6]) *Assume $\kappa = \lambda^+$, $\lambda$ regular, $\lambda^{<\lambda} = \lambda$, and $2^\lambda = \kappa$. Then there is a forcing extension which preserves all cardinals and in the forcing extension $2^\lambda = \kappa$ and for all $\kappa^+, \kappa$-trees $t$ there is stationary $A \subseteq \{\alpha \in \kappa \mid \mathrm{cf}(\alpha) = \omega\}$, such that $B = \{\alpha \in \kappa \mid \mathrm{cf}(\alpha) = \omega\} - A$ is stationary and $t(\kappa - A) \not\leq t$ and $t(\kappa - B) \not\leq t$.*

**Proposition 6.5** *Let $\tau = \{P, Q, <\}$. In the forcing extension of Theorem* 6.4 *the $\mathcal{M}_{\kappa+\kappa}(\tau)$-sentences $\varphi = \rho(P) \wedge \varphi_\kappa$ and $\psi = \rho(Q) \wedge \varphi_\kappa$ do not have a separant in $\mathcal{M}_{\kappa+\kappa}^n(\tau)$.*

*Proof:* Note that in the extension $\kappa^{<\kappa} = \kappa$. Clearly $\varphi$ and $\psi$ do not have a common $\tau$-model. Assume for a contradiction $\theta \in \mathcal{M}_{\kappa+\kappa}^n(\tau)$ is a separant. Let

$$T_1 = \{t(\|\mathfrak{M}\| - P^{\mathfrak{M}}) \mid \mathfrak{M} \text{ a } \tau\text{-model and } \mathfrak{M} \models \theta \wedge \varphi_\kappa\}$$

and

$$T_2 = \{t(\|\mathfrak{M}\| - Q^{\mathfrak{M}}) \mid \mathfrak{M} \text{ a } \tau\text{-model and } \mathfrak{M} \models \neg\theta \wedge \varphi_\kappa\}.$$

It is not hard to see that both $T_1$ and $T_2$ are RPC in $\mathcal{M}_{\kappa+\kappa}$. Thus also $T = T_1 \cup T_2$ is RPC in $\mathcal{M}_{\kappa+\kappa}$.

If $t \in T_1$ then $t$ cannot have a $\kappa$-branch because then $Q^{\mathfrak{M}}$ would contain an $\omega$-cub subset and $\mathfrak{M} \models \psi \wedge \varphi_\kappa$. Similarly for $T_2$. Let $t$ be an arbitrary $\kappa^+, \kappa$-tree.

Let $A, B$ be from Theorem 6.4. Now it is easy to see that either $t(\kappa - A) \in T_1$ or $t(\kappa - B) \in T_2$. Thus $T$ contains a tree $t'$ such that $t' \not\leq t$. This contradicts Proposition 6.2.

By Lemma 3.7 $\mathrm{Mod}^{\tau}(\varphi)$ and $\mathrm{Mod}^{\tau}(\psi)$ in Proposition 6.5 are PC in $\mathcal{L}_{\kappa+\kappa}$, and they cannot be separated by any class EC in $\mathcal{M}^{n}_{\kappa+\kappa} \equiv \Delta\mathcal{L}_{\kappa+\kappa}$. So we get the following corollary.

**Corollary 6.6** *Let $\tau$, $\varphi$, and $\psi$ be as in Proposition* 6.5. *In the forcing extension of Theorem* 6.4:
(i) $\varphi, \psi \in \mathcal{M}_{\kappa+\kappa}(\tau)$ *do not have a negation in* $\mathcal{M}_{\kappa+\kappa}(\tau)$;
(ii) $\mathcal{M}_{\kappa+\kappa}$ *allows separation for* $\mathcal{L}_{\kappa+\kappa}$;
(iii) $\mathcal{M}^{n}_{\kappa+\kappa}$ *does not allow separation for* $\mathcal{L}_{\kappa+\kappa}$;
(iv) $\Delta\mathcal{L}_{\kappa+\kappa}$ *does not allow separation for* $\mathcal{L}_{\kappa+\kappa}$.

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